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Master Thesis

Abstract Tangles as an Inverse Limit, and a Tangle Compactification for Topological Spaces

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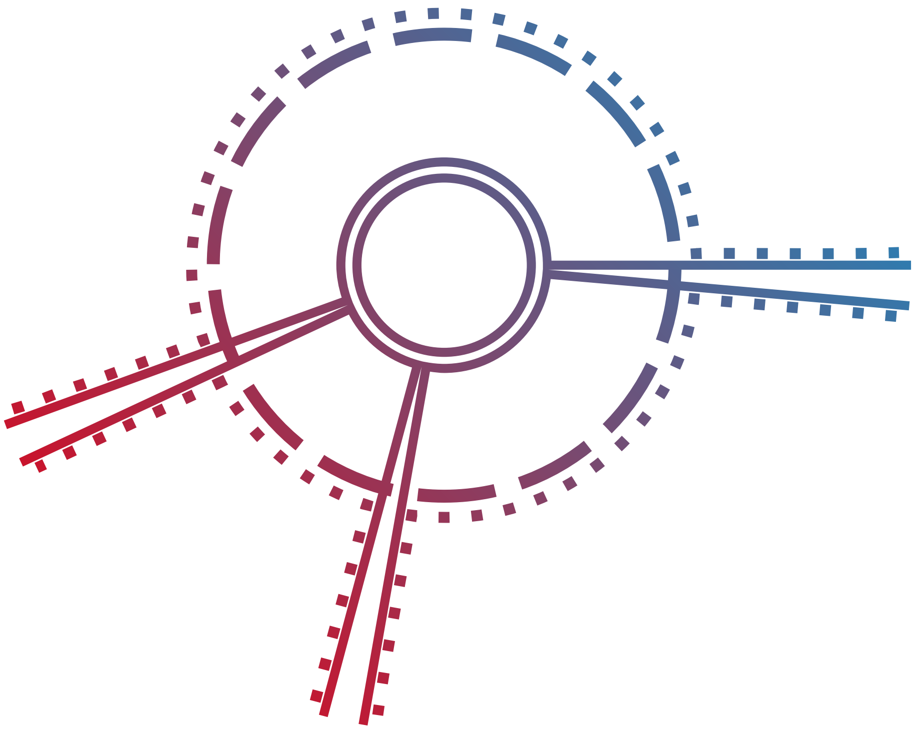

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1 Introduction

The notion of a k-*tangle* was first introduced by Robertson and Seymour [9] as an orientation of the vertex separations of a graph. The idea was that a structure, such as a k-block, that could not be broken up by deleting fewer than k vertices would lie on one side of every separation of order $< k$. It would thus induce a k-tangle, a consistent orientation of the low order vertex separations.

While the original interest was in tree decompositions of finite graphs, tangles have since proven useful for infinite graphs: One structure of this type are the *directions* in a graph G, a consistent choice of one component of $G - S$ for every finite set of vertices S. These directions correspond precisely to the ends of a graph [5], and induce $\aleph_0$-tangles.

The space $|G|$ obtained by adding the ends of a graph G as compactification points is extensively studied in topological infinite graph theory [3]. When applied to locally finite graphs, it is tantamount to a compactification introduced for more general topological spaces by Hans Freudenthal [5]. And while the topological space of a graph together with its ends still is useful for graphs that are not locally finite, it is not compact. The construction of the *tangle compactification* of a graph in [2] mended this by not only adding ends, but all the $\aleph_0$-tangles in the graph as compactification points. Moreover it characterises the tangles not induced by an end as non-principal ultrafilters on the set of components of $G - S$ for some finite S. Ends on the other hand are not characterised by a single ultrafilter but a choice of one principal ultrafilter for every such S, as given by the corresponding direction.

In Section 4 we attempt to adapt this local characterisation of tangles, so that it can be expressed solely in the language of *abstract separations* without mentioning vertices. This lays some groundwork towards the question posed by Diestel in [2], whether the distinction between tangles induced by an end and those characterised by non-principal ultrafilters could be applied to this more abstract setting. This might give an abstract notion of an end of an abstract separation system. If this were to be accomplished, some of the work on ends in graphs might be applicable to other structures such as matroids.

In Section 2, we will first give an introduction to the terminology and prove a key utility lemma about which sets extend to tangles. In Section 3, we will then generalise the tangle compactification to arbitrary topological spaces, in such a way that the tangle compactification of graphs from [2] is a special case. We will show that this construction yields the Freudenthal compactification if the space we start with is locally compact and Hausdorff. Finally in Section 4 we will, after an explanation of Diestel's construction [2], emulate the representation of the tangles as an inverse limit for certain abstract separations systems.

2 Tangles in a large universe

What follows is a concise introduction to the terminology needed for our purpose. For a more exhaustive introduction to abstract separation systems refer to [1], from where the definitions given here are taken.

2.1 Universes of separations

A *separation system* is a partially ordered set $(\vec{S}, \leqslant)$ of *(orientated) separations* together with an order-reversing involution *. A subset $\vec{S'} \subseteq \vec{S}$ is a *separation system in* $\vec{S}$ if it is closed under the involution. We will denote the elements of $\vec{S}$ as $\vec{r}, \vec{s}, \vec{t}$ etc. and use $\overleftarrow{s}$ to mean $\vec{s}^*$. We write s for $\{\vec{s}, \overleftarrow{s}\}$ and call $\overleftarrow{s}$ and $\vec{s}$ the *orientations* of the (unorientated) *separation* s. The set of all unorientated separations of $\vec{S}$ is denoted as S. We say that two separations (orientated or unorientated) are *nested* if they have orientations that are comparable, otherwise they are said to *cross*. We refer to both orientated and unorientated separations as just 'separations' in cases where the meaning is clear.

A separation $\vec{s} \in \vec{S}$ is *small* if $\vec{s} \leqslant \overleftarrow{s}$, and *co-small* if $\vec{s} \geqslant \overleftarrow{s}$. The set of all small separations of $\vec{S}$ is denoted as $\vec{S}^-$, the set of all co-small separations as $\vec{S}^+$. Note, that $\vec{S}^-$ is downward closed, and likewise $\vec{S}^+$ is upward closed. If a separation is both small and co-small we call it *degenerate*.

A *universe* $(\vec{U}, \leqslant, ^*, \wedge, \vee)$ of separations is a separation system in which every pair $\vec{r}, \vec{s} \in \vec{U}$ of separations has an infimum (or *meet*) $\vec{r} \wedge \vec{s}$ and a supremum (or *join*) $\vec{r} \vee \vec{s}$, making it a lattice. Since * is order-reversing these satisfy DeMorgan's laws

$$(\vec{r} \vee \vec{s})^* = \overleftarrow{r} \wedge \overleftarrow{s} \text{ and } (\vec{r} \wedge \vec{s})^* = \overleftarrow{r} \vee \overleftarrow{s}.$$

A separations system $\vec{U'} \subseteq \vec{U}$ in a universe $\vec{U}$ is a *subuniverse* of $\vec{U}$ if, additionally, it is closed under taking suprema and infima. By DeMorgan's law, if a subset $\vec{U'} \subseteq \vec{U}$ is closed under involution and taking suprema it is also closed under taking infima. Given a finite set $S \subseteq \vec{U}$ we write $\bigvee S$ for the supremum, $\bigwedge S$ for the infimum of S, and $S^* := \{\overleftarrow{s} \mid \vec{s} \in S\}$ as shorthand. Note that suprema of infinite subsets need not always exist, in fact in many cases they won't.

Lemma 1. *Let $\vec{U}$ be a universe of separations and $\vec{s} \in \vec{U}$. Then $\vec{s} \wedge \overleftarrow{s}$ is small.*

Proof. This is a simple calculation: $\vec{s} \wedge \overleftarrow{s} \leqslant \vec{s} \leqslant \overleftarrow{s} \vee \vec{s} = (\vec{s} \wedge \overleftarrow{s})^*$. □

The prototypical universe which motivated these definitions is the universe $\vec{S_\omega}(G)$ of the *finite order vertex separations* of a graph $G = (V, E)$. Its orientated separations are the pairs (A, B) of sets $A, B \subseteq V$, where $A \cap B$ is finite and no edge runs between $B \setminus A$ and $A \setminus B$. We understand the separation (A, B) to be *pointing towards* the *right side* B, and away from the *left side* A. The involution is defined as $(A, B)^* := (B, A)$ and the partial order is $(A, B) \leqslant (C, D) :\Leftrightarrow A \subseteq C, B \supseteq D$. The operations turning $\vec{S_\omega}(G)$ into a lattice are $(A, B) \vee (C, D) = (A \cup C, B \cap D)$ and $(A, B) \wedge (C, D) = (A \cap C, B \cup D)$.

More generally, given just a set X, we refer to the universe of the pairs (A, B) where $A \cup B = X$ with the operations and partial order as above as the *universe of separations* *of the set* X. The universe of finite order vertex separations of a graph G is a subuniverse of the universe of separations of its vertex set $V(G)$. Another notable subuniverse of the universe of separations of a set X, is the universe of *finite order separations of the set* X: It contains only those pairs (A, B) where the intersection $A \cap B$ is finite.

The small separations of the universe of separations of the set X are precisely the separations of the form (A, X) where $A \subseteq X$. This is easy to see: Since $A \subseteq X$ and, hence, also $X \supseteq A$ we have $(A, X) \leqslant (X, A) = (A, X)^*$. For the converse let (A, B) be a separation of the set X and $(A, B) \leqslant (B, A)$. Then $A \subseteq B$ and thus, $B = A \cup B = X$.

2.2 Tangles, profiles and large universes

For $\vec{s} \leqslant \vec{t}$ we say that $\vec{s}$ *points towards* t and its orientations and that $\overleftarrow{s}$ *points away from* t and its orientations. A separation system $\vec{S} \subseteq \vec{U}$ in a universe $\vec{U}$ is called *submodular* if for every $\vec{s}, \vec{t} \in \vec{S}$ at least one of $\vec{s} \vee \vec{t}$ and $\vec{s} \wedge \vec{t}$ is in $\vec{S}$. A subset $O \subseteq \vec{S}$ of a separation system is called a *partial orientation* of S if it is antisymmetric, i.e., if for every $s \in S$ there is at most one of its orientations in O. A partial orientation O is an *orientation* of S if for every $s \in S$ exactly one of its orientations is in O. A partial orientation O is called *consistent* if for $\vec{s} \leqslant \vec{t}$ we don't have both $\overleftarrow{s} \in O$ and $\vec{t} \in O$, unless $\overleftarrow{s} = \vec{t}$, i.e., no two distinct orientated separations in O point away from each other. If O is an orientation of S, this is equivalent to saying that it is closed under $\geqslant$.

For a set $\mathcal{F}$ of sets of orientated separations an $\mathcal{F}$*-tangle* of a separation system $\vec{S}$ is a consistent orientation of S including no subset $F \in \mathcal{F}$. An abstract *tangle* of a separation system in a universe $\vec{S} \subseteq \vec{U}$ is a $\mathcal{T}$-tangle of $\vec{S}$ where $\mathcal{T}$ is the set of all subsets of $\vec{U}$ of size at most 3 whose supremum is co-small. Original research into abstract tangles primarily covered $\mathcal{T}_*$-tangles, where $\mathcal{T}_*$ is the set of only the *stars* in $\mathcal{T}$, that is to say, only those sets σ of separations that all point towards each other: $\vec{s} \leqslant \overleftarrow{t}$ for all distinct $\vec{s}, \vec{t} \in \sigma$. For the most part of this thesis, we will work with tangles of a universe $\vec{U}$, i.e., in the above $\vec{S} = \vec{U}$.

For the universes $\vec{U}$ considered in this thesis the $\mathcal{T}$-tangles of $\vec{U}$ are precisely its $\mathcal{T}_*$-tangles, this is the subject of Proposition 5. A universe $\vec{U}$ is *distributive*, if it is so as a lattice, i.e., for all $\vec{r}, \vec{s}, \vec{t} \in \vec{U}$ we have $(\vec{r} \wedge \vec{s}) \vee \vec{t} = (\vec{r} \vee \vec{t}) \wedge (\vec{s} \vee \vec{t})$ and $(\vec{r} \vee \vec{s}) \wedge \vec{t} = (\vec{r} \wedge \vec{t}) \vee (\vec{s} \wedge \vec{t})$. We call a universe $\vec{U}$ *large* if it is distributive, has no degenerate element and has the property that $\vec{s} \vee \vec{t}$ is small whenever $\vec{s}, \vec{t} \in \vec{U}$ are.

For instance, the universe $\vec{U}$ of finite order separations of an infinite set X is large. Note that subuniverses of large universes are in turn large. Many known examples of universes are subuniverses of a universe of finite order separations of a set.

The idea behind this notion of 'large' is that it cannot be exhausted by finitely many small separations. For example in the universe of finite order separations of an infinite set V, the small separation have the form (A, V) where A is finite. If we think of (A, V) as 'exploring V to the extent of A', then taking only finitely many $A_1, \ldots, A_n$ will leave the largest part of V unexplored; the separation $(A_1 \cup \ldots \cup A_n, V)$ is still small. In particular, since V is infinite, this separations is not co-small.

There is another useful characterisation of the property, that the supremum of two small separations is small.

Lemma 2. *Let $\vec{U}$ be a universe of separations and $\vec{s}, \vec{t} \in \vec{U}$ small. Then $\vec{s} \vee \vec{t}$ is small if, and only if $\vec{s} \leqslant \overleftarrow{t}$, i.e., $\vec{s}$ and $\vec{t}$ point towards each other.*

Proof. If $\vec{s} \vee \vec{t}$ is small, then $\vec{s} \leqslant \vec{s} \vee \vec{t} \leqslant \overleftarrow{s} \wedge \overleftarrow{t} \leqslant \overleftarrow{t}$.

For the backwards direction, let $\vec{s} \leqslant \overleftarrow{t}$ be given. Then

$$\vec{s} \vee \vec{t} \leqslant \vec{s} \vee \overleftarrow{s} = \overleftarrow{s} \text{ and also}$$
$$\vec{s} \vee \vec{t} \leqslant \overleftarrow{t} \vee \vec{t} = \overleftarrow{t}.$$

Together this gives

$$\vec{s} \vee \vec{t} \leqslant \overleftarrow{s} \wedge \overleftarrow{t} = (\overleftarrow{s} \vee \vec{t})^*. \qquad \square$$

Equivalently one might say, that every small separation is smaller than every co-small separation.

The property that the supremum of small separations should be small is not arbitrary. We shall see in a moment, it is related to the concept of profiles. A *profile* of a separation system $\vec{S}$ in a universe $\vec{U}$ is an orientation P of $\vec{S}$ such that $(\vec{s} \vee \vec{t})^* \notin P$ whenever $\vec{s}, \vec{t} \in P$. A profile is called *regular* if it contains no co-small separation. Profiles of abstract separation systems are further discussed by Diestel in [4].

Proposition 3. *Let $\vec{S} \subseteq \vec{U}$ be a submodular separation system in a large universe. Then every $\mathcal{T}_*$-tangle of $\vec{S}$ is a profile of $\vec{S}$.*

Proof. Let $\vec{s}, \vec{t}$ be contained in a $\mathcal{T}_*$-tangle ϑ of $\vec{S}$. We have to show that if $\vec{s} \vee \vec{t}$ is in $\vec{S}$, it is in ϑ. So, we assume $\vec{s} \vee \vec{t} \in \vec{S}$. If it is the case that $(\overleftarrow{s} \wedge \vec{t})^* = \vec{t}$ we know that $\vec{s} \leqslant \vec{t}$ and thus $\vec{s} \vee \vec{t} = \vec{t} \in \vartheta$. If $(\overleftarrow{s} \wedge \vec{t})^* \neq \vec{t}$, then the consistency of ϑ gives $\overleftarrow{s} \wedge \vec{t} \in \vartheta$, since $\overleftarrow{s} \wedge \vec{t} \leqslant \vec{t} \in \vartheta$. Since ϑ is an orientation of $\vec{S}$ it must contain some orientation of $\vec{s} \vee \vec{t}$.

Now suppose $(\vec{s} \vee \vec{t})^* = (\overleftarrow{s} \wedge \overleftarrow{t}) \in \vartheta$. Since $\vec{S}$ is submodular, either $\overleftarrow{s} \wedge \vec{t}$ or $\overleftarrow{s} \vee \vec{t} = (\vec{s} \wedge \overleftarrow{t})^*$, and thus also $\vec{s} \wedge \overleftarrow{t}$, is in $\vec{S}$. Without loss of generality $\overleftarrow{s} \wedge \vec{t} \in \vec{S}$ (otherwise swap $\vec{s}$ and $\vec{t}$).

The set $\{\vec{s}, (\overleftarrow{s} \wedge \vec{t}), (\overleftarrow{s} \wedge \overleftarrow{t})\} \subseteq \vartheta$ is a star with a co-small supremum. Indeed

$$\vec{s} \vee (\overleftarrow{s} \wedge \vec{t}) \vee (\overleftarrow{s} \wedge \overleftarrow{t}) = \vec{s} \vee \Big(\overleftarrow{s} \wedge (\vec{t} \vee \overleftarrow{t})\Big) = (\vec{s} \vee \overleftarrow{s}) \wedge (\vec{s} \vee \vec{t} \vee \overleftarrow{t}),$$

which is the infimum of two co-small separations. Now ϑ includes a star with co-small supremum – a contradiction. So $(\vec{s} \vee \vec{t})^*$ can not be in ϑ and, thus, $(\vec{s} \vee \vec{t}) \in \vartheta$. $\square$

The converse, that every distributive universe without degenerate elements where every $\mathcal{T}_*$-tangle is a profile is large, is not true. With the assistance of a computer we found a counterexample of size $|\vec{U}| = 12$. The curious reader may find the example in the appendix, but it is not very enlightening.

A profile of a large universe that has a finite subset with co-small supremum, also has one of size 3.

Corollary 4. *A $\mathcal{T}_*$-tangle of a submodular separation system in a large universe includes no finite star with co-small supremum.*

As remarked earlier, the tangles of a large universe are precisely its $\mathcal{T}_*$-tangles:

Proposition 5. *Let $\vec{U}$ be a large universe. A set $\vartheta \subseteq \vec{U}$ is a $\mathcal{T}_*$-tangle of $\vec{U}$ if, and only if it is a $\mathcal{T}$-tangle of $\vec{U}$.*

Proof. The backward implication holds because $\mathcal{T}_* \subseteq \mathcal{T}$.

For the forward direction suppose a $\mathcal{T}_*$-tangle ϑ of $\vec{U}$ includes some finite subset with a co-small supremum. Let $S \subseteq \vartheta$ be such a subset of minimum size. Then the separations in S are nested: For crossing $\vec{s}, \vec{t} \in S$ we have $(\vec{s} \vee \vec{t}) \in \vartheta$ as ϑ is a profile, but then replacing $\vec{s}$ and $\vec{t}$ in S with $(\vec{s} \vee \vec{t})$ yields a smaller subset of ϑ which still has a co-small supremum. By the consistency of ϑ no two elements of S point away from each other, and by minimality of S no two separations in S are comparable. Hence S is a finite star with co-small supremum, in contradiction to Corollary 4. $\square$

We will use the following corollary of the above again and again, to prove that if a tangle ϑ of a large universe $\vec{U}$ contains two separations $\vec{s}$ and $\vec{t}$, it also contains their supremum $\vec{s} \vee \vec{t}$.

Corollary 6. *Let $\vec{S} \subseteq \vec{U}$ be a submodular separation system in a large universe. The tangles of $\vec{S}$ are regular profiles of $\vec{S}$.*

Proof. Since $\mathcal{T}_* \subseteq \mathcal{T}$, every ($\mathcal{T}$-)tangle of $\vec{S}$ is a $\mathcal{T}_*$-tangle of $\vec{S}$. By Proposition 3 they are profiles. Tangles may not contain any co-small separation, so they are regular profiles. $\square$

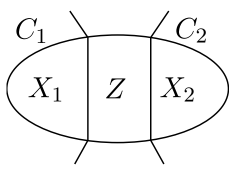
Lemma 7. *Let $\vec{U}$ be a universe. Every regular profile of $\vec{U}$ is a tangle of $\vec{U}$.*

Proof. Let P be a regular profile of $\vec{U}$. Then P includes no subset with co-small supremum: Profiles of a universe are closed under taking finite suprema, and regular profiles contain no co-small separation. A profile of $\vec{U}$ is an orientation of U, it only remains to show that P is consistent. Suppose there are $\vec{s} \leqslant \vec{t}$ with $\overleftarrow{s}, \vec{t} \in P$, witnessing its inconsistency. Then $\overleftarrow{s} \vee \vec{t} \in P$, but $\overleftarrow{s} \vee \vec{t} \geqslant \overleftarrow{s} \vee \vec{s}$, which is co-small by Lemma 1, so $\overleftarrow{s} \vee \vec{t} \in P$ is co-small, contradicting the regularity. $\square$

2.3 The Extension Lemma

Suppose that some finitely many separations $\vec{s_1}, \dots, \vec{s_n} \in \vec{U}$ represent between them all the tangles of a large universe $\vec{U}$, i.e., that every tangle of $\vec{U}$ contains one of the $\vec{s_i}$. Then, clearly, $\vec{s} = \vec{s_1} \wedge \dots \wedge \vec{s_n}$ lies in every tangle. Does this imply that $\vec{s}$ is small?

More fundamentally, recall that tangles, by definition, contain no co-small separations. Thus if $\vec{s} \in \vec{U}$ is small, then $\vec{s}$ lies in every tangle of $\vec{U}$. Can there be separations $\vec{s} \in \vec{U}$ which are not small but still lie in every tangle of $\vec{U}$?



Perhaps surprisingly, the answer is yes. As an example, the figure above illustrates finite order vertex separations of a connected infinite graph $G = (V, E)$, say. C_1, C_2, X_1, X_2 and Z for a partition of V into disjoint sets, X_1, X_2 and Z are finite. C_1 and C_2 are the vertex sets of the components of $G - (X_1 \cup Z \cup X_2)$. Edges only run within the sets and between sets, which are drawn adjacently. The separation

$$(V \setminus Z, X_1 \cup Z \cup X_2)$$

is not co-small, but it also is not in any tangle of $\vec{S}_\omega(G)$: Every tangle ϑ of $\vec{S}_\omega(G)$ must contain all the small separations of $\vec{S}_\omega(G)$. In particular, $(Z, V) \in \vartheta$. The universe $\vec{S}_\omega(G)$ is large, so ϑ is closed under taking finite suprema. Now, if $(V \setminus Z, X_1 \cup Z \cup X_2) \in \vartheta$ then so is

$$(V \setminus Z, X_1 \cup Z \cup X_2) \vee (Z, V) = (V, X_1 \cup Z \cup X_2),$$

which is co-small, contradicting the fact, that ϑ is a tangle of $\vec{S}_\omega(G)$.

However, there is an only slight weakening of this converse that does holds in every large universe $\vec{U}$: every separation $\vec{s}$ that lies in every tangle of $\vec{U}$ is *nearly small* in the sense, that there exists a co-small separation $\vec{t} \in \vec{U}$ such that $\vec{s} \wedge \vec{t}$ is small. If a separation $\vec{s}$ is nearly small, we say that $\overleftarrow{s}$ is *nearly co-small.*

The question of whether a given separation $\vec{s}$ lies in every tangle can be rephrased as asking whether the set $\{\vec{s}\} \subseteq \vec{U}$ extends to a tangle of $\vec{U}$. Our main goal in this section is to prove a key lemma which answers this question more generally. First, we prove a sufficient condition for when a subset of $\vec{U}$ is a tangle of $\vec{U}$.

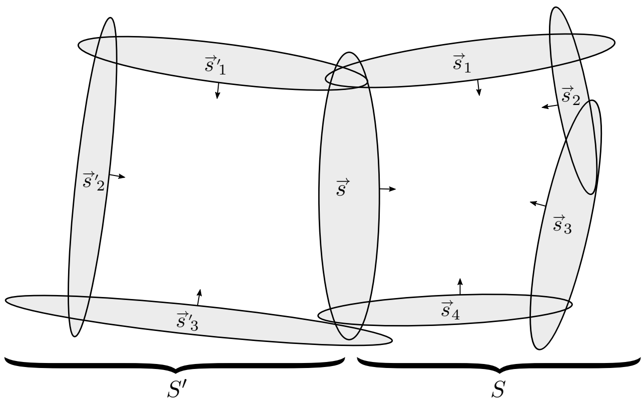
Lemma 8. *Let $\vec{U}$ be a universe of separations. A set $\vartheta \subseteq \vec{U}$ which meets every separation $s = \{\vec{s}, \overleftarrow{s}\} \in U$ is a tangle of $\vec{U}$ if it includes no subset of at most three separations that has a co-small supremum.*

Proof. By the definition of a tangle all we need to show is that ϑ is a consistent partial orientation of U. Recall that $\{\vec{s}, \overleftarrow{s}\}$ is a set with co-small supremum, so ϑ is indeed an orientation of U. Furthermore, for any $\vec{s}, \vec{t} \in \vartheta$ that witness the inconsistency, the set $\{\vec{s}, \vec{t}\}$ has co-small supremum and therefore cannot be a subset of ϑ. Thus, ϑ is consistent. □

Lemma 9 (Extension Lemma)**.** *Let $\vec{U}$ be a large universe and let $\vartheta \subseteq \vec{U}$ be a set with $\vec{U}^- \subseteq \vartheta$ such that no finite subset of ϑ has a co-small supremum. Then ϑ can be extended to a tangle $\vartheta' \supseteq \vartheta$ of $\vec{U}$.*

Proof. By a routine application of Zorn's Lemma there is a maximal set $\vartheta' \subseteq \vec{U}$ extending ϑ that has no finite subset with a co-small supremum. By Lemma 8 it suffices to show that ϑ' meets every $s \in U$.

Suppose, for a contradiction, that $\vec{s}, \overleftarrow{s} \notin \vartheta'$ for some $s \in U$. Then, by the maximality of ϑ', there are finite subsets $S, S' \subseteq \vartheta'$ such that $S \cup \{\vec{s}\}$ and $S \cup \{\overleftarrow{s}\}$ respectively have co-small suprema.



Since $\vec{s} \wedge \overleftarrow{s}$ is small, we have $(\vec{s} \wedge \overleftarrow{s}) \in \vartheta'$. Hence $S \cup S' \cup \{\vec{s} \wedge \overleftarrow{s}\}$ is a finite subset of ϑ', its supremum

$$\vec{r} := \bigvee (S \cup S' \cup \{\vec{s} \wedge \overleftarrow{s}\}) = \bigvee (S \cup S' \cup \{\vec{s}\}) \wedge \bigvee (S \cup S' \cup \{\overleftarrow{s}\})$$

cannot be co-small. But, as displayed above, $\vec{r}$ is the infimum of two co-small separations $\bigvee (S \cup S' \cup \{\vec{s}\})$ and $\bigvee (S \cup S' \cup \{\overleftarrow{s}\})$. Since $\vec{U}$ is large, and these separations are co-small by assumption, so must be $\vec{r}$ – a contradiction. □

If $\vec{U}$ is large, the assumptions made in the premise of the Extension Lemma are both necessary: a tangle of $\vec{U}$ must contain every small separation of $\vec{U}$, and by Corollary 4 no tangle of $\vec{U}$ includes a finite subset with co-small supremum. In particular we have:

Corollary 10. *Every large universe $\vec{U}$ has a tangle.*

Proof. If $\vec{r} = \bigvee \mathcal{S}$ is small it cannot be co-small: $\vec{r}$ would then be degenerate, which is ruled out in the definition of 'large'. So, the set $\vec{S}^-$ of the small separations of a large universe includes no subset with a co-small supremum. Hence, it satisfies the premise of the Extension Lemma. By the Extension Lemma $\vec{S}^-$ extends to a tangle of $\vec{U}$. □

Coming back to our original question, we can now phrase the answer as a corollary.

Corollary 11. *Let $\vec{U}$ be a large universe and $\Theta = \Theta(\vec{U})$ the set of tangles of $\vec{U}$. Write $\Theta(\vec{s})$ for the set of all tangles of $\vec{U}$ that contain $\vec{s}$, that is $\Theta(\vec{s}) := \{\vartheta \in \Theta \mid \vec{s} \in \vartheta\}$. Let $\vec{s_1}, \dots, \vec{s_n} \in \vec{U}$ be separations such that $\bigcup_i \Theta(\vec{s_i}) = \Theta$. Then there exists a small separation $\vec{t}$ such that $\vec{t} \vee \bigvee_i \overleftarrow{s_i}$ is co-small i.e. $\bigvee_i \overleftarrow{s_i}$ is nearly co-small.*

Proof. Suppose not. Then in particular $\bigvee_i \overleftarrow{s_i}$ is not co-small, and neither is $\overleftarrow{s_i} \vee \vec{t}$ for any small $\vec{t} \in \vec{U}$ and $i \in \{1, \dots, n\}$. Since $\vec{U}$ is large any suprema $\vec{t}$ of finite subsets of $\vec{U}^-$ are also small. Hence, $\vec{U}^- \cup \{\overleftarrow{s_1}, \dots, \overleftarrow{s_n}\}$ has no finite subset with a co-small supremum. By the Extension Lemma it extends to a tangle ϑ of $\vec{U}$. But then $\vartheta \notin \Theta(\vec{s_i})$ for every $i \in \{1, \dots, n\}$ – a contradiction. □

3 A tangle compactification for topological spaces

Diestel [2] used the tangles of the finite order vertex separations of an infinite graph G, called the $\aleph_0$-tangles in G, to compactify G viewed as a 1-complex. It turns out that this construction can be done with just the topological space of the 1-complex, without considering vertices. In fact, we can generalise the construction to arbitrary topological spaces. Instead of finite sets of vertices we consider compact subsets of the topological space; for a graph viewed as a 1-complex these meet only finitely many edges and vertices, as we will see.

Given a topological space X let $\vec{U}_T(X)$ be the universe of all pairs of closed sets (Y, Z) where $Y \cup Z = X$ and $Y \cap Z$ is compact, and where $\leqslant, ^*, \wedge$ and $\vee$ are as for the separations of the set X. We call this the *universe of compact intersection separations* on X.

Proposition 12. *The universe of compact intersection separations $\vec{U}_T(X)$ on a topological space X is a subuniverse of the universe of separations of the set X. If X is compact, then $\vec{U}_T(X)$ is a large universe of separations.*

Proof. It is plain to see that $\vec{U}_T(X)$ is closed under involution. To show that $\vec{U}_T(X)$ is closed under suprema take $(Y, Z), (Y', Z') \in \vec{U}_T(X)$. Since finite intersections and finite unions of closed sets are closed, the sets $Y \cup Y', Z \cap Z'$ and $(Y \cup Y') \cap (Z \cap Z')$ are closed. Since finite unions of compact sets are compact, the set $(Y \cap Z) \cup (Y' \cap Z')$ is compact. Since closed subsets of compact sets are compact, the set $(Y \cup Y') \cap (Z \cap Z') \subseteq (Y \cap Z) \cup (Y' \cap Z')$ is also compact. Therefore $(Y, Z) \vee (Y', Z') = (Y \cup Y', Z \cap Z')$ is a pair of closed sets with compact intersection, it is thus an element of $\vec{U}_T(X)$.

It follows that $\vec{U}_T(X)$ is a subuniverse of the universe of separations of the set X. As a subuniverse it inherits the distributivity and the property, that the supremum of small separations is small. If X is not compact then $(X, X) \notin \vec{U}_T(X)$, and (X, X) is the only degenerate separation in the universe of separations of the set X. Thus, if X is not compact then $\vec{U}_T(X)$ has no degenerate element, so it is large. $\square$

Note that the small separations of $\vec{U}_T(X)$ are precisely those of the form (C, X) where C is compact. If X carries the discrete topology $\vec{U}_T(X)$ coincides with the universe of finite order separations of the set X.

Take some fixed topological space X and call its universe of compact intersection separations $\vec{U} := \vec{U}_T(X)$. We now consider its tangles $\Theta := \Theta(\vec{U})$. We view Θ as a topological space endowed with the subspace topology of $2^{\vec{U}}$. The topology on $2^{\vec{U}}$ is the product topology

$$2^{\vec{U}} = \prod_{\vec{s} \in \vec{U}} \{0, 1\},$$

where $\{0, 1\}$ carries the discrete topology. A subbasis of this topology is formed by the sets

$$\underbrace{\{S \subseteq \vec{U} \mid \vec{s} \in S\}}_{= \pi_{\vec{s}}^{-1}(1)}, \underbrace{\{S \subseteq \vec{U} \mid \vec{s} \notin S\}}_{= \pi_{\vec{s}}^{-1}(0)} \text{ for } \vec{s} \in \vec{U}.$$

Here, $\pi_{\vec{s}}$ is the projection to the $\vec{s}$-component. The intersections of these sets with Θ form a subbasis of the subspace $\Theta \subseteq 2^{\vec{U}}$. Moreover, since tangles of $\vec{U}$ are profiles of $\vec{U}$ by Corollary 6, these sets form a basis: First observe that

$$\pi_{\vec{s}}^{-1}(0) \cap \Theta = \{\vartheta \in \Theta \mid \vec{s} \notin \vartheta\} = \{\vartheta \in \Theta \mid \overleftarrow{s} \in \vartheta\} = \pi_{\overleftarrow{s}}^{-1}(1) \cap \Theta,$$

since tangles of $\vec{U}$ are orientations of $\vec{U}$. Since tangles of $\vec{U}$ are profiles and downward closed,

$$\pi_{\vec{s}}^{-1}(1) \cap \pi_{\vec{t}}^{-1}(1) \cap \Theta = \{\vartheta \in \Theta \mid \vec{s}, \vec{t} \in \vartheta\} = \{\vartheta \in \Theta \mid \vec{s} \vee \vec{t} \in \vartheta\} = \pi_{\vec{s} \vee \vec{t}}^{-1}(1) \cap \Theta.$$

This gives us that the sets of the form $\Theta(Y, Z) := \{\vartheta \in \Theta \mid (Y, Z) \in \vartheta\}$ form a basis of Θ.

With this topology Θ is a totally-disconnected Hausdorff space. Indeed, for any two distinct $\vartheta, \vartheta' \in \Theta$ there exists some $\{(Y, Z), (Z, Y)\} \in U$ where they disagree. Hence ϑ and ϑ' have disjoint open neighbourhoods $\Theta(Y, Z)$ and $\Theta(Z, Y)$. Moreover, since $\Theta(Y, Z)$ together with $\Theta(Z, Y)$ covers all of Θ, the topological space Θ is totally disconnected. Because $\Theta(Y, Z) = \Theta(Z, Y)^{\complement}$ the basis of Θ is one of open-and-closed sets. A topological space with such a basis is called *0-dimensional*.

By Tychonoff's Theorem $2^{\vec{U}}$ is a compact space. The subspace Θ is closed and therefore also compact: Consider $S \in 2^{\vec{U}} \setminus \Theta$. If S isn't a partial orientation we have $S \in \pi_{\vec{s}}^{-1}(1) \cap \pi_{\overleftarrow{s}}^{-1}(1) \subseteq \Theta^{\complement}$ for some $s \in U$. If S is a partial orientation but not an orientation we have $S \in \pi_{\vec{s}}^{-1}(0) \cap \pi_{\overleftarrow{s}}^{-1}(0) \subseteq \Theta^{\complement}$ for some $s \in U$. If S is an orientation but not consistent we have $S \in \pi_{\vec{s}}^{-1}(1) \cap \pi_{\vec{t}}^{-1}(1) \subseteq \Theta^{\complement}$ for some $\overleftarrow{t} \leqslant \vec{s} \in \vec{U}$. If S is a consistent orientation but not a tangle of $\vec{U}$ we have $S \in \bigcap_{\vec{s} \in F} \pi_{\vec{s}}^{-1}(1) \subseteq \Theta^{\complement}$ for some $F \in \mathcal{T}(\vec{U})$. In any case $S \in \Theta^{\complement}$ has an open neighbourhood included in $\Theta^{\complement}$. This makes Θ a closed subspace of $2^{\vec{U}}$.

We define the *tangle compactification* ϑX of X to be the topological space on $X \cup \Theta$ where we declare as basic open sets all open subsets of X as well as the sets

$$\mathcal{O}(Y, Z) := \Theta(Y, Z) \cup Y^{\complement} \text{ for } (Y, Z) \in \vec{U}.$$

In particular X is an open subspace of ϑX and Θ a compact subspace. These sets do indeed form a basis of a topological space, in fact, they are closed under finite intersections:

Lemma 13. *For all $(Y, Z), (Y', Z') \in \vec{U}$ it is true that*

$$\mathcal{O}(Y, Z) \cap \mathcal{O}(Y', Z') = \mathcal{O}(Y \cup Y', Z \cap Z')$$

Proof. If X is compact, then $(X, X) \in \vec{U}$ is a degenerate separation. Thus $\Theta = \emptyset$ and $\vartheta X = X$. Then $\mathcal{O}(Y, Z) = Y^{\complement}$ and $\mathcal{O}(Y', Z) = Y'^{\complement}$, and hence,

$$\mathcal{O}(Y, Z) \cap \mathcal{O}(Y', Z') = Y^{\complement} \cap Y'^{\complement} = (Y \cup Y')^{\complement} = \mathcal{O}(Y \cup Y', Z \cap Z'),$$

as desired.

In the case that X is not compact, the universe $\vec{U}$ is large, so every tangle of $\vec{U}$ is a profile. Hence, every tangle containing both (Y, Z) and (Y', Z') also contains $(Y \cup Y', Z \cap Z')$. Conversely, since tangles are consistent orientations, every tangle containing $(Y \cup Y', Z \cap Z')$ also contains (Y, Z) and (Y', Z'). Hence, $\Theta(Y, Z) \cap \Theta(Y', Z') = \Theta(Y \cup Y', Z \cap Z')$. What remains is simple computation:

$$\begin{aligned}\mathcal{O}(Y, Z) \cap \mathcal{O}(Y', Z') &= (\Theta(Y, Z) \cup Y^{\complement}) \cap (\Theta(Y', Z) \cup Y'^{\complement}) \\ &= (\Theta(Y, Z) \cap \Theta(Y', Z')) \cup (Y^{\complement} \cap Y'^{\complement}) \\ &= \ \Theta(Y \cup Y', Z \cap Z') \cup (Y \cup Y')^{\complement} \\ &= \ \mathcal{O}(Y \cup Y', Z \cap Z')\end{aligned}$$ □

Now, let's show, that ϑX actually is a compactification of X.

Theorem 14. *ϑX is a compact space.*

Proof. If X is compact, then $\vartheta X = X$, which is compact. So let X be non-compact. In that case $\vec{U}$ is a large universe. Let $\mathcal{V}$ be a basic open cover of X. Since Θ is a compact subspace of X there is a finite subset $\{\mathcal{O}(Y_1, Z_1), \dots, \mathcal{O}(Y_n, Z_n)\} \subseteq \mathcal{V}$ which covers Θ. By Corollary 11 the set $\vartheta X \setminus \bigcup_{i=1}^{n} \mathcal{O}(Y_i, Z_i)$ is compact. □

Furthermore X is dense in ϑX, that is, every open neighbourhood of every $\vartheta \in \vartheta X \setminus X$ meets X.

3.1 The tangle compactification of a graph

We will now consider an infinite graph G with the usual topology of a 1-complex: The edges correspond to copies of the unit interval, identified at the vertices.

Recall that $\vec{S_\omega}(G)$ is the universe of all pairs (A, B) with $A \cup B = V(G)$, where no edge of G runs between $A \setminus B$ and $B \setminus A$ and $|A \cap B| < \infty$.

In [2] Diestel extends G to a compact topological space $|G| = G \cup \Theta_G$ where Θ_G are the $\aleph_0$-tangles in G, the tangles of the universe $\vec{S_\omega}(G)$ of finite order vertex separations of the graph G. Analogously to before we use $\Theta_G(A, B)$ to mean the set of all $\aleph_0$-tangles of G containing the vertex separation (A, B).

The basic open sets on $|G|$ are the the open sets of G as well as the sets of the form

$$\mathcal{O}_G(S, \mathcal{C}) = \bigcup \mathcal{C} \cup \mathring{E}(S, \bigcup \mathcal{C}) \cup \Theta_G\Big(V(G) \setminus V(\bigcup \mathcal{C}), S \cup V(\bigcup \mathcal{C})\Big),$$

for every finite $S \subset V(G)$ and set $\mathcal{C}$ of components of $G - S$. The expression $\mathring{E}(S, \bigcup \mathcal{C})$ means the set of all inner points of edges running between S and $\bigcup \mathcal{C}$.

As [5] mentions, a compact subset of a graph viewed as a 1-complex meets only finitely many edges and vertices. This is a simple exercise on 1-complices.

Lemma 15. *Let G be a graph as a 1-complex and $Z \subseteq G$ a compact subset. Then Z meets only finitely many edges and only finitely many vertices of G.*

Proof. Take a closed set $Z \subseteq G$ that meets infinitely many vertices. We define an open cover of all of G: Around each vertex v take the open neighbourhood

$$\{v\} \cup \mathring{E}(\{v\}, V(G) \setminus \{v\}).$$

This way each vertex is covered by exactly one of the open sets. Hence, this open cover of $Z \subseteq G$ has no finite subcover.

Now take a closed set $Z \subseteq G$ that meets infinitely many edges in inner points. We construct a closed but not compact set $S \subseteq Z$ as follows: From every edge that Z meets in an inner point pick precisely one inner point. Now $S \subseteq Z$ is a set of just inner points of vertices.

We will show that S is closed, by showing that its complement is open: Let $x \in S^{\complement}$ be given. If x is an inner point of an edge e, say. Then $\mathring{e} \setminus S$ is an open neighbourhood of x. If x is a vertex, we choose an open neighbourhood of x by taking half open intervals on every incident edge e. By construction of S, every edge e can meet S in at most one point. We choose every one of these intervals small enough, that if an edge e meets S in an inner point, the interval should not contain that point.

Now the set $\{\mathring{e} \mid e \in E(G)\}$ is an open cover of S, which has no finite subcover. Hence, S is not compact and neither is Z. □

Every subgraph induced by a finite subset of vertices forms a compact subset of the 1-complex. In this sense we may consider the universe of vertex separations to be a separation system in the universe of compact intersections. More precisely the embedding

$$\iota : \vec{S}_{\omega}(G) \to \vec{U}_T(G),\ (A, B) \mapsto (G[A], G[B])$$

is an order isomorphism onto its image and commutes with the involution. Here, $G[A]$ is the subgraph induced by the set A of vertices, interpreted as a subset of G as a 1-complex. We will use this embedding to show that the tangles of $\vec{U}_T(G)$ and the tangles of $\vec{S}_{\omega}(G)$ correspond to each other.

Lemma 16. *Let ϑ be a tangle of $\vec{U}_T(G)$, then the pre-image of ϑ under ι is a tangle of $\vec{S}_{\omega}(G)$.*

Proof. Let $\vartheta' := \iota^{-1}[\vartheta]$ be said pre-image. Since ι is an order isomorphism onto its image, ϑ' is a consistent orientation of $\vec{S}_{\omega}(G)$.

Suppose some finite subset $\{(A_1, B_1), \dots, (A_n, B_n)\} \subseteq \vartheta'$ has a co-small supremum:

$$\bigvee_{i=1}^{n} (A_i, B_i) = (V(G), \bigcap_{i=1}^{n} B_i).$$

Then in ϑ we have the subset $\{\iota(A_1, B_1), \dots, \iota(A_n, B_n)\}$ with the supremum

$$\bigvee_{i=1}^{n} \iota(A_i, B_i) = \left(\bigcup_{i=1}^{n} G[A_i],\ \bigcap_{i=1}^{n} G[B_i] \right) = \left(\bigcup_{i=1}^{n} G[A_i],\ G\left[\bigcap_{i=1}^{n} B_i \right] \right).$$

The right hand side then is a subgraph induced by finite vertex set, and hence compact. This supremum is thus nearly co-small, so $\{\iota(A_1, B_1), \ldots, \iota(A_n, B_n)\}$ is not a subset of a tangle of $\vec{U_T}(G)$ – a contradiction. □

Since ι is not surjective, the image of a tangle of $\vec{S_\omega}(G)$ under ι cannot be a tangle. However, this image does extend to a tangle with the hep of the Extension Lemma, and it orients enough separations to extend uniquely:

Lemma 17. *Let ϑ be a tangle of $\vec{S_\omega}(G)$, then the image of ϑ under ι extends to a unique tangle $\widehat{\iota[\vartheta]}$ of $\vec{U_T}(G)$.*

Proof. Take $\vartheta' := \iota[\vartheta] \cup \vec{U_T^-}(G)$; we will first show that ϑ' satisfies the premise of the Extension Lemma.

Given a finite subset $F = \{(A_1, B_1), \ldots, (A_n, B_n)\} \subseteq \vartheta$ and any small separation (X, G), we need to show that $\bigvee_{i=1}^n \iota(A_i, B_i) \vee (X, G)$ is not co-small. First consider the vertex separation

$$(A, B) := \bigvee_{i=1}^{n} (A_i, B_i) = \left(\bigcup_{i=1}^{n} A_i,\ \bigcap_{i=1}^{n} B_i \right).$$

Since F is a finite subset of a tangle of $\vec{S_\omega}(G)$ the supremum (A, B) of F is not nearly co-small, so the set B of vertices is infinite.

The separation in $\vec{U_T}(G)$

$$(Y, Z) := \bigvee_{i=1}^{n} \iota(A_i, B_i) = \left(\bigcup_{i=1}^{n} G[A_i],\ \bigcap_{i=1}^{n} G[B_i] \right)$$

agrees with (A, B) on the vertices, i.e.,

$$V(Y) = \bigcup_{i=1}^{n} A_i = A \quad \text{and} \quad V(Z) = \bigcap_{i=1}^{n} B_i = B.$$

In particular Z meets infinitely many vertices, precisely those in B. Now the right hand side of $\bigvee \iota[F] \vee (X, G) = (Y \cup X, G \cap Z) = (Y \cup X, Z)$ meets infinitely many vertices. Thus it cannot be compact and $(Y \cup X, Z)$ is not co-small, as desired.

Now, by the Extension Lemma, ϑ' extends to a tangle $\hat{\vartheta}'$ of $\vec{U_T}(G)$. Let's see that it does so uniquely.

Consider $(X, Y) \in \vec{U_T}(G)$ and let S be the set of those vertices and end-vertices of those edges that $X \cap Y$ meets. Since $X \cap Y$ is compact, S is finite. Let A be the set of vertices in X and B the set of vertices in Y. We have $X \subseteq G[A \cup S]$ and $Y \subseteq G[B \cup S]$: Let some $x \in X$ be given. If x is a vertex, then $x \in A$ and thus $x \in G[A] \subseteq G[A \cup S]$. If x is an inner point of an edge vw, say, and both v and w are in X, then both v and w are in A, so $x \in G[A] \subseteq G[A \cup S]$. If at least one of v or w is not in X there must be some inner point of vw which is in $X \cap Y$. In this case $v, w \in S$ and thus $x \in G[S] \subseteq G[A \cup S]$. The same argument proves $Y \subseteq G[B \cup S]$.

The separation $(A \cup S, B \cup S)$ is a finite order vertex separation of G, so either it or its inverse lies in ϑ. In the case, that $(A \cup S, B \cup S) \in \vartheta$ we have $(G[A \cup S], G[B \cup S]) \in \vartheta'$. Then

$$(Y, X) \vee (G[A \cup S], G[B \cup S]) = (Y \cup G[A \cup S], X \cap G[B \cup S]),$$

and its right hand side is compact:

$$X \cap G[B \cup S] \subseteq G[A \cup S] \cap G[B \cup S].$$

We cannot have $(Y, X) \in \hat{\vartheta}'$, since the supremum

$$(Y, X) \vee (G[A \cup S], G[B \cup S]) \vee (G[A \cup S] \cap G[B \cup S], G)$$

is co-small, so instead $(X, Y) \in \hat{\vartheta}'$.

Symmetrically, if $(B \cup S, A \cup S) \in \vartheta$ then $(Y, X) \in \hat{\vartheta}'$. The orientation that $\{(X, Y), (Y, X)\}$ receives in $\hat{\vartheta}'$ is thus uniquely determined by ϑ. □

This gives us a bijection between $|G|$ and ϑG:

$$\Phi : |G| \to \vartheta G,\ x \mapsto \begin{cases} x, & \text{for } x \in G, \\ \widehat{\iota[x]}, & \text{the unique tangle from Lemma 17, for } x \in \Theta_G. \end{cases}$$

Proposition 18. *The bijection Φ is a homeomorphism.*

Proof. First, to prove that Φ is continuous, let a basic open subset $O \subseteq \vartheta G$ be given. If $O \subseteq G$ then $O = \Phi^{-1}(O)$ is open in $|G|$, since G is an open subspace of $|G|$. In the other case that $O = \mathcal{O}(Y, Z) = \Theta(Y, Z) \cup Y^{\complement}$, let S be the set of all vertices and end-vertices of all edges met by $Y \cap Z$, just as above. We have

$$\Theta(Y, Z) = \Theta(Y \cup G[S], Z \cup G[S]).$$

Let's take the set $\mathcal{C}$ of all components of $G - S$, that are included in Z. By our choice of S these are even inside $Y^{\complement}$. Now

$$\Phi^{-1}(\Theta(Y \cup G[S], Z \cup G[S])) = \Theta_G(V(G \setminus \bigcup \mathcal{C}), V(\bigcup \mathcal{C}) \cup S),$$

and we have

$$\mathcal{O}_G(S, \mathcal{C}) = \underbrace{\bigcup \mathcal{C} \cup \mathring{E}(S, \bigcup \mathcal{C})}_{\subseteq Y^{\complement}} \ \cup\ \underbrace{\Theta_G(V(G \setminus \bigcup \mathcal{C}), S \cup V(\bigcup \mathcal{C}))}_{= \Phi^{-1}(\Theta(Y,Z))}.$$

So $\Phi^{-1}(\mathcal{O}(Y, Z)) = \mathcal{O}_G(S, \mathcal{C}) \cup Y^{\complement}$, which is open in $|G|$.

Now, to show that Φ is open, let a basic open set $O \subseteq |G|$ be given. Again, we need only consider the sets of the form $O = \mathcal{O}_G(S, \mathcal{C})$. It is plain to see that

$$\Phi(\mathcal{O}_G(S, \mathcal{C})) = \mathcal{O}(G[V(\bigcup \mathcal{C}) \cup S], G[V(G \setminus \bigcup \mathcal{C})]).$$ □

3.2 Tangle compactification versus Freudenthal compactification

In general topology the term 'compactification' usually refers to Hausdorff compactifications. One such Hausdorff compactification is the *Freudenthal compactification*. The general construction of the Freudenthal compactification relies on the theory of uniform spaces and proximity spaces (cf. [12]), which is beyond the scope of this thesis.

The underlying concept of the Freudenthal compactification is however best understood from the original definition [6]. The Freudenthal compactification of a second-countable, locally compact, connected and locally connected Hausdorff space X is obtained by adding as compactification points the *topological ends*, i.e., equivalence classes of decreasing sequences $U_1 \supseteq U_2 \supseteq \dots$ of connected, open sets with compact boundary such that $\bigcap_{i\in\mathbb{N}} \overline{U_i} = \emptyset$. Two such sequences $U_1 \supseteq U_2 \supseteq \dots$ and $U'_1 \supseteq U'_2 \supseteq \dots$ are equivalent if for every $i \in \mathbb{N}$ there exist $j, k \in \mathbb{N}$ such that $U_j \subseteq U'_i$ and $U'_k \subseteq U_i$. Let $\mathcal{E}(X)$ be the set of these equivalence classes. For $e \in \mathcal{E}(X)$ and $U \subseteq X$ open, write $e \subset U$ if for every $(U_1, U_2, \dots) \in e$, there exists a $k \in \mathbb{N}$ such that $U_k \subseteq U$. The Freudenthal compactification εX of X then is the topological space on $X \cup \mathcal{E}(X)$ with, in addition to the open sets of X, an open set $U \cup \{e \in \mathcal{E}(\mathcal{X}) \mid e \subset U\}$ for every open $U \subseteq X$.

We will use another characterisation of the Freudenthal compactification to prove that the tangle compactification of a locally compact Hausdorff space is its Freudenthal compactification: A compactification $\hat{X}$ of a topological space X is called *perfect* if for every disjoint pair of open sets $V, W \subseteq X$ the equation $O(V \cup W) = O(V) \cup O(W)$ holds where $O(U)$ is the largest open subset of $\hat{X}$ such that $O(U) \cap X = U$. That is to say $O(U) = \bigcup\{U' \subseteq \hat{X} \mid U' \cap X = U\} = \bigcup\{U' \subseteq \hat{X} \mid U' \cap X \subseteq U\}$. The set $\hat{X} \setminus X$ is called the *remainder* of the compactification $\hat{X}$ of X. A result by Sklyarenko says that the Freudenthal compactification is the unique perfect Hausdorff compactification with 0-dimensional remainder. The proof can be found in [11], a summary in [7, p. 682f].

We have already seen that the tangle compactification has 0-dimensional remainder Θ. It remains to see, that it is in fact a Hausdorff compactification, and that it is a perfect compactification.

Lemma 19. *If X is a locally compact Hausdorff space, then ϑX is a Hausdorff space.*

Proof. We have shown that $\Theta(\overrightarrow{U}_T(X))$ is a Hausdorff subspace of ϑX. From the premise, we know that X is a Hausdorff subspace of ϑX. To show that ϑX is Hausdorff it remains to show that any $x \in X$ and $\vartheta \in \Theta(\overrightarrow{U}_T(X))$ can be distinguished topologically by open neighbourhoods in ϑX. Let $V \subseteq X$ be an open neighbourhood of x with compact closure in X. The separation $(\overline{V}, X)$ is small and thus in ϑ. The points x and ϑ are distinguished by their disjoint neighbourhoods V and $\mathcal{O}(\overline{V}, X)$. □

The Lemma above is not true for just any Hausdorff space X. In fact consider any Hausdorff space X which is not locally compact but rim-compact, that is, for every $x \in X$ and every open neighbourhood $U \ni x$ there is an open neighbourhood $V \subseteq U$ of x with compact boundary. Let $x \in X$ witness, that X is not locally compact, then no open neighbourhood of x is included in a compact subset of X. One can easily verify that the set of the separations $(U^C, \overline{U})$, where U are arbitrarily small open neighbourhoods of x with compact boundary, together with $\vec{U_T}(X)$ satisfy the premise of the Extension Lemma. One can also check, that the tangle thus obtained cannot be distinguished topologically from x in ϑX. Hence, in this case, the compactification ϑX is not a Hausdorff compactification. In particular it is not the Freudenthal compactification.

As promised, here is the last piece of the equivalence:

Lemma 20. *The tangle compactification ϑX of any topological space X is a perfect compactification.*

Proof. For compact X we have $\vartheta X = X$, so $O(V) = V$ for every open set $V \subseteq X$ and the claim is trivial. We may thus assume that X is not compact.

It is plain to see that for open sets $V \subseteq X$ we have $O(V) = V \cup \Theta(V)$ where $\Theta(V) := \{\vartheta \in \Theta(\vec{U}_T(X)) \mid \exists\, (A,B){\in}\vartheta :\ A^{\complement} \subseteq V\}$.

So for open and disjoint $V, W \subseteq X$, we need to show that $\Theta(V \cup W) = \Theta(V) \cup \Theta(W)$. The $\supseteq$–part is obvious. For '$\subseteq$' let $\vartheta \in \Theta(V \cup W)$ be given and pick any $(A,B) \in \vartheta$ with $A^{\complement} \subseteq V \cup W$.

Consider $(A \cup W^{\complement}, B \cap V^{\complement})$. We will first show that this is a separation in $\vec{U}_T(X)$. We have

$$(A \cup W^{\complement}) \cup (B \cap V^{\complement}) = (\underbrace{A \cup B}_{X} \cup W^{\complement}) \cap (A \cup \underbrace{W^{\complement} \cup V^{\complement}}_{X}) = X.$$

The sets $A \cup W^{\complement}$ and $B \cap V^{\complement}$ are closed, since they are a finite union and a finite intersection of closed sets, respectively. Since $A^{\complement}$ is a subset of the disjoint union $V \cup W$, we have $A \cup W^{\complement} = A \cup V$. The intersection of $A \cup W^{\complement}$ and $B \cap V^{\complement}$ therefore is $(A \cup V) \cap (B \cap V^{\complement}) = A \cap B \cap V^{\complement}$ which is a closed subset of the compact set $A \cap B$ and hence compact. Thus, $(A \cup W^{\complement}, B \cap V^{\complement})$ is indeed a separation in $\vec{U}_T(X)$, so either it or its inverse must lie in ϑ.

If $(A \cup W^{\complement}, B \cap V^{\complement})$ lies in ϑ, then $(A \cup W^{\complement})^{\complement} = A^{\complement} \cap W \subseteq W$, so $\vartheta \in \Theta(W)$. We may thus assume that

$$(B \cap V^{\complement}, A \cup W^{\complement}) \in \vartheta. \tag{$*$}$$

Let us show that, also $(A \cup V^{\complement}, B \cap W^{\complement}) \in \vartheta$, and hence $\vartheta \in \Theta(V)$ as above.

As earlier, $(A \cup V^{\complement}, B \cap W^{\complement}) = (A \cup W, B \cap W^{\complement})$ is a separation in $\vec{U}_T(X)$. If $(B \cap W^{\complement}, A \cup W)$ is in ϑ then, by $(*)$, so is $(B \cap W^{\complement}, A \cup W) \vee (B \cap V^{\complement}, A \cup V)$, since ϑ is a tangle of $\vec{U}_T(X)$ and thus, by Corollary 6, a profile. But

$$(B \cap W^{\complement}, A \cup W) \vee (B \cap V^{\complement}, A \cup V) = (B \cap (V^{\complement} \cup W^{\complement}), A \cup (W \cap V)) = (B, A),$$

which contradicts the assumption that the tangle ϑ contains (A,B). □

Corollary 21. *The tangle compactification ϑX of a locally compact Hausdorff space X is its Freudenthal compactification.*

Proof. By Lemmas 19 and 20 the tangle compactification ϑX of a locally compact Hausdorff space is a perfect Hausdorff compactification. As remarked on page 9 the remainder $\Theta(\vec{U}_T(X))$ of ϑX is a 0-dimensional space. By Sklyarenko's theorem [11] there is a unique compactification with these properties: The Freudenthal compactification. □

3.3 Alternative universes of separations on a topological space

Of course, the universe of separations $\vec{U}_T(X)$ discussed here is not the only possible choice of a universe representing the structure of a topological space X. For instance, we might not want to consider all $(A, B) \in \vec{U}_T(X)$, but only the set of those, where $A \cap B$ is not only compact, but also thin in some sense. For example, we could work with

$$\vec{U}'_T(X) = \{(A, B) \in \vec{U}_T(X) \mid A \cap B \text{ has empty interior}\}.$$

Lemma 22. *Let X be a topological space. $\vec{U}'_T(X)$ is a subuniverse of $\vec{U}_T(X)$.*

Proof. $\vec{U}'_T(X)$ is closed under involution. Let's show that it is closed under taking suprema. Take $(A, B), (C, D) \in \vec{U}'_T(X)$, we need to show that $(A \cup B) \cap (C \cap D)$, which is the same as $(A \cap C \cap D) \cup (B \cap C \cap D)$, has empty interior. The sets $A' := (A \cap C \cap D)$ and $B' := (B \cap C \cap D)$ are closed with empty interior, i.e., no non-empty open set of X is a subset of either. Let any non-empty open set $U \subseteq X$ be given. Since A' has empty interior, $U \cap A'^{\complement}$ is a non-empty open set. Now, since B' has empty interior, $(U \cap A'^{\complement}) \cap B'^{\complement}$ is a non-empty open set, i.e., $U \not\subseteq A' \cup B'$.

So $A' \cup B'$ has non-empty interior, which means that $(A, B) \vee (C, D) \in \vec{U}'_T(X)$. □

This is a reasonable choices of a universe. The universe $\vec{U}_T(X)$ however has a merit, which makes it simple to work with: The fact that there is a small separation (C, X) for every compact $C \subseteq X$ gives, that no tangle of $\vec{U}_T(X)$ has a finite subset $\{(A_1, B_1), \ldots, (A_n, B_n)\}$ where $\bigcap_{i=1}^n B_n$ is compact. $\vec{U}'_T(X)$ is quite the opposite: The universe $\vec{U}'_T(X)$ is large, even for non-compact topological spaces $X \neq \emptyset$. Distributivity and small suprema of small separations are inherited from the universe of separations of the set X, and the only possible degenerate separation is (X, X), but the interior of $X \cap X$ is X, so non-empty, always. So, $(X, X) \notin \vec{U}'_T(X)$. Since $\vec{U}'_T(X)$ is large even for compact X it does have a tangle as a consequence of Corollary 10. And not only one, it is immediate from the Extension Lemma, that for every open set $U \subseteq X$ with compact boundary there is a tangle of $\vec{U}'_T(X)$ containing $(U^{\complement}, \overline{U})$. It might be possible to ignore these tangles, however, it seems like that would make working with $\vec{U}'_T(X)$ a more complicated matter.

Alternatively, we could restrict ourselves to measure spaces and only consider those separations (A, B) where $A \cap B$ is a null set. The study of separations of measure spaces gives a lot more structure: Given a measure space X with measure λ we can define an order function

$$|\cdot| : \vec{U}_T(X) \to \mathbb{R}^+ \cup \{\infty\}, \; (A, B) \mapsto \lambda(A \cap B).$$

Not only the subuniverse $\{(A, B) \in \vec{U}(X) \mid |A, B| = 0\}$, but also the subuniverse of all separations with finite order are interesting, and should be studied in the future.

4 Tangles as an inverse limit of stars

4.1 Background: Classification of the $\aleph_0$-tangles in a graph

In [2] Diestel constructed the $\aleph_0$-tangles of a graph as an inverse limit of ultrafilters. We will now give an introduction to the construction. Given an infinite connected graph G consider, for every finite set of vertices X, the set $\mathcal{C}_X$ of components of $G \setminus X$. Picking for each X a component $f(X) \in \mathcal{C}_X$ in a consistent way, i.e., such that for $X' \supset X$ the component $f(X')$ is included in $f(X)$, defines a *direction* in G.

The directions of a graph G are precisely its ends, a fact observed by Robertson, Seymour and Thomas [10]. A full proof of the equivalence is provided by Diestel and Kühn [5].

Every end ω defines an $\aleph_0$-tangle: Every separation $(A, B) \in \vec{S}_\omega(G)$ has a unique side, A or B, such that every (equivalently: one) ray $R \in \omega$ has a tail whose vertices are included in that side. The end's $\aleph_0$-tangle in G is defined as

$$\vartheta_\omega := \{(A, B) \in \vec{S}_\omega(G) \mid \text{every } R \in \omega \text{ has a tail in } G[B]\}.$$

The same $\aleph_0$-tangle may be defined using the respective direction: Let f_ω be the direction corresponding to ω, it easy to check that

$$\vartheta_\omega = \{(A, B) \in \vec{S}_\omega(G) \mid f_\omega(A \cap B) \subseteq G[B]\}. \tag{$*$}$$

Now let's shift our perspective on vertex separations of a graph G: Let $\mathcal{X}$ be the set of finite subsets of $V(G)$. A separation $(A, B) \in \vec{S}_\omega(G)$ is nothing but a bipartition of the set of components $\mathcal{C}_{A \cap B}$ of $G \setminus (A \cap B)$, and $\vec{S}_\omega(G)$ corresponds to the set of such bipartitions for every finite $A \cap B \in \mathcal{X}$. An $\aleph_0$-tangle ϑ orients every such bipartition, and we say that the set of components which this bipartition points towards is 'big', and its complement is 'small' in ϑ. The fact that $\aleph_0$-tangles are profiles ensures that if, for some $X \in \mathcal{X}$, the sets $C, D \subseteq \mathcal{C}_X$ are big in this sense, then so is $C \cap D$. Indeed, every $\aleph_0$-tangle ϑ in G induces an ultrafilter on every $\mathcal{C}_X$, specifically the ultrafilter

$$U(\vartheta, X) := \left\{\mathcal{C} \subseteq \mathcal{C}_X \;\middle|\; \left(\bigcup(\mathcal{C}_X \setminus \mathcal{C}) \cup X, \bigcup \mathcal{C} \cup X\right) \in \vartheta\right\}.$$

In the case of a tangle defined by an end ω these ultrafilters are always principal: For every separation, which does bipartition $\mathcal{C}_X$ for some $X \in \mathcal{X}$, the $\aleph_0$-tangle ϑ_ω picks the orientation that points towards $f_\omega(X)$. This is just a rephrasing of $(*)$.

Every tangle that induces a principal ultrafilter on every $\mathcal{C}_X$ comes from an end, so every tangle ϑ that is not defined by an end induces a non-principal ultrafilter $U(\vartheta, X)$ for some $X \in \mathcal{X}$. In that case for every $X' \in \mathcal{X}$ with $X \subseteq X'$ the induced ultrafilter $U(\vartheta, X')$ is non-principal as well, and it is, in fact, uniquely defined by $U(\vartheta, X)$.

Diestel shows that, given an $\aleph_0$-tangle ϑ and $X, X' \in \mathcal{X}$ with $X \subseteq X'$, the ultrafilter $U(\vartheta, X')$ determines $U(\vartheta, X)$ in the following way:

$$U(\vartheta, X) = \{\mathcal{C} \subseteq \mathcal{C}_X \mid \exists \mathcal{C}' \in U(\vartheta, X') : \bigcup \mathcal{C}' \subseteq \bigcup \mathcal{C}\}.$$

He goes on to construct an inverse system on $(\mathcal{U}_X \mid X \in \mathcal{X})$, the family of the sets $\mathcal{U}_X$ of ultrafilters on $\mathcal{C}_X$, where the bonding maps $f_{X',X} : \mathcal{U}_{X'} \to \mathcal{U}_X$ map ultrafilters on $\mathcal{C}_{X'}$ to ultrafilters on $\mathcal{C}_X$, for $X \subseteq X'$, in the way displayed above.

In the inverse limit $\mathcal{U} = \varprojlim(\mathcal{U}_X \mid X \in \mathcal{X})$ it is then true that for every element $U = (U_X \mid X \in \mathcal{X}) \in \mathcal{U}$ there exists an $\aleph_0$-tangle ϑ such that $U_X = U(\vartheta, X)$ for all $X \in \mathcal{X}$, and conversely for every ϑ there exists such a U. This makes the classification of $\aleph_0$-tangles, into those defined by an end and those inducing a non-principal ultrafilter, explicit: Write $\mathcal{U}_{\mathcal{X}}$ to mean $\bigcup_{X\in\mathcal{X}} \mathcal{U}_X$; a central theorem of [2] states:

Theorem 23 (Diestel, [2])**.** *Every $\aleph_0$-tangle τ in G satisfies exactly one of the following:*

- *$\exists$ ray $R \subseteq G$ such that $\tau = \{(A, B) \in \vec{S} : G[B]$ contains a tail of $R\}$;*
- *$\exists$ ultrafilter $U \in \mathcal{U}_{\mathcal{X}}$ such that $\tau = \{(A, B) \in \vec{S} : \exists \mathcal{C} \in U : \bigcup\mathcal{C} \subseteq B \setminus A\}$.*

In this thesis we will emulate the construction from [2] of the tangles as an inverse limit, using only terms of abstract separation systems. Instead of sets $\mathcal{C}_X$ of components left by deleting a finite set of vertices X, we consider finite stars of separations which have some fixed co-small separation as a supremum. In the case of finite order vertex separations of a graph $G = (V, E)$ the co-small separations are the (V, X), where X is finite. Note that even if a set X of vertices leaves behind infinitely many components, the stars that we shall consider will be finite and, as such, correspond to the finite partitions of the set of components.

We will then take common refinements of pairs of such stars and consider consistent choices of a separation from each star. These choices, which we dub 'compasses', will correspond to ultrafilters on the set of components of a graph. We will then construct an inverse system of the compasses, akin to the construction of $\mathcal{U}$ in [2].

A similar construction for graphs has independently been used by Kurkofka [8] to obtain the compactification $|G|$ of a graph G from [2] as an inverse limit of finite graphs. For each finite $X \subseteq V(G)$ he looked at the partitions of $\mathcal{C}_X$ into finitely many classes. These partitions, ordered by refinement, form a directed poset $(\mathcal{P}_X, \leqslant)$. For each partition P, he contracted the classes into a single vertex each, to obtain the graph G/P. He then took an inverse limit $G_X = \varprojlim(G/P \mid P \in \mathcal{P}_X)$. Those elements of G_X which are always dummy vertices correspond precisely to the ultrafilters on $\mathcal{C}_X$. He then proved that $|G| \cong \varprojlim(G_X \mid X \in \mathcal{X})$.

4.2 The local picture: Stars at some fixed co-small separation

Throughout this section we work with a fixed large universe of separations $\vec{U}$. In this subsection, let's also fix a co-small separation $\vec{r} \in \vec{U}$. Recall that a star of separations σ is a set $\sigma \subseteq \vec{U}$ of orientated separations which all point towards each other, i.e., $\vec{t} \leqslant \overleftarrow{s}$ for all $\vec{s}, \vec{t} \in \sigma$. A star σ is *proper* if $\vec{s}$ and $\vec{t}$ are nested in only this way, i.e., additionally $\vec{t} \nleqslant \vec{s}$, $\vec{t} \ngeqslant \vec{s}$, and $\vec{t} \ngeqslant \overleftarrow{s}$ for all $\vec{s}, \vec{t} \in \sigma$.

We say that a finite star $\sigma \subseteq \vec{U}$ is a *star at* $\vec{r}$, if $\bigvee \sigma = \vec{r}$. We will define what it means for one such star to refine another, and take common refinements of such stars. We will then define a 'compass' to be a compatible choice of separations, one from every such star. These compasses will emulate the ultrafilters on the set of components from [2]. To define such compatible choices, we will construct an inverse system from some of the stars at $\vec{r}$.

More formally, write $\Sigma_{\vec{r}}$ for the set of all those stars of $\vec{U}$ which are inclusion-minimal with respect to being at $\vec{r}$. Let's characterise these stars a bit more.

Lemma 24. *Every $\sigma \in \Sigma_{\vec{r}}$ is inclusion minimal with co-small supremum.*

Proof. Take some $\vec{t} \in \sigma$ and let $\tau = \sigma \setminus \{\vec{t}\}$. Now suppose, for a contradiction, that $\bigvee \tau$ is co-small. By the star property, $\vec{t} \leqslant \bigwedge_{\vec{s} \in \tau} \overleftarrow{s} = (\bigvee \tau)^*$ and, by supposition, $(\bigvee \tau)^* \leqslant \bigvee \tau$. So $\vec{t} \leqslant \bigvee \tau$, and thus,

$$\bigvee \sigma = \vec{t} \vee \bigvee \tau = \bigvee \tau,$$

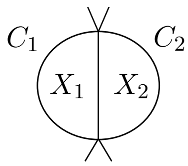
which contradicts the minimality of σ. □

Lemma 25. *Every $\sigma \in \Sigma_{\vec{r}}$ is either a proper star or of the form $\{\vec{s}, \overleftarrow{s}\}$.*

Proof. Take a star $\sigma \in \Sigma_{\vec{r}}$. If σ is not proper there are separations $\vec{s}, \vec{t} \in \sigma$, which are nested in some additional way other than $\vec{s} \leqslant \overleftarrow{t}$.

If $\vec{t} \leqslant \vec{s}$, then $\bigvee \sigma = \bigvee(\sigma \setminus \{\vec{t}\})$, and if $\vec{t} \geqslant \vec{s}$, then $\bigvee \sigma = \bigvee(\sigma \setminus \{\vec{s}\})$. Both contradict the minimality of σ. So, it must be the case that $\vec{t} \geqslant \overleftarrow{s}$, which gives $\vec{t} \geqslant \overleftarrow{s} \geqslant \vec{t}$, i.e., $\vec{t} = \overleftarrow{s}$. Then, by Lemma 1, the set $\{\vec{s}, \vec{t}\} = \{\vec{s}, \overleftarrow{s}\} \subseteq \sigma$ has co-small supremum and Lemma 24 gives $\sigma = \{\vec{s}, \overleftarrow{s}\}$. □

The converse, however, is not true: Not every proper star with supremum $\vec{r}$ is in $\Sigma_{\vec{r}}$. For an example the figure below, similar to the one before, illustrates a partition of the vertex set of an infinite connected graph $G = (V, E)$ into disjoint sets C_1, C_2. X_1 and X_2, where X_1 and X_2 are finite and C_1 and C_2 are the sets of vertices of the components of $G \setminus (X_1 \cup X_2)$.



In $\vec{S}_\omega(G)$, the universe of finite order vertex separations of this graph, the star

$$\sigma = \{(C_1 \cup X_1, V \setminus C_1), (C_2 \cup X_2, V \setminus C_2), (X_1 \cup X_2, V)\}$$

is proper, but it is not minimal with supremum $(X_1 \cup X_2, V)$.

For $\sigma, \tau \in \Sigma_{\vec{r}}$ we say that σ *refines* τ and write $\sigma \leqslant \tau$ if there exists some increasing map $f : \sigma \to \tau$. These maps are always surjective:

Lemma 26. *For $\sigma, \tau \in \Sigma_{\vec{r}}$ every increasing map $f : \sigma \to \tau$ is surjective.*

Proof. We have $\vec{r} = \bigvee \sigma \leqslant \bigvee f[\sigma] \leqslant \bigvee \tau = \vec{r}$, so the minimality of τ implies $\tau = f[\sigma]$. □

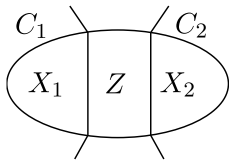
Corollary 27. *The relation $\leqslant$ is a partial order on $\Sigma_{\vec{r}}$.*

Proof. The relation is clearly reflexive and transitive, and its anti-symmetry follows immediately from Lemma 26, as follows: Take $\sigma, \tau \in \Sigma_{\vec{r}}$ with $\sigma \leqslant \tau \leqslant \sigma$, and let $f : \sigma \to \tau$ and $g : \tau \to \sigma$ be increasing maps. Then $g \circ f : \sigma \to \sigma$ is an increasing map. Since σ is minimal with supremum $\vec{r}$, this map is the identity. Then for every $\vec{s} \in \sigma$

$$\vec{s} \leqslant f(\vec{s}) \leqslant g(f(\vec{s})) = \vec{s},$$

so $f(\vec{s}) = \vec{s}$. Since f is surjective, $\sigma = \tau$. $\square$

Remark 28. *The maps $f : \sigma \to \tau$ for $\sigma \leqslant \tau$ may not be unique. Here is an example:*



The set-up is the same as for the example in Section 2.3. The figure displayed above outlines a a partition of the vertex set of an infinite connected graph $G = (V, E)$ into disjoint sets of vertices. X_1, Z and X_2 are finite, and edges run only within the sets and between sets drawn adjacently. In $\vec{S}_\omega(G)$ consider the stars

$$\begin{aligned}
\tau &= \{(X_1 \cup C_1 \cup Z, V \setminus C_1), (X_2 \cup C_2 \cup Z, V \setminus C_2)\},\\
\sigma &= \{(X_1 \cup C_1 \cup Z, V \setminus C_1), (X_2 \cup C_2, V \setminus C_2)\} \text{ and}\\
\nu &= \{(X_1 \cup C_1, V \setminus C_1), (X_2 \cup C_2, V \setminus C_2), (Z, V)\}.
\end{aligned}$$

Then the increasing maps $\nu \to \sigma$ and $\sigma \to \tau$ are unique, but $\nu \to \tau$ may map (Z, V) to either element. The star ν is also an example of a star in $\Sigma_{\vec{r}}$ which contains a small separation.

The increasing maps are, however, unique on the separations that matter, the non-small ones. For a star σ at $\vec{r}$ we write $\mathring{\sigma}$ for the set of non-small separations in σ. It turns out that for $\sigma, \tau \in \Sigma_{\vec{r}}$ with $\sigma \leqslant \tau$, there is only one increasing map from $\mathring{\sigma}$ to τ:

Lemma 29. *For stars σ, τ all increasing maps $\sigma \to \tau$ agree on $\mathring{\sigma}$. The restriction of such a map to $\mathring{\sigma}$ takes its values in $\mathring{\tau}$.*

Proof. Suppose there are $\vec{s} \in \sigma$ and distinct $\vec{t}_1, \vec{t}_2 \in \tau$ with $\vec{s} \leqslant \vec{t}_1, \vec{t}_2$. Then $\vec{s} \leqslant \vec{t}_1 \leqslant \overleftarrow{t}_2 \leqslant \overleftarrow{s}$ by the star property of τ, so $\vec{s}$ is small. Hence for each $\vec{s} \in \mathring{\sigma}$ there is at most one $\vec{t} \in \tau$ with $\vec{s} \leqslant \vec{t}$.

The 'moreover'-part is immediate from the fact that if $f(\vec{s})$ is small then so must be $\vec{s}$, since $\vec{s} \leqslant f(\vec{s})$. $\square$

Since we are working with a large universe $\vec{U}$, a finite star at some $\vec{r} \in \vec{U}^+$ cannot consist entirely of small separations. In that case its supremum $\vec{r}$ would be small and hence degenerate. So the $\mathring{\sigma}$ for $\sigma \in \Sigma_{\vec{r}}$ are non-empty. Since $\{\vec{r}\}$ is a star at $\vec{r}$ the set $\Sigma_{\vec{r}}$ is non-empty. For $\Sigma_{\vec{r}}$ to form an inverse system the partial order $\geqslant$ has to be directed.

Lemma 30. *The poset $(\Sigma_{\vec{r}}, \geqslant)$ is directed: For all $\sigma, \tau \in \Sigma_{\vec{r}}$ there exists a $\mu \in \Sigma_{\vec{r}}$ such that $\mu \leqslant \sigma$ and $\mu \leqslant \tau$.*

Proof. Given $\sigma, \tau \in \Sigma_{\vec{r}}$, take

$$\mu := \{\vec{s} \wedge \vec{t} \mid \vec{s} \in \sigma,\ \vec{t} \in \tau\}.$$

By distributivity, this μ has $\vec{r}$ as its supremum:

$$\bigvee \mu = \bigvee_{\vec{s} \in \sigma} \bigvee_{\vec{t} \in \tau} (\vec{s} \wedge \vec{t}) = \bigvee_{\vec{s} \in \sigma} (\vec{s} \wedge \bigvee \tau) = \bigvee \sigma \wedge \vec{r} = \vec{r}.$$

Furthermore μ is a star: Distinct separations $\vec{q}_1, \vec{q}_2 \in \mu$ lie either below two distinct elements of σ or below two distinct elements of τ and, since σ and τ are stars, point towards each other.

Take an inclusion-minimal subset μ' of μ with $\bigvee \mu' = \vec{r}$. Then $\mu' \in \Sigma_{\vec{r}}$ and $\mu' \leqslant \sigma, \tau$ by construction. □

For $\vec{r} \in \vec{U}^+$ we can now define a *compass at* $\vec{r}$ as an element of the inverse limit

$$\mathcal{U}_{\vec{r}} := \varprojlim \left(\mathring{\sigma} \mid \sigma \in \Sigma_{\vec{r}}\right),$$

of the inverse system $(\mathring{\sigma} \mid \sigma \in \Sigma_{\vec{r}})$ with bonding maps $f_{\sigma\tau}$ for $\sigma \leqslant \tau$, which are the unique increasing maps $\mathring{\sigma} \to \mathring{\tau}$ from Lemma 29. Lemma 29 also implies that these maps are compatible.

Recall that a $\Sigma_{\vec{r}}$-tangle of $\vec{U}$ is a consistent orientation of $\vec{U}$ including no $\sigma \in \Sigma_{\vec{r}}$ as a subset. Every $\Sigma_{\vec{r}}$-tangle of $\vec{U}$ (in particular, every tangle of $\vec{U}$) induces a compass at $\vec{r}$ in the following way:

Lemma 31. *Let ϑ be a $\Sigma_{\vec{r}}$-tangle of $\vec{U}$. Then in every $\sigma \in \Sigma_{\vec{r}}$ there is a unique $\vec{s}_\sigma \in \sigma$ such that $\overleftarrow{s}_\sigma \in \vartheta$. The family $(\vec{s}_\sigma \mid \sigma \in \Sigma_{\vec{r}})$ is a compass at $\vec{r}$. We say that ϑ* induces *this compass at $\vec{r}$.*

Proof. Let ϑ be such tangle. For every $\sigma \in \Sigma_{\vec{r}}$ there is some $\vec{s} \in \sigma \cap \vartheta^*$, since ϑ avoids σ. Since σ is a star and ϑ a consistent orientation, this $\vec{s} =: \vartheta^*(\sigma)$ is unique. It remains to show that this gives a compatible choice of separations. Let $\sigma \leqslant \tau$ and $f : \sigma \to \tau$ be increasing. Suppose $\vartheta^*(\tau) \neq f(\vartheta^*(\sigma))$. Then, since τ is a star, we have $\vartheta^*(\tau)^* > f(\vartheta^*(\sigma)) \geqslant \vartheta^*(\sigma)$. But ϑ is a consistent orientation, so this cannot be the case. □

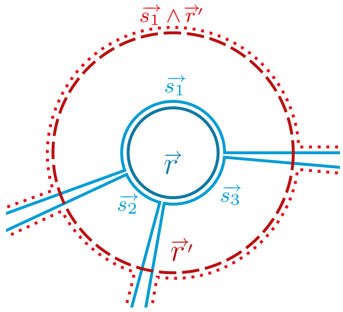
Conversely, every tangle ϑ of $\vec{U}$, which avoids all $\Sigma_{\vec{r}}$ for co-small $\vec{r}$, contains one element from every 2-star $\{\vec{s}, \overleftarrow{s}\}$, since $\{\vec{s}, \overleftarrow{s}\} \in \Sigma_{\vec{s} \vee \overleftarrow{s}}$. Every tangle ϑ of $\vec{U}$ is a set consisting of one of $\vec{s}$ or $\overleftarrow{s}$ for every $s \in U$, we have the following converse to Lemma 31:

Theorem 32. *Every tangle ϑ of $\vec{U}$ is uniquely determined by the collection of the compasses it induces, one for each $\vec{r} \in \vec{U}^+$.*

Proof. Let ϑ be a tangle and $s \in U$ any separation. By Lemma 1 $\vec{r} := \vec{s} \vee \overleftarrow{s}$ is co-small. The set $\sigma = \{\vec{s}, \overleftarrow{s}\}$ is a star at $\vec{r}$. If $\overleftarrow{s} \leqslant \vec{s}$ then $\overleftarrow{s}$ is small and must be in every tangle. Likewise if $\overleftarrow{s} \geqslant \vec{s}$ then $\vec{s}$ is small and must be in every tangle.

Otherwise $\sigma \in \Sigma_{\vec{r}}$. Let u be the compass, that ϑ induces at $\vec{r}$. Now $u(\sigma)$ is either $\vec{s}$ or $\overleftarrow{s}$, whichever is in ϑ^*. So, the orientation that s receives in ϑ is uniquely determined by the compass, that ϑ induces at $\vec{r}$. $\square$

4.3 From local to global: How to choose compasses at different separations $\vec{r}$ compatibly



The dotted star at $\vec{r}'$ is the stretch of the solid star at $\vec{r}$.
(All the pictured separations point towards the centre.)

We are now equipped to imitate the construction of an inverse limit of ultrafilters in [2]. Instead of ultrafilters we have compasses, and instead of larger and larger sets X of vertices we take smaller and smaller co-small $\vec{r}$. We will move from stars at $\vec{r}$ to stars at $\vec{r}' < \vec{r}$. In $\vec{S}_\omega(G)$ of a graph $G = (V, E)$, this means we go from stars with supremum (V, X) to stars with supremum (V, X') for finite $X' \supset X$. Every star $\sigma = \{(A_1, B_1), \ldots, (A_n, B_n)\}$ with supremum (V, X) induces the star $\sigma' = \{(A_1, B_1 \cup X'), \ldots, (A_n, B_n \cup X')\}$ with supremum (V, X').

In the abstract, given a star $\sigma \in \Sigma_{\vec{r}}$ we can find a star at $\vec{r}'$ that refines σ by just taking infima with $\vec{r}'$. We call this star the *stretch of σ to $\vec{r}'$*:

$$\sigma \uparrow \vec{r}' := \{\vec{s} \wedge \vec{r}' \mid \vec{s} \in \sigma\} \leqslant \sigma.$$

We will use these stars to have compasses at $\vec{r}'$ induce compasses at $\vec{r}$, analogously to how ultrafilters on the set of components do in [2]. This will allow us to build an inverse system. We will then show that its inverse limit corresponds to the tangles.

Lemma 33. *For co-small $\vec{r}' < \vec{r}$ the stretch $\sigma \uparrow \vec{r}'$ of any $\sigma \in \Sigma_{\vec{r}}$ is in $\Sigma_{\vec{r}'}$.*

Proof. Clearly $\sigma \uparrow \vec{r}'$ is a star: If $\vec{s} \leqslant \overleftarrow{t}$ then $\vec{s} \wedge \vec{r}' \leqslant \overleftarrow{t} \vee \overleftarrow{r}' = (\vec{t} \wedge \vec{r}')^*$. Its supremum comes out as

$$\bigvee(\sigma \uparrow \vec{r}') = \bigvee\{\vec{s} \wedge \vec{r}' \mid \vec{s} \in \sigma\} = \vec{r}' \wedge \bigvee \sigma = \vec{r}' \wedge \vec{r} = \vec{r}'.$$

Suppose for some proper subset $\tau \subset \sigma$ we have that

$$\bigvee\{\vec{s} \wedge \vec{r'} \mid \vec{s} \in \tau\} = \vec{r'}.$$

Without loss of generality $\sigma = \tau \cup \{\vec{t}\}$. By Lemma 24 $\bigvee \tau$ is not co-small, so neither is the smaller separation

$$\vec{r'} \wedge \bigvee \tau = \bigvee\{\vec{s} \wedge \vec{r'} \mid \vec{s} \in \tau\} = \vec{r'}.$$

A contradiction to the premise that $\vec{r'}$ is co-small.

Hence, the stretch $\sigma \uparrow \vec{r'}$ is inclusion-minimally at $\vec{r'}$ and thus indeed $\sigma \uparrow \vec{r'} \in \Sigma_{\vec{r'}}$. $\square$

In order to define bonding maps $\mathcal{U}_{\vec{r'}} \to \mathcal{U}_{\vec{r}}$ we need to choose, given an element $u' = (\vec{s'} \in \mathring{\sigma}' \mid \sigma \in \Sigma_{\vec{r'}})$ of $\mathcal{U}_{\vec{r'}}$, one separation $\vec{s}$ from every star $\mathring{\sigma}$ with $\sigma \in \Sigma_{\vec{r}}$. We determine $\vec{s}$ by enquiring about the separation $\vec{s'}$ which u' chooses for $\sigma' := \sigma \uparrow \vec{r'}$, as follows:

Lemma 34. *For co-small $\vec{r'} < \vec{r}$, $\sigma \in \Sigma_{\vec{r}}$ and $\vec{s'} \in \mathring{\sigma}'$ where $\sigma' = \sigma \uparrow \vec{r'}$ there is a unique $\vec{s} \in \sigma$ with $\vec{s'} \leqslant \vec{s}$. This $\vec{s}$ is the unique $\vec{s} \in \sigma$ such that $\vec{s'} = \vec{s} \wedge \vec{r'}$. Moreover, $\vec{s} \in \mathring{\sigma}$.*

Proof. By definition of $\sigma \uparrow \vec{r'}$ there is some $\vec{s} \in \sigma$ such that $\vec{s'} = \vec{s} \wedge \vec{r'}$ and hence $\vec{s'} \leqslant \vec{s}$. Now if $\vec{s'} \leqslant \vec{t}$ for some $\vec{t} \in \sigma$ with $\vec{t} \neq \vec{s}$ then $\vec{s'}$ would be small (because $\vec{s} \leqslant \overleftarrow{t}$ since σ is a star, as earlier). Hence, $\vec{s}$ is unique in σ with $\vec{s'} \leqslant \vec{s}$. As $\vec{s'}$ is not small, neither is $\vec{s} \leqslant \vec{s'}$. Thus, $\vec{s} \in \mathring{\sigma}$ as claimed. $\square$

The above lemma justifies the following notation. For co-small $\vec{r'} < \vec{r}$, a star $\sigma \in \Sigma_{\vec{r}}$, $\sigma' = \sigma \uparrow \vec{r'}$, and a separation $\vec{s'} \in \mathring{\sigma}'$ write $\vec{s'} \downarrow \sigma$ for the unique $\vec{s} \in \sigma$ for which $\vec{s'} = \vec{s} \wedge \vec{r'}$.

Given co-small $\vec{r'} < \vec{r}$ and a compass $u' \in \mathcal{U}_{\vec{r'}}$ we define

$$u' \downarrow \vec{r} := \left(u'(\sigma \uparrow \vec{r'}) \downarrow \sigma \,\middle|\, \sigma \in \Sigma_{\vec{r}}\right),$$

where $u'(\sigma \uparrow \vec{r'})$ is the separation $\vec{s'} \in \sigma \uparrow \vec{r'}$ which the map u' assigns to $\sigma \uparrow \vec{r'} \in \Sigma_{\vec{r'}}$. In words, for each $\sigma \in \Sigma_{\vec{r}}$ the family $u' \downarrow \vec{r}$ picks the unique $\vec{s} \in \sigma$ for which u' picks $\vec{s} \wedge \vec{r'}$ from the stretch of σ to $\vec{r'}$.

Lemma 35. *For co-small $\vec{r'} < \vec{r}$ and a compass $u' \in \mathcal{U}_{\vec{r'}}$ the family $u = u' \downarrow \vec{r}$ is indeed a compass at $\vec{r}$, an element of $\mathcal{U}_{\vec{r}}$. We say that u' induces u.*

Proof. By definition the family u consists of a choice of some $\vec{s} \in \mathring{\sigma}$ for each $\sigma \in \Sigma_{\vec{r}}$. Given $\sigma < \tau \in \Sigma_{\vec{r}}$ we need to show that

$$\vec{s} := u'(\sigma') \downarrow \sigma \;\leqslant\; u'(\tau') \downarrow \tau =: \vec{t},$$

where $\sigma' := \sigma \uparrow \vec{r'}$ and $\tau' := \tau \uparrow \vec{r'}$. Since u' is a compass and $\sigma' \leqslant \tau'$, we have

$$\vec{s'} := u'(\tau') \leqslant u'(\sigma') =: \vec{t'}.$$

By Lemma 34 and the definitions of $\vec{s}$ and $\vec{t}$ we deduce

$$\vec{s} \wedge \vec{r'} = \vec{s'} \leqslant \vec{t'} = \vec{t} \wedge \vec{r'}.$$

It remains to show that this implies $\vec{s} \leqslant \vec{t}$.

Since $\sigma < \tau$, there is a unique $\vec{q} \in \tau$ with $\vec{s} \leqslant \vec{q}$. Then

$$\vec{s'} = \vec{s} \wedge \vec{r'} \leqslant \vec{q} \wedge \vec{r'} \in \tau'.$$

From $\vec{s'} \in \mathring{\sigma}'$, $\vec{s'} \leqslant \vec{q} \wedge \vec{r'}$, $\vec{s'} \leqslant \vec{t'}$ and the uniqueness of the increasing map $\mathring{\sigma}' \to \tau'$ (Lemma 29) we deduce that

$$\vec{q} \wedge \vec{r'} = \vec{t'}.$$

In particular, $\vec{t'} \leqslant \vec{q}$. The uniqueness in Lemma 34 now gives us that $\vec{q} = \vec{t}$, as desired. $\square$

We may now define

$$\mathcal{U} := \varprojlim \, (\mathcal{U}_{\vec{r}} \mid \vec{r} \in \vec{U}^{+}).$$

as an inverse limit with bonding maps $f_{\vec{r'},\vec{r}} : \mathcal{U}_{\vec{r'}} \to \mathcal{U}_{\vec{r}}$, $u' \mapsto u' \downarrow \vec{r}$ for $\vec{r'} < \vec{r}$. The largeness of $\vec{U}$ ensures that $(\vec{U}^{+}, >)$ is a directed poset.

For $\upsilon = (u_{\vec{r}} \in \mathcal{U}_{\vec{r}} \mid \vec{r} \in \vec{U}^{+}) \in \mathcal{U}$ we write $\upsilon(\sigma)$ to mean $u_{\bigvee \sigma}(\sigma)$. It is easy to see that every tangle induces some unique element of $\mathcal{U}$. Conversely, via the 2-stars every element of $\mathcal{U}$ induces some orientation. We'll have to check that such orientation is consistent. It will then be a tangle.

Lemma 36. *For every $\upsilon \in \mathcal{U}$ the $\upsilon(\{\vec{s}, \overleftarrow{s}\})^{*}$ form a tangle of $\vec{U}$.*

Proof. It suffices to show, that the orientation they form is consistent. Let $\tau = \{\vec{t}, \overleftarrow{t}\}$ and $\sigma = \{\vec{s}, \overleftarrow{s}\}$ where $\vec{s} \leqslant \vec{t}$ and $\upsilon(\tau)^{*} = \vec{t}$. We need to prove that indeed $\upsilon(\sigma)^{*} = \vec{s}$. Suppose it is not, i.e., $\upsilon(\sigma) = \vec{s}$.

We stretch both σ and τ to

$$(\vec{s} \vee \overleftarrow{s}) \wedge (\vec{t} \vee \overleftarrow{t}) = (\overleftarrow{s} \wedge \vec{t}) \vee (\overleftarrow{s} \wedge \vec{t})^{*} = (\overleftarrow{s} \wedge \vec{t}) \vee (\vec{s} \vee \overleftarrow{t}).$$

The stretch of σ is

$$\begin{aligned}
&\sigma \uparrow ((\vec{s} \vee \overleftarrow{s}) \wedge (\vec{t} \vee \overleftarrow{t})) \\
&\quad = \{\vec{s} \wedge (\vec{s} \vee \overleftarrow{s}) \wedge (\vec{t} \vee \overleftarrow{t}),\, \overleftarrow{s} \wedge (\vec{s} \vee \overleftarrow{s}) \wedge (\vec{t} \vee \overleftarrow{t})\} \\
&\quad = \{\vec{s} \wedge ((\overleftarrow{s} \wedge \vec{t}) \vee (\vec{s} \vee \overleftarrow{t})),\, \overleftarrow{s} \wedge ((\overleftarrow{s} \wedge \vec{t}) \vee (\vec{s} \vee \overleftarrow{t}))\} \\
&\quad = \{(\vec{s} \wedge \overleftarrow{s} \wedge \vec{t}) \vee (\vec{s} \wedge \vec{s}) \vee (\vec{s} \wedge \overleftarrow{t}),\, (\overleftarrow{s} \wedge \overleftarrow{s} \wedge \vec{t}) \vee (\overleftarrow{s} \wedge \vec{s}) \vee (\overleftarrow{s} \wedge \overleftarrow{t})\} \\
&\quad = \{(\vec{s} \wedge \overleftarrow{s} \wedge \vec{t}) \vee \vec{s} \vee (\vec{s} \wedge \overleftarrow{t}),\, (\overleftarrow{s} \wedge \vec{t}) \vee (\overleftarrow{s} \wedge \vec{s}) \vee (\overleftarrow{s} \wedge \overleftarrow{t})\} \\
&\quad = \{\vec{s},\, (\overleftarrow{s} \wedge \vec{t}) \vee (\overleftarrow{s} \wedge \vec{s}) \vee \overleftarrow{t}\},
\end{aligned}$$

and, similarly, that of τ is

$$\begin{aligned}
&\tau \uparrow ((\vec{s} \vee \overleftarrow{s}) \wedge (\vec{t} \vee \overleftarrow{t})) \\
&\quad = \{\vec{t} \wedge (\vec{s} \vee \overleftarrow{s}) \wedge (\vec{t} \vee \overleftarrow{t}),\, \overleftarrow{t} \wedge (\vec{s} \vee \overleftarrow{s}) \wedge (\vec{t} \vee \overleftarrow{t})\} \\
&\quad = \{(\vec{t} \wedge \overleftarrow{s} \wedge \vec{t}) \vee (\vec{t} \wedge \vec{s}) \vee (\vec{t} \wedge \overleftarrow{t}),\, (\overleftarrow{t} \wedge \overleftarrow{s} \wedge \vec{t}) \vee (\overleftarrow{t} \wedge \vec{s}) \vee (\overleftarrow{t} \wedge \overleftarrow{t})\} \\
&\quad = \{(\overleftarrow{s} \wedge \vec{t}) \vee (\vec{t} \wedge \vec{s}) \vee (\vec{t} \wedge \overleftarrow{t}),\, (\overleftarrow{t} \wedge \overleftarrow{s} \wedge \vec{t}) \vee (\overleftarrow{t} \wedge \vec{s}) \vee \overleftarrow{t}\} \\
&\quad = \{(\overleftarrow{s} \wedge \vec{t}) \vee \vec{s} \vee (\vec{t} \wedge \overleftarrow{t}),\, \overleftarrow{t}\}.
\end{aligned}$$

Now a common refinement of both stretches is $\nu := \{\vec{s}, \overleftarrow{t}, \overleftarrow{s} \wedge \vec{t}\}$ and we have

$$\upsilon(\tau) = \overleftarrow{t} \Rightarrow \upsilon(\nu) = \overleftarrow{t}, \text{ but } \upsilon(\sigma) = \vec{s} \Rightarrow \upsilon(\nu) = \vec{s}.$$

A contradiction. □

We get the following correspondence between $\mathcal{U}$ and Θ:

Theorem 37. *Let $F : \Theta \to \mathcal{U}$ be the function which maps every tangle $\vartheta \in \Theta$ to the family $\upsilon_\vartheta = (u_{\vec{r}} \mid \vec{r} \in \vec{U}^+)$, where $u_{\vec{r}}$ is the compass that ϑ induces at $\vec{r}$ according to Lemma 31. F is a bijection.*

Proof. The fact that for every $\vartheta \in \Theta$ the family υ_ϑ is in $\mathcal{U}$, i.e., $f_{\vec{r}', \vec{r}}(u_{\vec{r}'}) = u_{\vec{r}}$ for $\vec{r}' < \vec{r}$, is immediate from the fact, that tangles of $\vec{U}$ are downward closed and Lemma 29.

The function F is injective, since if two tangles ϑ_1, ϑ_2 are different they disagree on the orientation of some $s \in U$. Then $\upsilon_{\vartheta_1}(\{\vec{s}, \overleftarrow{s}\}) \neq \upsilon_{\vartheta_2}(\{\vec{s}, \overleftarrow{s}\})$.

We will use Lemma 36 to show that F is surjective: Given $\upsilon \in \mathcal{U}$, take the tangle from Lemma 36

$$\vartheta := \{\upsilon(\{\vec{s}, \overleftarrow{s}\})^* \mid s \in U\},$$

we will show $\upsilon = \upsilon_\vartheta$. Suppose they differ, then for some star σ

$$\vec{s} := \upsilon(\sigma) \neq \upsilon_\vartheta(\sigma) := \vec{t}.$$

We then also have

$$\upsilon_\vartheta(\{\vec{s}, \overleftarrow{s}\}) = \overleftarrow{s} \neq \vec{s} = \upsilon(\{\vec{s}, \overleftarrow{s}\}),$$

which contradicts the construction of υ_ϑ. □

4.4 Conclusion

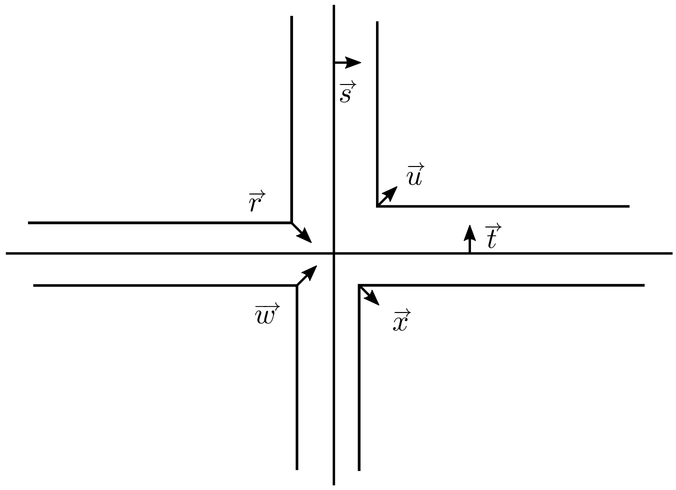
As we discussed before, the construction of the $\aleph_0$-tangles in a graph as an inverse limit of ultrafilters is a generalisation of the directions. The characterisation of a tangle via the ultrafilters that it induces allows us to differentiate end tangles from the remaining $\aleph_0$-tangles.

We have not found any way to abstract the notion of principality which could hold up to basic scrutiny. It isn't entirely clear, what properties 'end tangles' should fullfill compared to the rest. The most apparent problem is, that in the abstract there is no such thing as 'components of $G \setminus X$' for $X \subseteq V(G)$ in a graph G.

Towards the end of writing this thesis, I was hinted at some personal notes of Nathan Bowler. They prove that every distributive universe where the supremum of two small separations is small can be represented as a subuniverse of a universe of separations of a suitably constructed set. This, of course, gives a lot of additional structure. So much of it, that some of the proofs we did in the abstract may become trivial. Our abstract construction however highlights, how much of a key role this property has for the construction of an inverse system: Otherwise the co-small separations would not form a directed set. We conclude this thesis, dissastisfied.

Appendix

A non-large distributive universe, where every tangle is a profile



On the universe $\vec{U} := \{\vec{r}, \vec{s}, \vec{t}, \vec{u}, \vec{w}, \vec{x}, \overleftarrow{r}, \overleftarrow{s}, \overleftarrow{t}, \overleftarrow{u}, \overleftarrow{w}, \overleftarrow{x}\}$ consider the order relation

$$\begin{aligned}\leqslant = \{&(\overleftarrow{r},\overleftarrow{r}),(\overleftarrow{r},\overleftarrow{t}),(\overleftarrow{r},\overleftarrow{w}),(\overleftarrow{r},\vec{x}),(\vec{r},\overleftarrow{r}),(\vec{r},\vec{r}),(\vec{r},\overleftarrow{s}),(\vec{r},\vec{s}),(\vec{r},\overleftarrow{t}),(\vec{r},\overleftarrow{u}),\\
&(\vec{r},\vec{u}),(\vec{r},\overleftarrow{w}),(\vec{r},\vec{x}),(\overleftarrow{s},\overleftarrow{r}),(\overleftarrow{s},\overleftarrow{s}),(\overleftarrow{s},\overleftarrow{t}),(\overleftarrow{s},\overleftarrow{w}),(\overleftarrow{s},\vec{x}),(\vec{s},\overleftarrow{r}),(\vec{s},\overleftarrow{s}),\\
&(\vec{s},\vec{s}),(\vec{s},\overleftarrow{t}),(\vec{s},\vec{u}),(\vec{s},\overleftarrow{w}),(\vec{s},\vec{x}),(\overleftarrow{t},\overleftarrow{t}),(\overleftarrow{t},\overleftarrow{w}),(\overleftarrow{t},\vec{x}),(\vec{t},\overleftarrow{r}),(\vec{t},\vec{r}),\\
&(\vec{t},\overleftarrow{s}),(\vec{t},\vec{s}),(\vec{t},\overleftarrow{t}),(\vec{t},\vec{t}),(\vec{t},\overleftarrow{u}),(\vec{t},\vec{u}),(\vec{t},\overleftarrow{w}),(\vec{t},\vec{x}),(\overleftarrow{u},\overleftarrow{r}),(\overleftarrow{u},\overleftarrow{s}),\\
&(\overleftarrow{u},\overleftarrow{t}),(\overleftarrow{u},\overleftarrow{u}),(\overleftarrow{u},\overleftarrow{w}),(\overleftarrow{u},\vec{x}),(\vec{u},\overleftarrow{r}),(\vec{u},\overleftarrow{t}),(\vec{u},\vec{u}),(\vec{u},\overleftarrow{w}),(\vec{u},\vec{x}),(\overleftarrow{w},\overleftarrow{w}),\\
&(\overleftarrow{w},\vec{x}),(\vec{w},\overleftarrow{r}),(\vec{w},\vec{r}),(\vec{w},\overleftarrow{s}),(\vec{w},\vec{s}),(\vec{w},\overleftarrow{t}),(\vec{w},\vec{t}),(\vec{w},\overleftarrow{u}),(\vec{w},\vec{u}),(\vec{w},\overleftarrow{w}),\\
&(\vec{w},\vec{w}),(\vec{w},\vec{x}),(\overleftarrow{x},\overleftarrow{r}),(\overleftarrow{x},\vec{r}),(\overleftarrow{x},\overleftarrow{s}),(\overleftarrow{x},\vec{s}),(\overleftarrow{x},\overleftarrow{t}),(\overleftarrow{x},\vec{t}),(\overleftarrow{x},\overleftarrow{u}),(\overleftarrow{x},\vec{u}),\\
&(\overleftarrow{x},\overleftarrow{w}),(\overleftarrow{x},\vec{w}),(\overleftarrow{x},\overleftarrow{x}),(\overleftarrow{x},\vec{x}),(\vec{x},\vec{x})\}.\end{aligned}$$

This is the transitive closure of

$$\begin{aligned}&\{(\vec{a},\vec{a}) \mid \vec{a} \in \vec{U}\} \cup \{(\overleftarrow{r},\overleftarrow{t}),(\vec{r},\vec{s}),(\vec{r},\overleftarrow{u}),(\overleftarrow{s},\overleftarrow{r}),(\vec{s},\overleftarrow{s}),(\vec{s},\vec{u}),(\overleftarrow{t},\overleftarrow{w}),\\
&(\vec{t},\vec{r}),(\overleftarrow{u},\overleftarrow{s}),(\vec{u},\overleftarrow{r}),(\overleftarrow{w},\vec{x}),(\vec{w},\vec{t}),(\overleftarrow{x},\vec{w})\}.\end{aligned}$$

One can check, that $\vec{U}$ forms a distributive lattice. The small separations are

$$\vec{r}, \vec{s}, \vec{t}, \vec{w} \text{ and } \overleftarrow{x}.$$

In particular, $\vec{r}, \vec{s}$ are small separations with non-small supremum $\vec{u}$, witnessing that $\vec{U}$ is not a large universe.

Since $\vec{r}, \vec{s}, \vec{t}, \vec{w}$ and $\overleftarrow{x}$ are small, there are only 2 potential candidates for tangles of $\vec{U}$:

$$\begin{aligned}\vartheta_1 &= \{\vec{r}, \vec{s}, \vec{t}, \vec{w}, \overleftarrow{x}, \vec{u}\}, \text{ and}\\
\vartheta_2 &= \{\vec{r}, \vec{s}, \vec{t}, \vec{w}, \overleftarrow{x}, \overleftarrow{u}\}.\end{aligned}$$

Since ϑ_2 has the finite subset $\{\vec{s}, \vec{t}, \overleftarrow{u}\}$ and the supremum

$$\vec{s} \vee \vec{t} \vee \overleftarrow{u} = \overleftarrow{s},$$

is co-small, ϑ_2 is not a tangle of $\vec{U}$. The orientation ϑ_1 of $\vec{U}$ is a regular profile of $\vec{U}$ and a tangle of $\vec{U}$.

Here is the code we used to check these claims:

```
#! python

sv, vs = 's<', 's>'
tv, vt = 't<', 't>'
rv, vr = 'r<', 'r>'
uv, vu = 'u<', 'u>'
wv, vw = 'w<', 'w>'
xv, vx = 'x<', 'x>'

def writeSetOfPairs(theset):
    return '{ ' + ', '.join('(%s,%s)' % (a,b) for (a,b) in sorted(theset)) + ' }'

universe = set([sv,vs,tv,vt,rv,vr,uv,vu,wv,vw,xv,vx])
print('Universe: {',', '.join(sorted(universe)), '}')

invs = [(sv,vs), (tv,vt), (rv,vr), (uv,vu), (wv,vw), (xv,vx)]
invs = invs + list((y,x) for (x,y) in invs)

def ast(x):
    return [b for (a,b) in invs if a == x][0]

order = [(vs, sv), (vt, tv),
         (vs, vu), (vt, vu), (vu, rv), (vu, wv), (vu, vx),
         (vs, vx), (tv, vx),
         (sv, rv), (vt, rv), (rv, tv), (wv, vx),
         (sv, wv), (tv, wv)]

order = order + list((ast(b), ast(a)) for (a,b) in order)


def transitiveHull(x):
    succ = lambda y: (b for (a, b) in x if a == y)

    newX = set(x)
    prevX = set()
    while not newX == prevX:
        prevX = frozenset(newX)
        for (a,b) in prevX:
            newX |= set((a,c) for c in succ(b))
    return newX

hull = transitiveHull(order)
LE = hull.union( (a,a) for a in universe )

print('Order:', writeSetOfPairs(LE))
```

```
def isSmall(a):
    return (a, ast(a)) in LE

def sup(a,b):
    candidates = list(c for c in universe if (a,c) in LE and (b,c) in LE)
    leasts = [l for l in candidates if all((l,c) in LE for c in candidates)]
    if len(leasts) == 1:
        return leasts[0]
    else:
        raise Exception(a + ' v ' + b + ' does not exist. Candidates are: ' + ',
        ↪  '.join(candidates))

def inf(a,b):
    return ast(sup(ast(a), ast(b)))


for a in universe:
    for b in universe:
        assert( (sup(a,b) == b) == ((a,b) in LE) ) # LE matches sup
        assert( (inf(a,b) == b) == ((b,a) in LE) ) # LE matches inf
        assert( ((a,b) in LE) == ((ast(b),ast(a)) in LE) ) # Involution
        for c in universe:
            assert( sup(sup(a,b),c) == sup(a, sup(b,c)) ) # Associative
            assert( inf(inf(a,b),c) == inf(a, inf(b,c)) ) # Associative
            assert( inf(sup(a,b),c) == sup(inf(a,c), inf(b,c)) ) # Distributive
        assert( sup(a,b) == sup(b,a) ) # Commutative
        assert( inf(a,b) == inf(b,a) ) # Commutative
        assert( sup(a, inf(a,b)) == a) # Absorption
        assert( inf(a, sup(a,b)) == a) # Absorption
print('This does in fact form a universe.')

skeleton = []
for (a,b) in LE: # We throw out every pair (a,b) where a c inbeetween exists. Thus we
↪  can reconstruct LE as a transitive hull. We also remove all (a,a).
    if a != b:
        direct = True
        for c in universe:
            if c != a and c!=b and (a,c) in LE and (c,b) in LE:
                direct = False
                break
        if direct:
            skeleton.append((a,b))
skeleton = set(skeleton)
assert(transitiveHull(skeleton) == hull)
print('Which is the transitive hull of:', writeSetOfPairs(skeleton))

print('The small separations are:')
for a in sorted(universe):
    if isSmall(a):
        print(a)
        if isSmall(ast(a)):
            raise Exception(a, 'is degenerate!')
```

```
print('No degenerate elements.')

print('s> sup t> sup u< = ', sup(sup(vs,vt),uv))

theta = [vr,vs,vt,vw,xv,vu]
for a in theta:
    for b in theta:
        assert(sup(a,b) in theta)
print('theta = ', theta,' is a profile')

# There are more clever ways to do this...
for a in theta:
    for b in theta:
        for c in theta:
            for d in theta:
                for e in theta:
                    for f in theta:
                        s = sup(sup(sup(sup(sup(a,b),c),d),e),f)
                        assert(not isSmall(ast(s)))
print('theta is a tangle')
```

And the output:

```
Universe: { r<, r>, s<, s>, t<, t>, u<, u>, w<, w>, x<, x> }
Order: { (r<,r<), (r<,t<), (r<,w<), (r<,x>), (r>,r<), (r>,r>), (r>,s<), (r>,s>),
 ↪  (r>,t<), (r>,u<), (r>,u>), (r>,w<), (r>,x>), (s<,r<), (s<,s<), (s<,t<), (s<,w<),
 ↪  (s<,x>), (s>,r<), (s>,s<), (s>,s>), (s>,t<), (s>,u>), (s>,w<), (s>,x>), (t<,t<),
 ↪  (t<,w<), (t<,x>), (t>,r<), (t>,r>), (t>,s<), (t>,s>), (t>,t<), (t>,t>), (t>,u<),
 ↪  (t>,u>), (t>,w<), (t>,x>), (u<,r<), (u<,s<), (u<,t<), (u<,u<), (u<,w<), (u<,x>),
 ↪  (u>,r<), (u>,t<), (u>,u>), (u>,w<), (u>,x>), (w<,w<), (w<,x>), (w>,r<), (w>,r>),
 ↪  (w>,s<), (w>,s>), (w>,t<), (w>,t>), (w>,u<), (w>,u>), (w>,w<), (w>,w>), (w>,x>),
 ↪  (x<,r<), (x<,r>), (x<,s<), (x<,s>), (x<,t<), (x<,t>), (x<,u<), (x<,u>), (x<,w<),
 ↪  (x<,w>), (x<,x<), (x<,x>), (x>,x>) }
This does in fact form a universe.
Which is the transitive hull of: { (r<,t<), (r>,s>), (r>,u<), (s<,r<), (s>,s<),
 ↪  (s>,u>), (t<,w<), (t>,r>), (u<,s<), (u>,r<), (w<,x>), (w>,t>), (x<,w>) }
The small separations are:
r>
s>
t>
w>
x<
No degenerate elements.
s> sup t> sup u< =  s<
theta =  ['r>', 's>', 't>', 'w>', 'x<', 'u>']  is a profile
theta is a tangle
```

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Eidesstattliche Erklärung

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Hamburg, den 17. August 2017

Jean Maximilian Teegen