Universität Hamburg
Fachbereich Mathematik

Masterarbeit

Infinite tree sets and their representations

vorgelegt von

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October, 2016

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Contents

1 Introduction

Separations of graphs and sets have been studied for a long time. For instance tree-decompositions have both theoretical and algorithmic applications when searching for dense objects in a given graph. While any tree-decomposition of a graph into small parts witnesses that a graph has low tree-width, there are various dense objects that force high tree-width in a graph. Among these are large cliques and clique minors, large grids and grid minors as well dense brambles.

In order to fit all these different objects into a unifying framework one can turn to graph separations. All these dense objects in a graph have the property that they orient its low-order separations by lying mostly on one side of the separation. For any given dense structure in a graph these orientations of separations are consistent with each other: for two nested low-order separations the dense object cannot lie on the left side of the left separation *and* to the right of the right separation. Thus if one imagines the oriented separations as pointing towards the dense object no two of them 'disagree' about where the dense object lies by pointing away from each other. Furthermore if for instance a large complete minor M orients two crossing low-order separations as 'right' and 'up' respectively, then M orients the top-right corner separation towards the top right, provided it is of low enough order to be oriented at all. One can now attempt to study the dense structures in a graph indirectly by focusing on these consistent orientations they induce on the low-order separations.

In [8] Robertson and Seymour proposed the notion of tangles, which are such families of consistently oriented separations up to a certain order. These tangles can be studied in their own right, instead of any dense objects that may induce them. By varying the strength of the consistency conditions one can model different kinds of dense objects, and the resulting consistent orientations give rise to different types of tangle.

To talk about these separations systems one does not even need an underlying graph structure or ground set: they can be formulated in a purely axiomatic way. Such a separation system is simply a poset with an order-reversing involution, and possibly a submodular order function. The notions of consistency of separations that come from large structures in graphs can be translated into this setting as well. The tangles of graphs then become abstract tangles, and the tree-like structures become nested systems of separations. This abstract framework turns out to be no less powerful, even for graphs alone, than ordinary graph separations. In [5] Diestel and Oum established a unifying duality theorem for separation systems which easily implies all the classical duality results from graph- and matroid theory, such as the tree-width duality theorem by Seymour and Thomas [9]. Furthermore this abstract notion of separation systems can be applied in fields outside of graph theory, for instance in image compression [6].

In an abstract separation system there might be separations which, for basic consistency reasons, get oriented the same way by every abstract tangle. These elements do not add any structural information, and we call them *trivial*. Surprisingly the trivial elements of a separation system have a simple characterization that uses only the partial order and the involution and does not refer to any particular notion of consistency depending on the kind of tangle considered.

A subset of non-trivial separations which are pairwise comparable is called a *tree set*. These tree sets play the role that tree-decompositions played in graphs: the unified duality theorem asserts that for any sensible notion of consistency a separation system contains either an abstract tangle or a tree set witnessing that no such tangle exists. In this sense, tree sets are dual to tangles and therefore important.

For graphs, Robertson and Seymour proved in [8] that every finite graph has a tree-decomposition which distinguishes all its distinguishable tangles. This theorem, too, can be proved – and even strengthened – in the setting of abstract separations systems: In [4] Diestel, Hundertmark and Lemanczyk established an abstract tangle-tree theorem, which (roughly) says that, for every set of certain abstract tangles in a finite separation system, there is a nested set of separations which distinguishes all these tangles. As trivial separations cannot distinguish abstract tangles this nested set of separations will be a tree set in most natural applications.

Both the unified duality theorem and the tangle-tree theorem address finite separations systems only. A natural stepping stone on the way towards the long-term goal of proving infinite versions of these theorems is the study of *profinite* separation systems: for an infinite ground set V a pair $\{A, B\}$ is a separation of V if and only if $\{A \cap H,\, B \cap H\}$ is a separation of H for every finite subset H of V, and the same is true for separations of graphs. Moreover, a separation of an infinite graph or set is uniquely determined by the family of its restrictions to the finite subgraphs or subsets. Therefore one can study the separations of an infinite graph without direct reference to the graph itself, but work instead with these finite separations.

This reduction can be formulated in the setting of abstract separations systems as well: we call a separation system *profinite* if it arises as the inverse limit of a system of finite separation systems. If all the finite projections of a profinite separation system are tree sets then that system, too, is a tree set. The converse is far from true, because finite projections of nontrivial separations can be trivial. Since tree sets are at the centre of both the unified duality theorem and the tangle-tree theorem we therefore need a thorough study of how infinite tree sets are related to their finite projections before we can attempt to lift those theorems from finite to profinite separation systems.

Tree sets are also interesting objects in their own right: they are flexible enough to model a whole range of other 'tree-like' structures in discrete mathematics, such as ordinary graph trees, order trees and nested systems of bipartitions of sets. For instance in [3] Diestel showed how to recover a tree-decomposition of a graph from the tree set of separations it induces, and explored the relationship between nested bipartitions of sets and abstract tree sets.

Continuing these investigations we shall analyse which tree sets are induced by certain tree-like structures.

First, in Chapter 3, we characterize the tree sets that are induced by graph-theoretical trees. Tree sets arising from graph-theoretical trees have no non-trivial limits, and this property characterizes them: an abstract tree set comes from a suitable graph if and only if it has no non-trivial limits.

In the remainder of Chapter 3 we study the connection between tree sets and nested bipartitions of sets. Every *regular* tree set (that is, in which no element is comparable to its own inverse) can be viewed as a nested set of bipartitions of a suitable ground set, much in the same way as the edges of a tree T can be represented by the components of $T - e$. Theorem 2 describes a straightforward and easy-to-prove way to do so. However Theorem 2 is not very efficient in terms of the size of the ground set it uses. In many cases it is possible to use a much smaller ground set without losing any structural information; we explore a few different possibilities for this.

Chapter 4 is dedicated to the study of profinite tree sets. We prove a characterization of the profinite tree sets and show that every profinite tree set can be obtained as an inverse limit of finite tree sets. As seen in Chapter 3, regular tree sets are especially interesting due to their connections to other tree-like structures. We also give a characterization of the regular profinite tree sets. As it turns out, regularity is a very strong restriction for profinite tree sets. Our characterization in Chapter 4 shows that regular profinite tree sets are, somewhat surprisingly, only slightly richer that finite tree sets. As a first application of this study of profinite tree sets we lift one of the theorems from Chapter 4 to the class of regular profinite tree sets.

In the Chapter 5, finally, we turn our attention to topological limit objects of graphs that are not themselves graphs: so-called graph-like spaces. We prove a theorem about the connectivity of compact graph like spaces which enables us to say precisely when a graph-like space is tree-like. Finally we show that every regular tree set can be represented as the edge tree set of such a tree-like space.

2 Separation systems

Abstract separations systems model two common types of separations at once: graph separations and separations of sets. In order to offer the reader some intuition for abstract separation systems, we therefore briefly discuss these examples first. We shall then move on to establish basic facts about abstract separation systems, which will be needed later.

For a graph $G = (V, E)$ a *graph separation* is a set $\{A, B\}$ with $A \cup B = V$ such that there is no edge $e \in E$ with one end-vertex in $A \backslash B$ and one in $B \backslash A$. A separation $\{A, B\}$ of G has the two orientations (A, B) and (B, A), and we call $\{A, B\}$ an unoriented separation and (A, B) and (B, A) oriented separations of G.

For a ground set V a *bipartition* of V, which we also call an oriented separation of the set V, is an ordered pair (A, B) of disjoint subsets $A, B \subseteq V$ with $A \dot{\cup} B = V$.

We write $\vec{S}(G)$ for the set of all oriented separations of G, and $\mathcal{B}(V)$ for the set of all bipartitions of a set V. On both these sets the map mapping (A, B) to its *inverse* (B, A) is an involution. We can define a partial order on $\vec{S}(G)$ and $\mathcal{B}(V)$ in the following way:

$$(A, B) :\leqslant (C, D) \qquad \Leftrightarrow \qquad A \subseteq C \text{ and } D \subseteq B.$$

Then the involution is *order-reversing*, that is

$$(A, B) \leqslant (C, D) \qquad \Leftrightarrow \qquad (B, A) \geqslant (D, C)$$

for all oriented separations (A, B) and (C, D).

Large structures in a graph define an orientation of the low-order separations. For example let $G = (V, E)$ be a graph, K a clique of size k in G and $\vec{S}_k = \vec{S}_k(G)$ the set of all $(A, B) \in \vec{S}(G)$ with $|A \cap B| < k$ for some $k \in \mathbb{N}$. Then either $K \subseteq A$ or $K \subseteq B$ for each $(A, B) \in \vec{S}_k$, but not both. Let O be the set of all $(A, B) \in \vec{S}_k$ with $K \subseteq B$; then O is an *orientation* of $\vec{S}_k$ in the sense that it contains exactly one of (A, B) and (B, A) for each low-order separation $\{A, B\}$. This orientation is *consistent* is the following sense: if $(A, B) \in O$ and $(C, D) \leqslant (A, B)$ for some $(C, D) \in \vec{S}_k$ with $\{A, B\} \neq \{C, D\}$, then $(C, D) \in O$ too. Thus there are no two separations $(A, B), (C, D) \in O$ with $\{A, B\} \neq \{C, D\}$ and $(B, A) \leqslant (C, D)$.

To work with sets of separations such as $\vec{S}(G)$ or $\mathcal{B}(V)$ we do not really need to know G or V. All the information we need are the partial order on them and the involution map; this environment is still rich enough to meaningfully talk about consistent orientations, as the example above demonstrates.

Formally, an abstract *separation system* $(\vec{S}, \leqslant, *)$ is a partially ordered set with an order-reversing involution $*$. An element $\vec{s} \in \vec{S}$ is called an *oriented separation*, and its *inverse* $(\vec{s})^*$ is denoted as $\overleftarrow{s}$, and vice versa. The pair $s = \{\vec{s}, \overleftarrow{s}\}$ is an *unoriented separation*[1], with *orientations* $\vec{s}$ and $\overleftarrow{s}$, and the set of all such pairs is denoted as S. The assumption that $*$ is order-reversing means that for all $\vec{s}, \vec{r} \in \vec{S}$

$$\vec{s} \leqslant \vec{r} \qquad \Leftrightarrow \qquad \overleftarrow{s} \geqslant \overleftarrow{r}.$$

[1]To improve readability 'oriented' and 'unoriented' will often be omitted if the type of separation follows from the context.

If S' is a set of unoriented separations, we write $\vec{S'}$ for the set $\bigcup S'$ of all orientations of separations in S'.

A separation $\vec{s}$ is *small* and its inverse $\overleftarrow{s}$ *co-small* if $\vec{s} \leqslant \overleftarrow{s}$. If $\vec{s}$ is both small and co-small, that is if $\vec{s} = \overleftarrow{s}$, then $\vec{s}$ and s are *degenerate*. If neither $\vec{s}$ nor $\overleftarrow{s}$ is small s is *regular*, and we call both $\vec{s}$ and $\overleftarrow{s}$ regular too.

A separation $\vec{s} \in \vec{S}$ is *trivial in* $\vec{S}$ and its inverse $\overleftarrow{s}$ is *co-trivial in* $\vec{S}$ if there is some $\vec{r} \in \vec{S}$ with $\vec{s} \leqslant \vec{r}, \overleftarrow{r}$ and $s \neq r$. In this case r is the *witness* of the triviality of $\vec{s}$. If neither $\vec{s}$ nor $\overleftarrow{s}$ is trivial in $\vec{S}$ we call s *nontrivial*.

As shorthand notation we write $\vec{s} \lneqq \vec{r}$ to mean $\vec{s} \leqslant \vec{r}$ and $s \neq r$ for $\vec{s}, \vec{r} \in \tau$. Note that this differs from $\vec{s} < \vec{r}$, which just means $\vec{s} \leqslant \vec{r}$ and $\vec{s} \neq \vec{r}$.

If $\vec{s}$ is a trivial separation with witness r then $\vec{s}$ is small as $\vec{s} \leqslant \vec{r} \leqslant \overleftarrow{s}$. Conversely if $\vec{r} \lneqq \vec{s}$ and $\vec{s}$ is small, then $\vec{r}$ is trivial as $\vec{r} < \vec{s} \leqslant \overleftarrow{s}$.

Two unoriented separations s, r are *nested* if they have comparable orientations. Otherwise r and s *cross*. A set S' of separations is nested if all of its elements are pairwise nested.

A *tree set* is a nested separation system with no trivial elements. It is *regular* if none of its elements is small.

An *orientation* of a set $\vec{S'}$ or S' of separations is a set $O \subseteq \vec{S'}$ with $|O \cap s| = 1$ for every $s \in S'$. An orientation is *consistent* if $\overleftarrow{s} \leqslant \vec{r}$ implies $r = s$ for all $\vec{r}, \vec{s} \in O$. A *partial orientation* of $\vec{S}$ is an orientation of a subset of $\vec{S}$. A partial orientation P *extends* another partial orientation Q if $Q \subseteq P$.

For a tree set τ an orientation O of τ is a *splitting* orientation if it is consistent and has the property that for every $\vec{r} \in O$ there is some maximal element $\vec{s}$ of O with $\vec{r} \leqslant \vec{s}$. These splitting orientations can be thought of as the 'vertices' of a tree set, an idea that we will make more precise in the next chapter.

A subset $\sigma \subseteq \tau$ is a *star* if $\vec{r} \leqslant \overleftarrow{s}$ for all $\vec{r}, \vec{s} \in \sigma$ with $\vec{r} \neq \vec{s}$. For example, the set of maximal elements of a consistent orientation of a tree set is always a star:

Lemma 2.1. *Let O be a consistent orientation of a tree set τ. Then the set σ of the maximal elements of O is a star.*

Proof. Let $\vec{r}, \vec{s} \in \sigma$ with $\vec{r} \neq \vec{s}$ be given. Then neither $\vec{r} \leqslant \vec{s}$ nor $\vec{r} \geqslant \vec{s}$ as both are maximal elements of O. The consistency of O implies that $\vec{r} \neq\geqslant \overleftarrow{s}$, so $\vec{r} \leqslant \overleftarrow{s}$ is the only possible relation and hence σ is a star. □

A star $\sigma \subseteq \tau$ *splits* τ, or is a *splitting star* of τ, if it is the set of maximal elements of a splitting orientation of τ. If a splitting star has more than two elements we also call it a *branching* star and its elements *branching points*. Note that every element of a finite tree set lies in a splitting star as we show in the next chapter, but infinite tree sets can have elements that lie in no splitting star; see Example 2.3 and Lemma 2.4 below.

More generally, given a partial orientation P of τ, is it possible to extend it to a consistent orientation of τ? Of course P needs to be consistent itself for this to be possible. The next Lemma shows that under this necessary assumption it is always possible to extend a partial orientation to all of τ. In particular, every element of a tree set induces a consistent orientation in which it is a maximal element. This orientation is in fact unique:

Lemma 2.2 (Extension Lemma)**.** [3] *Let S be a set of separations, and let P be a consistent partial orientation of S.*

(i) *P extends to a consistent orientation O of S if and only if no element of P is co-trivial in S.*

(ii) *If $\vec{p}$ is maximal in P, then O in* (i) *can be chosen with $\vec{p}$ maximal in O if and only if $\vec{p}$ is nontrivial in $\vec{S}$.*

(iii) *If S is nested, then the orientation O in* (ii) *is unique.*

The last part of the Extension Lemma implies that any element $\vec{s}$ of a tree set τ is maximal in exactly one consistent orientation O of τ. Hence $\vec{s}$ lies in a splitting star if and only if this O is splitting.

In an infinite tree set there might be elements that do not lie in a splitting star:

Example 2.3. Let τ be the tree set with ground set

$$\{\,\vec{s}_n \mid n \in \mathbb{N}\} \cup \{\,\overleftarrow{s}_n \mid n \in \mathbb{N}\} \cup \{\,\vec{t}\,, \overleftarrow{t}\,\},$$

where $\vec{s}_i \leqslant \vec{s}_j$ and $\overleftarrow{s}_i \geqslant \overleftarrow{s}_j$ whenever $i \leqslant j$, as well as $\vec{s}_n \leqslant \vec{t}$ and $\overleftarrow{s}_n \geqslant \overleftarrow{t}$ for all $n \in \mathbb{N}$. The separation $\overleftarrow{t}$ is maximal in the orientation

$$O = \{\,\vec{s}_n \mid n \in \mathbb{N}\} \cup \{\,\overleftarrow{t}\,\},$$

which is not splitting as no $\vec{s}_n$ lies below a maximal element of O. Hence $\overleftarrow{t}$ does not lie in a splitting star of τ.

In the above example the chain $C = \{\,\vec{s}_n \mid n \in \mathbb{N}\} \cup \{\,\vec{t}\,\}$ has order-type $\omega+1$. In fact if a tree set τ does not contain a chain of this order type then every separation of τ lies in a splitting star, and vice-versa:

Lemma 2.4. *For a tree set τ every element if τ lies in a splitting star of τ if and only if τ contains no chain of order-type $\omega + 1$.*

Proof. For the forward direction let C be a chain of order-type ω with supremum $\vec{t}$ in τ and set $P = C \cup \{tv\}$. Then P is a consistent partial orientation of τ and $\overleftarrow{t}$ is maximal in P, so by the Extension Lemma P extends to a unique orientation O of τ in which $\overleftarrow{t}$ is maximal. This O is not splitting: pick any $\vec{c} \in C$. Suppose there is a maximal element $\vec{m}$ of O with $\vec{c} \leqslant \vec{m}$. Then $\vec{m} \neq \overleftarrow{t}$ as $\vec{c} \lneqq \vec{t}$, so $\vec{m} \lneqq \vec{t}$ by Lemma 2.1 and the star-property. As $\vec{t}$ is the supremum of C the separation $\vec{m}$ cannot be an upper bound for C, so there is some $\vec{d} \in C$ with $\vec{d} \nleqslant \vec{m}$. Hence $\vec{d} \leqslant \overleftarrow{m}$ by the maximality of $\vec{m}$ and the consistency of O, but this implies $\vec{c} \leqslant \vec{d} \leqslant \overleftarrow{m}$, so $\vec{c}$ would be trivial with witness m. Therefore O is not splitting and $\overleftarrow{t}$ does not lie in a splitting star of τ.

For the converse assume τ contains no chain of order-type $\omega + 1$. Let $\vec{s} \in \tau$ be any separation and O the unique consistent orientation of τ in which $\vec{s}$ is maximal. We claim that O is splitting. To this end let $\vec{t} \in O$. If $\vec{t} \leqslant \vec{s}$ then $\vec{t}$ lies below a maximal element of O and there is nothing to show. Otherwise $\vec{t} \leqslant \overleftarrow{s}$ by the consistency of O. Let C be a maximal chain in O with minimum $\vec{t}$. Due to consistency $\vec{c} \lneqq \overleftarrow{s}$ for all $\vec{c} \in C$. Thus if C is infinite the chain $C \cup \{\,\overleftarrow{s}\}$ contains a chain of order-type $\omega + 1$, contrary to assumption. Hence C is finite. By the maximality of C the maximum $\vec{m}$ of C is a maximal element of O with $\vec{t} \leqslant \vec{m}$. Therefore O is splitting, as claimed. □

A direct consequence of Lemma 2.4 is that every element of a finite tree set lies in a splitting star.

Given two separation systems R, S, a map $f\colon R \to S$ is a *homomorphism* of separation systems if it *commutes with the involution* and is *order-preserving*. The map f commutes with the involution if $\big(f(\vec{r})\big)^* = f(\overleftarrow{r})$ for all $\vec{r} \in R$, and f is order-preserving if $f(\vec{r}_1) \leqslant f(\vec{r}_2)$ whenever $\vec{r}_1 \leqslant \vec{r}_2$ for all $\vec{r}_1, \vec{r}_2 \in R$. Please note that the condition for f to be order-preserving is not 'if and only if': it is allowed that $f(\vec{r}_1) \leqslant f(\vec{r}_2)$ for incomparable $\vec{r}_1, \vec{r}_2 \in R$. Furthermore f need not be injective.

As all trivial separations are small every regular nested separation system is a tree set. These two properties, regular and nested, are preserved by homomorphisms of separations systems, albeit in different directions: the image of nested separations is nested, and the pre-image of regular separations is regular.

Lemma 2.5. *Let $f\colon R \to S$ be a homomorphism of separation systems. If S is regular then so is R; and if R is nested then so is its image in S.*

Proof. First suppose some $\vec{r} \in R$ is small, that is $\vec{r} \leqslant \overleftarrow{r}$. But then

$$f(\vec{r}) \leqslant f(\overleftarrow{r}) = \big(f(\vec{r})\big)^*,$$

so S contains a small element. Therefore if S is regular then R must be too.

Now consider two unoriented separations $s, s' \in S$. If there are $r, r' \in R$ with $s = f(r)$ and $s' = f(r')$ and R is nested, then, say, $\vec{r} \leqslant \vec{r}'$ and thus $\vec{s} := f(\vec{r}) \leqslant f(\vec{r}') =: \vec{s}'$. Hence if R is nested its image in S is nested too. □

A bijection $f\colon R \to S$ is an *isomorphism* of separation systems if both f and its inverse map are homomorphisms of separation systems. Two separation systems R, S are *isomorphic*, denoted as $R \cong S$, if there is an isomorphism $f\colon R \to S$ of separation systems. If one of R and S (and thus both) is a tree set we call f an *isomorphism of tree sets*.

Lemma 2.5 makes it possible to show that a homomorphism $f\colon R \to S$ of separation systems is an isomorphism of tree sets without knowing beforehand that either R or S is a tree set:

Lemma 2.6. *Let $f\colon R \to S$ be a bijective homomorphism of separation systems. If R is nested and S regular then f is an isomorphism of tree sets.*

Proof. From Lemma 2.5 it follows that both R and S are regular and nested, which means they are regular tree sets. Therefore all we need to show is that the inverse of f is order-preserving, i.e. that $\vec{r}_1 \leqslant \vec{r}_2$ whenever $f(\vec{r}_1) \leqslant f(\vec{r}_2)$. Let $\vec{r}_1, \vec{r}_2 \in R$ with $f(\vec{r}_1) \leqslant f(\vec{r}_2)$ be given. As R is nested r_1 and r_2 have comparable orientations.

If $\vec{r}_1 \geqslant \vec{r}_2$ then $f(\vec{r}_1) = f(\vec{r}_2)$, implying $\vec{r}_1 = \vec{r}_2$ and hence the claim.

If $\vec{r}_1 \leqslant \overleftarrow{r}_2$ then $f(\vec{r}_1) \leqslant f(\vec{r}_2), f(\overleftarrow{r}_2)$, contradicting the fact that S is a regular tree set.

Finally if $\vec{r}_1 \geqslant \overleftarrow{r}_2$ then $f(\overleftarrow{r}_2) \leqslant f(\vec{r}_2)$, contradicting the fact that S is regular.

Hence $\vec{r}_1 \leqslant \vec{r}_2$, as desired. □

In our applications we sometimes already know that S is a tree set, but not that S is regular. The proof of Lemma 2.6 still goes through though if we know that the pre-images of small separations are small:

Lemma 2.7. *Let $f: R \to S$ be a bijective homomorphism of separation systems. If R is nested, S is a tree set and $\vec{r} \in R$ is small whenever $f(\vec{r})$ is small, then f is an isomorphism of tree sets.*

Proof. It suffices to show that the inverse of f is order-preserving. Let $\vec{r}_1, \vec{r}_2 \in R$ with $f(\vec{r}_1) \leqslant f(\vec{r}_2)$ be given. If $f(\vec{r}_1) = f(\vec{r}_2)$ or $f(\overleftarrow{r}_1) = f(\vec{r}_2)$ we have $\vec{r}_1 \leqslant \vec{r}_2$ by assumption Therefore we may assume that $f(\vec{r}_1) \lneqq f(\vec{r}_2)$ and hence $r_1 \neq r_2$. As R is nested r_1 and r_2 have comparable orientations.

If $\vec{r}_1 \geqslant \vec{r}_2$ then $f(\vec{r}_1) = f(\vec{r}_2)$ contradicting $f(\vec{r}_1) \lneqq f(\vec{r}_2)$.

If $\vec{r}_1 \lneqq \overleftarrow{r}_2$ then $f(\vec{r}_1) \lneqq f(\vec{r}_2), f(\overleftarrow{r}_2)$, contradicting the fact that S contains no trivial element.

Finally if $\vec{r}_1 \geqslant \overleftarrow{r}_2$ then $f(\overleftarrow{r}_2) \lneqq f(\vec{r}_1), f(\overleftarrow{r}_2)$, again contradicting the fact that S contains no trivial element.

Hence $\vec{r}_1 \leqslant \vec{r}_2$, as desired. □

For a non-empty set $X \subseteq \mathbb{R}$ of real numbers the *natural tree set on X* is the tree set

$$\tau(X) := \left(\bigcup_{x \in X} \{\vec{x}, \overleftarrow{x}\}, \leqslant, * \right),$$

where $(\vec{x})^* := \overleftarrow{x}$ as well as $\vec{x} :\leqslant \vec{y}$ and $\overleftarrow{x} :\geqslant \overleftarrow{y}$ if and only if $x \leqslant y$.

3 Representations of tree sets

Tree sets are flexible enough to model a whole range of tree-like structures. For example in [3] Diestel showed that every order tree can be represented by a tree set together with a consistent orientation. Furthermore every graph-theoretical tree T gives rise to a tree set, its *edge tree set* $\tau(T)$ (see below for a formal definition). However, while every tree gives rise to a tree set, not every tree set comes from a tree. In the following two sections we give a characterization of those tree sets that can be represented by a tree.

Every nested set of non-trivial bipartitions of a fixed ground set is a regular tree set. Conversely, in Theorem 2 in Section 3.3, we show that every regular tree set can be represented by nested bipartitions of a suitable ground set. However Theorem 2 uses a very large ground set. We give multiple ways of representing both finite and infinite regular tree sets by nested bipartitions of a set that improve on Theorem 2 by only using certain subsets of its ground set.

3.1 Tree sets and trees – Introduction

As tree sets are an abstraction of tree-decompositions of graphs it is natural to study their relation to graph-theoretical trees. In this section we characterize those infinite tree sets that arise from graph-theoretical trees. First we need to make precise the way in which a tree gives rise to a tree set.

Let $T = (V, E)$ be a graph-theoretical tree, finite or infinite. Let $\vec{E}(T)$ be the set of oriented edges of T, that is

$$\vec{E}(T) = \{(x, y) \mid \{x, y\} \in E(T)\}.$$

Then the *edge tree set* $\tau(T) = (\vec{E}(T), \leqslant, *)$ is defined by setting $(x, y)^* := (y, x)$ and $(x, y) < (v, w)$ for edges $xy, vw \in E(T)$ if and only if $\{x, y\} \neq \{v, w\}$ and the unique $\{x, y\}$-$\{v, w\}$-path in T joins y to v. It is straightforward to check that $\tau(T)$ is a regular tree set.

If T is the decomposition tree of a tree-decomposition of a graph G, then the tree set $\tau(T)$ is isomorphic to the tree set formed by the separations of G that correspond to the edges of T (with some pathological exceptions). This relationship between tree-decompositions and tree sets was further explored in [3].

3.2 Tree sets and trees – Characterization

For which tree sets τ is there a graph-theoretical tree T such that $\tau \cong \tau(T)$? For any tree T its edge tree set $\tau(T)$ is regular, so such a τ must be regular too. If τ is finite, it has been shown in [3] that this assumption alone is enough to ensure the existence of a tree T with $\tau \cong \tau(T)$. But what about infinite tree sets?

If T is a (possibly infinite) tree then $\tau(T)$ is regular and contains no chain of order-type $\omega + 1$: every maximal chain in $\tau(T)$ is a directed path, ray or double ray in T. Thus, for τ to to come from some tree T it is a necessary condition that τ is regular and contains no chain of order-type $\omega + 1$. We show that this necessary condition is also sufficient.

From now on let τ be a regular tree set with no chain of order-type $\omega + 1$.

Our aim is to find a tree $T = T(\tau)$ with $\tau \cong \tau(T)$. Recall that a consistent orientation O of τ is called splitting if every element of O lies below some maximal element of O. By the uniqueness part of the Extension Lemma every splitting star extends to exactly one splitting orientation. Write $\overline{\mathcal{O}}$ for the set of all splitting orientations of τ. We will later use $\overline{\mathcal{O}}$ as the vertex set of T.

Let us show first that, for any two splitting stars, each of them contains exactly one element that is inconsistent with the other star. We will later use this little fact when we define the edges of our tree.

Lemma 3.1. *Let σ_1, σ_2 be two distinct splitting stars of τ and $O_2 \in \overline{\mathcal{O}}$ the orientation inducing σ_2. Then there is exactly one $\vec{s} \in \sigma_1$ with $\overleftarrow{s} \in O_2$.*

Proof. There is at least one such $\vec{s}$ as O_2 does not induce σ_1. For any two $\vec{r}, \vec{s} \in \sigma$ the set $\{\overleftarrow{r}, \overleftarrow{s}\}$ is inconsistent, so there is at most one $\vec{s} \in \sigma_1$ with $\overleftarrow{s} \in O_2$. □

Note that this Lemma holds for every tree set as the proof did not use any assumptions on τ.

Our assumption that τ does not contain a chain of order type $\omega + 1$ implies the following sufficient condition for a consistent orientation to be splitting:

Lemma 3.2. *Let O be a consistent orientation of τ with at least one maximal element. Then O splits τ.*

Proof. Let $\vec{m} \in O$ be a maximal element of O. Assume some $\vec{s} \in O$ does not lie below any maximal element of O. Then in particular $\vec{s} \nleqslant \vec{m}$, so $\vec{s} \leqslant \overleftarrow{m}$ by consistency. Let C be a maximal chain in O with minimum $\vec{s}$. If any $\vec{r} \in C$ would lie below a maximal element of O then so would $\vec{s}$, hence C must be infinite. Furthermore for $\vec{r} \in C$ neither $\vec{r} \leqslant \vec{m}$ nor $\vec{r} \geqslant \vec{m}$ by assumption, nor $\vec{r} \geqslant \overleftarrow{m}$ by the consistency of O, so $\vec{r} \leqslant \overleftarrow{m}$ for all $\vec{r} \in C$. But then C together with $\overleftarrow{m}$ is a chain of order type $\omega + 1$. □

Together with the Extension Lemma this immediately implies the following:

Corollary 3.3. *Every $\vec{s} \in \tau$ lies in exactly one splitting star of τ. Equivalently every $\vec{s} \in \tau$ is maximal in exactly one consistent orientation O and $O \in \overline{\mathcal{O}}$.*

Proof. For $\vec{s} \in \tau$ apply the Extension Lemma to $\{\vec{s}\}$ to obtain a unique consistent orientation O of τ in which $\vec{s}$ is a maximal element. It then follows from Lemma 3.2 that O is splitting. □

For $\vec{s} \in \tau$ write $O(\vec{s})$ for the unique consistent orientation of τ in which $\vec{s}$ is maximal. Then Lemma 3.1 together with Corollary 3.3 says that for distinct $O, O' \in \overline{\mathcal{O}}$ there is exactly one $\vec{s} \in O'$ with $O(\overleftarrow{s}) = O$.

Now we define the graph $T = T(\tau)$. Let $V(T) = \overline{\mathcal{O}}$ and

$$E(T) = \{\{O(\vec{s}), O(\overleftarrow{s})\} \mid \vec{s} \in \tau\}.$$

We call $T(\tau)$ the *tree of the splitting orientations of* τ, where τ is a regular tree set with no chain of order-type $\omega + 1$.

We need to check that $T = T(\tau)$ is a tree and that $\tau \cong \tau(T)$. First we show that T is acyclic. For $O \in V(T) = \overline{\mathcal{O}}$ the set of incoming edges is precisely the splitting star induced by O. If $\vec{s}_1, \ldots, \vec{s}_k$ are the edges of an oriented cycle in $\vec{T}$, then each of these and the inverse of its cyclic successor lie in a common splitting star. Hence $\vec{s}_1 \leqslant \vec{s}_2 \leqslant \cdots \leqslant \vec{s}_k \leqslant \vec{s}_1$ by the star property, a contradiction.

Proving that T is connected is more difficult. Our strategy is as follows. To find a path from $O \in \overline{\mathcal{O}}$ to $O' \in \overline{\mathcal{O}}$ we use Lemma 3.1 to find $\vec{s} \in O$ which is maximal in O with $\overleftarrow{s} \in O'$. Then we consider $O^* := O + \{\overleftarrow{s}\} - \{\vec{s}\}$. This orientation is again in $\overline{\mathcal{O}}$ and a neighbour of O in T. If $O^* = O'$ we are done; otherwise we can iterate the process with O^* and O'. Either this process terminates after finitely many steps, in which case we found a path from O to O', or it continues indefinitely. In the latter case the infinitely many separations we inverted form a chain with an upper bound in O', which would be a chain of order type $\omega + 1$.

The next short Lemma forms the basis of this iterative flipping process.

Lemma 3.4. *Let $\vec{s}_1, \ldots, \vec{s}_n, \vec{s}' \in \tau$ be distinct separations with $O(\overleftarrow{s}_{k+1}) = O(\vec{s}_k)$ for $1 \leqslant k < n$ and $\vec{s}_n < \vec{s}'$. Then there is a $\vec{s}_{n+1} \in \tau$ with $O(\overleftarrow{s}_{n+1}) = O(\vec{s}_n)$ and $\vec{s}_{n+1} \leqslant \vec{s}'$.*

Proof. Let $\vec{s}_{n+1}$ be the unique separation in $O(\vec{s}')$ with $O(\overleftarrow{s}_{n+1}) = O(\vec{s}_n)$. Then $\vec{s}_n \leqslant \vec{s}_{n+1}$ by the star property. Hence if $\vec{s}_{n+1} \leqslant \overleftarrow{s}'$ then $\vec{s}_n$ would be trivial, therefore $\vec{s}_{n+1} \leqslant \vec{s}'$ as desired. □

For $\vec{s}_1, \ldots, \vec{s}_n, \vec{s}'$ and $\vec{s}_{n+1}$ as in Lemma 3.4 there is an edge between $O(\vec{s}_k)$ and $O(\overleftarrow{s}_{k+1})$ for every $1 \leqslant k \leqslant n$. Additionally if $\vec{s}_{n+1} \neq \vec{s}'$ then $\vec{s}_1, \ldots, \vec{s}_{n+1}, \vec{s}'$ again fulfill the assumptions of the Lemma, so it can be used iteratively.

Furthermore note that $\vec{s}_1 \leqslant \vec{s}_2 \leqslant \cdots \leqslant \vec{s}_n \leqslant \vec{s}_{n+1}$, so if this iteration does not terminate the $\vec{s}_k$ form an infinite chain. From this we now prove that T is connected.

Lemma 3.5. *T as defined above is connected.*

Proof. Let $O, O' \in V(T) = \overline{\mathcal{O}}$ be distinct orientations. Let $\vec{s}_1$ be the unique separation in O' with $O = O(\overleftarrow{s}_1)$, and $\overleftarrow{s}'$ the unique separation in O with $O' = O(\vec{s}')$. Then $\vec{s}_1 \leqslant \vec{s}'$, and if $\vec{s}_1 = \vec{s}'$ then O and O' are joined by an edge in T. Otherwise the assumptions of Lemma 3.4 are met for $n = 1$. Applying Lemma 3.4 iteratively either $\vec{s}_{n+1} = \vec{s}'$ for some $n \in \mathbb{N}$, in which case we found a path in T joining O and O', or we obtain a strictly increasing sequence $(\vec{s}_n)_{n \in \mathbb{N}}$ with $\vec{s}_n \leqslant \vec{s}'$ for all $n \in \mathbb{N}$, that is, a chain of order type $\omega + 1$. □

It remains to show that $\tau \cong \tau(T)$. For this let $\varphi\colon \tau \to \tau(T)$ be the map $\varphi(\vec{s}) = (O(\overleftarrow{s}), O(\vec{s}))$. This is a bijection by Corollary 3.3. Note that for $\vec{s} \in \tau$ the orientations $O(\overleftarrow{s})$ and $O(\vec{s})$ differ only in s by consistency and are thus adjacent in T.

As τ and $\tau(T)$ are regular tree sets all we need to show is that φ is a homomorphism of separation systems. Then it follows from Lemma 2.6 that φ is an isomorphism of tree sets.

It is clear from the definition that φ commutes with the involution. Therefore it suffices to show that φ is order-preserving.

Let $\vec{s}, \vec{s}' \in \tau$ be two separations with $\vec{s} < \vec{s}'$. We need to show that the unique $\{O(\overleftarrow{s}), O(\vec{s})\}$-$\{O(\overleftarrow{s}'), O(\vec{s}')\}$-path in T joins $O(\vec{s})$ and $O(\overleftarrow{s}')$. Redoing the proof of Lemma 3.5 with $O = O(\vec{s})$ and $O' = O(\overleftarrow{s}')$ constructs a

$O(\vec{s})$-$O(\overleftarrow{s'})$-path every one of whose nodes contains $\vec{s}$ and $\overleftarrow{s'}$ by consistency. Hence $\varphi(\vec{s}) < \varphi(\vec{s'})$ as desired.

We have proved the following Theorem.

Theorem 1. *A tree set is isomorphic to the edge tree set of a tree if and only if it is regular and contains no chain of order-type $\omega + 1$. Indeed, if τ is such a tree set, then $\tau = \tau\,(T(\tau))$, where $T(\tau)$ is the tree of the splitting orientations of τ.*

3.3 Tree sets and nested bipartitions of sets

As seen in Chapter 2, abstract separation systems are modeled in part on bipartitions of some fixed sets. Indeed, given a non-empty ground set V, the set $\mathcal{B}(V)$ of its bipartitions is always a regular separation system with the exception of $\{\emptyset, V\}$. Thus a nested system of non-trivial bipartitions of a set is a regular tree set by our definition.

In this section we discuss how to represent regular tree sets as separation systems of bipartitions of sets.

For a tree set τ let $\mathcal{O} = \mathcal{O}(\tau)$ be the set of consistent orientations of τ, and for $\vec{s} \in \tau$ let $\mathcal{O}(\vec{s})$ be the set of all $O \in \mathcal{O}$ with $\vec{s} \in O$. It is fairly straightforward to show that $\mathcal{O}(\tau)$ is a suitable ground set to represent τ as a system of bipartitions:

Theorem 2. [3] *Let τ be a regular tree set. The map $\varphi\colon \tau \to \mathcal{B}(\mathcal{O})$ with $\varphi(\vec{s}) = (\mathcal{O}(\overleftarrow{s}), \mathcal{O}(\vec{s}))$ is an isomorphism of tree sets between τ and its image in $\mathcal{B}(\mathcal{O})$.*

Proof. For every $\vec{s} \in \tau$ we have that $\mathcal{O}(\vec{s})$ is non-empty by the Extension Lemma and that $\mathcal{O}(\tau) = \mathcal{O}(\overleftarrow{s}) \dot\cup \mathcal{O}(\vec{s})$, so φ is well-defined with a regular image. It clearly commutes with the involution. By consistency $\mathcal{O}(\overleftarrow{r}) \subseteq \mathcal{O}(\overleftarrow{s})$ for $\vec{r}, \vec{s} \in \tau$ with $\vec{r} \leqslant \vec{s}$ and hence $\varphi(\vec{r}) \leqslant \varphi(\vec{s})$. Furthermore for any two separations $\vec{r} \neq \vec{s}$ at least one of the sets $\{\vec{r}, \overleftarrow{s}\}$ and $\{\overleftarrow{r}, \vec{s}\}$ is consistent. From this and the Extension Lemma it follows that φ is injective and hence an isomorphism of tree sets between τ and $\varphi(\tau)$ by Lemma 2.6. □

Remark 3.6. If a regular tree set τ contains no chain of order type $\omega + 1$, then the image of φ from Theorem 2 is isomorphic to the edge tree set of the tree $T = T(\tau)$ of the splitting orientations of τ as defined in the above sections: for $\vec{s} \in \tau$ its image $\varphi(\vec{s})$ as a bipartition of $\mathcal{O}(\tau) = V(T)$ is the pair of components of $T - s$.

For finite regular tree sets it is possible to use a smaller ground set than $\mathcal{O}(\tau)$. For example, the edges of a tree T define a partition not only of $V(T)$ but also just of the set $L(T)$ of leaves of T. If T is finite and has no vertex of degree 2, this correspondence is injective. The tree set $\tau(T)$ can then be represented as a tree set of bipartitions of $L(T)$.

Similarly, an abstract tree set τ can be represented as a tree set not only of bipartitions of the set $\mathcal{O}(\tau)$ of abstract consistent orientations of τ, but also as a tree set of bipartitions of a smaller ground set. For a finite tree T the leaves of T correspond precisely to the consistent orientations of $\tau(T)$ that have a greatest element. Therefore the set $\mathcal{O}' = \mathcal{O}'(\tau)$ of consistent orientations of τ with a greatest element is a natural candidate for a smaller ground set.

For $\vec{s} \in \tau$ write $\mathcal{O}'(\vec{s})$ for the set of all $O \in \mathcal{O}'$ with $\vec{s} \in O$ for $\vec{s} \in \tau$. Then Theorem 2 adapts as follows.

Theorem 3. [3] *Let τ be a regular finite tree set. The map $\varphi\colon \tau \to \mathcal{B}(\mathcal{O}')$ with $\varphi(\vec{s}) = (\mathcal{O}'(\overleftarrow{s}), \mathcal{O}'(\vec{s}))$ is an isomorphism of tree sets between τ and its image in $\mathcal{B}(\mathcal{O}')$ if and only if it is injective, which it is if and only if τ has no splitting star of size two.*

In Chapter 4 we show in Theorem 9 that Theorem 3 extends to the class of profinite tree sets.

For infinite regular tree sets it is also possible to use a smaller ground set than $\mathcal{O}(\tau)$ as in Theorem 2 and still get an isomorphism of tree sets. For example, if T is an infinite tree, every edge defines a bipartition of the set $L(T) \cup \Omega(T)$ of the leaves and ends of T, and this correspondence is injective if T contains no vertex of degree 2. Furthermore one can drop the assumption that T contains no vertex of degree 2 by adding these vertices to the ground set, so that every edge defines a bipartition of the set of all leaves, ends and vertices of degree 2 of T.

For abstract tree sets τ, the same idea can be implemented as follows:

For a tree set τ let $\mathcal{O}^{<3} = \mathcal{O}^{<3}(\tau)$ be the set of all consistent orientations of τ with less than three maximal elements, possibly none. For $\vec{s} \in \tau$ let $\mathcal{O}^{<3}(\vec{s})$ denote the set of all $O \in \mathcal{O}^{<3}$ that contain $\vec{s}$.

Theorem 4. *Let τ be a regular tree set. The map $\varphi\colon \tau \to \mathcal{B}(\mathcal{O}^{<3})$ with $\varphi(\vec{s}) = \big(\mathcal{O}^{<3}(\overleftarrow{s}), \mathcal{O}^{<3}(\vec{s})\big)$ is an isomorphism of tree sets between τ and its image in $\mathcal{B}(\mathcal{O}^{<3})$.*

Proof. We verify the assumptions of Lemma 2.6. First we show that $\mathcal{O}^{<3}(\vec{s})$ is non-empty for all $\vec{s} \in \tau$. For a $\vec{s} \in \tau$ let C be a maximal chain containing τ. As C is consistent the Extension Lemma yields a consistent orientation O with $C \subseteq O$. By the maximality of C this orientation has at most one maximal element, namely the supremum of C. Thus $O \in \mathcal{O}^{<3}$ with $\vec{s} \in O$. Hence the image of φ in $\mathcal{B}(\mathcal{O}^{<3})$ is regular.

The map φ commutes with the involution by definition. It is also order-preserving: for $\vec{r}, \vec{s} \in \tau$ with $\vec{r} \leqslant \vec{s}$ every $O \in \mathcal{O}^{<3}(\overleftarrow{r})$ must contain $\overleftarrow{s}$ by consistency, so $\mathcal{O}^{<3}(\overleftarrow{r}) \subseteq \mathcal{O}^{<3}(\overleftarrow{s})$ and hence $\varphi(\vec{r}) \leqslant \varphi(\vec{s})$.

For the injectivity let $\vec{r}, \vec{s} \in \tau$ with $\vec{r} \neq \vec{s}$ be given. If $\vec{r} \leqslant \overleftarrow{s}$ any $O \in \mathcal{O}^{<3}(\overleftarrow{s})$ witnesses $\mathcal{O}^{<3}(\vec{r}) \neq \mathcal{O}^{<3}(\vec{s})$ as $\vec{r} \in O$ but $\vec{s} \notin O$. If $\overleftarrow{r} \leqslant \vec{s}$ any $O \in \mathcal{O}^{<3}(\vec{r})$ contains $\overleftarrow{s}$ and thus witnesses $\mathcal{O}^{<3}(\vec{r}) \neq \mathcal{O}^{<3}(\vec{s})$. Finally if $\vec{r}, \vec{s}$ are comparable with, say, $\vec{r} \leqslant \vec{s}$ then let O be the unique consistent orientation of τ in which $\vec{r}$ is maximal, which exists by the Extension Lemma. Then $\overleftarrow{s} \in O$, so if O has less than three maximal elements it witnesses $\mathcal{O}^{<3}(\vec{r}) \neq \mathcal{O}^{<3}(\vec{s})$. If O has three or more maximal elements there is a maximal element $\vec{t}$ of O with $\vec{r} \neq \vec{t}$ and $\overleftarrow{s} \nleqslant \vec{t}$. By the consistency of O we have $\vec{t} \leqslant \overleftarrow{r}, \vec{s}$. Pick any $O' \in \mathcal{O}^{<3}(\overleftarrow{t})$. Then $\vec{r}, \overleftarrow{s} \in O'$ by consistency, so O' witnesses that $\mathcal{O}^{<3}(\vec{r}) \neq \mathcal{O}^{<3}(\vec{s})$.

Thus the assumptions of Lemma 2.6 are met and τ is isomorphic to its image in $\mathcal{B}(\mathcal{O}^{<3})$ under φ. □

4 Profinite tree sets

4.1 Introduction

Every oriented separation (A, B) an infinite graph G is uniquely determined by the family of its restrictions to the finite subgraphs of G: if two separations of G differ, there is some finite subgraph of G which displays this. Furthermore, for $(A, B), (C, D) \in \vec{S}(G)$ we have $(A, B) \leqslant (C, D)$ if and only if $(A, B)_H \leqslant (C, D)_H$ for all finite subgraphs H of G, where

$$(A, B)_H := (A \cap V(H), B \cap V(H)) .$$

Therefore we can study the separations of G without touching $\vec{S}(G)$ at all, by instead working with the families of their finite restrictions. Separation systems for which this is possible are called *profinite*. (See below for a formal definition.)

An obvious goal in the study of separations systems is to extend the unified duality theorem and the tangle-tree theorem (see [5] and [4] respectively) to infinite separation systems. For profinite separation systems one can hope to achieve this by applying these theorems to all suitable finite projections. Both separation systems of infinite graphs and infinite sets are naturally profinite, so any profinite version of the unified duality theorem or the tangle-tree theorem will apply to them.

In this chapter we study profinite tree sets with the above goal in mind. We obtain a characterization of the regular profinite tree sets in purely combinatorial terms and expand this to a characterization of the profinite tree sets. As a byproduct of these characterizations we show that every profinite tree set can be obtained as an inverse limit of finite tree sets.

In the remainder of this section we define profinite abstract separation systems. Section 5.2 then lays the technical foundations needed for the characterizations. First we characterize the regular profinite tree sets in Section 5.3. It turns out that regularity is a very strong restriction for profinite tree sets, and not many infinite tree sets are both regular and profinite. Section 5.4 contains the main result of this chapter, the characterization of all profinite tree sets. In Section 5.5 we turn our attention to the inverse limit topology which profinite tree sets carry and study the topologically closed orientations of profinite tree sets. Finally in Section 5.6 we use the results of Section 5.3 and Section 5.4 to lift Theorem 3 from the previous chapter to the class of profinite tree sets.

To define abstract profinite separation systems we first have to set up a bit of notation; as the abstract separation systems do not have an underlying graph we need something else to say what 'finite restrictions' ought to be.

A partially ordered set P is a *directed set* if it is non-empty and for all $p, q \in P$ there is a $r \in P$ with $p, q \leqslant r$. This is our abstraction of the finite subgraphs of G, which form a directed set when ordered by inclusion.

For a directed set P an *inverse system* of separation systems is a family $(S_p \mid p \in P)$ of finite separation systems together with a family

$$(f_{qp} \colon S_q \to S_p \mid q, p \in P \text{ with } q > p)$$

of *bonding maps*: homomorphism of separations systems which are *compatible* with each other, that is $f_{rp} = f_{qp} \circ f_{rq}$ for all $p < q < r \in P$.

For an inverse system a family $(\vec{s}_p \mid \vec{s}_p \in S_p,\, p \in P)$ is a *compatible choice* if $f_{qp}(\vec{s}_q) = \vec{s}_p$ for all $p < q$. The set of all compatible choices of $(S_p \mid p \in P)$ is denoted as $\varprojlim (S_p \mid p \in P)$ and called the *inverse limit* of the inverse system $(S_p \mid p \in P)$.

If $(S_p \mid p \in P)$ is an inverse system of finite separation systems then $(\varprojlim (S_p \mid p \in P), \leqslant, *)$ becomes a separation system by setting

$$(\vec{r}_p \mid p \in P) \leqslant (\vec{s}_p \mid p \in P) \qquad :\Leftrightarrow \qquad \vec{r}_p \leqslant \vec{s}_p \text{ for all } p \in P$$

and

$$(\vec{s}_p \mid p \in P)^* := (\overleftarrow{s}_p \mid p \in P).$$

Note that the latter is again a compatible choice as all bonding maps are homomorphisms.

If $\vec{S}$ is the inverse limit of $(S_p \mid p \in P)$ we usually abbreviate a separation $(\vec{s}_p \mid p \in P) \in \vec{S}$ as $\vec{s}$ and use the elements $\vec{s}_p$ of this compatible choice without explicitly introducing $(\vec{s}_p \mid p \in P)$, and we use s_p to denote $\{\vec{s}_p, \overleftarrow{s}_p\}$.

A separation system $\vec{S}$ is *profinite* if it is isomorphic to the inverse limit $\varprojlim (S_p \mid p \in P)$ of an inverse system of finite separation systems.

If every S_p is endowed with the discrete topology then $\varprojlim (S_p \mid p \in P)$ with the subspace topology of $\prod_{p \in P} S_p$ is a compact space. This topology will not be the subject of our study except in section 5.5.

The aim of this chapter is to characterize those tree sets that are profinite. Clearly finite tree sets are also profinite.

It is not true per definitionem that a profinite tree set is an inverse limit of tree sets. However, the profinite tree sets are indeed precisely these separation systems that are inverse limits of finite tree sets:

Theorem 5.

(i) *Every inverse limit of finite tree sets is a tree set.*

(ii) *Every profinite tree set is an inverse limit of finite tree sets.*

Proof of (i). Let $(S_p \mid p \in P)$ be an inverse system of finite tree sets and $\vec{S} = \varprojlim (S_p \mid p \in P)$. Suppose some $\vec{s} = (\vec{s}_p \mid p \in P) \in \vec{S}$ is trivial in $\vec{S}$ with witness r. As P is a directed set there is a $p \in P$ such that $\vec{s}_p \neq \vec{r}_p$ and $\vec{s}_p \neq \overleftarrow{r}_p$. But $\vec{s} < \vec{r}, \overleftarrow{r}$ in $\vec{S}$ implies by the definition of $\leqslant$ on $\vec{S}$ that $\vec{s}_p < \vec{r}_p, \overleftarrow{r}_p$ in S_p, contrary to the assumption that S_p is a tree set.

Moreover if every S_p is nested then so is $\vec{S}$: for $r \neq s$ in $\vec{S}$ pick $p \in P$ with $r_p \neq s_p$. Then, say, $\vec{r}_p \lneqq \vec{s}_p$ as S_p is nested. As the bonding maps are homomorphisms also $\vec{r}_q \lneqq \vec{s}_q$ for all $q > p$ as any other relation would make r_p or s_p trivial in S_p, and hence $\vec{r}_q \lneqq \vec{s}_q$ for all $q \in P$ and thus $\vec{r} \leqslant \vec{s}$.

Therefore $\vec{S}$ is a tree set. □

We will postpone the proof of (ii) until the end of Section 5.4. It is straightforward to show that every profinite nested separation system is an inverse limit of finite nested separation systems. However it is possible that each separation system in an inverse system contains a trivial element, but its inverse limit does not. This makes it difficult to obtain a given profinite tree set as an inverse limit of finite tree sets. In fact it is even possible that every projection of a non-trivial separation is trivial:

Example 4.1. For $n \in \mathbb{N}_{\geqslant 2}$ let $\vec{S}_n$ be the tree set on $\{\vec{s}^{\,n}_1, \ldots, \vec{s}^{\,n}_n\} \cup \{\overleftarrow{s}^{\,n}_1, \ldots, \overleftarrow{s}^{\,n}_n\}$, where

1. $\vec{s}^{\,n}_i \leqslant \vec{s}^{\,n}_j$ and $\overleftarrow{s}^{\,n}_i \geqslant \overleftarrow{s}^{\,n}_j$ if and only if $i \leqslant j$;
2. $\overleftarrow{s}^{\,n}_n \leqslant \vec{s}^{\,n}_n$;
3. $\overleftarrow{s}^{\,n}_n \leqslant \vec{s}^{\,n}_{n-1}$ and $\vec{s}^{\,n}_n \geqslant \overleftarrow{s}^{\,n}_{n-1}$.

For $m > n$ define the bonding map $f_{mn} \colon \vec{S}_m \to \vec{S}_n$ as

$$f(\vec{s}^{\,m}_k) = \begin{cases} \vec{s}^{\,n}_k, & k \leqslant n \\ \vec{s}^{\,n}_n, & k > n \end{cases}$$

and

$$f(\overleftarrow{s}^{\,m}_k) = \begin{cases} \overleftarrow{s}^{\,n}_k, & k \leqslant n \\ \overleftarrow{s}^{\,n}_n, & k > n \end{cases}.$$

Then $(\vec{S}_n \mid n \in \mathbb{N}_{\geqslant 2})$ is an inverse system of finite nested separation systems whose inverse limit τ is a tree set. The separation $(\overleftarrow{s}^{\,n}_n \mid n \in \mathbb{N}_{\geqslant 2}) \in \tau$ is small but not trivial in τ, but every projection $\overleftarrow{s}^{\,n}_n$ of it is trivial in $\vec{S}_n$ with witness s^n_{n-1}.

For regular profinite tree sets a similar assertion to Theorem 5 is true, and in fact follows from Theorem 5:

Theorem 6.

(i) *Every inverse limit of regular finite tree sets is a regular tree set.*

(ii) *Every regular profinite tree set is an inverse limit of regular finite tree sets.*

Proof. For (i), notice that by definition a separation in an inverse limit is small if and only if each of its projections is small. Thus (i) follows from Theorem 5.

For (ii), let τ be a regular profinite tree set. By Theorem 5(ii) there is an inverse system $(S_p \mid p \in P)$ of finite tree sets whose inverse limit is isomorphic to τ. If no S_p is regular let S'_p be the sub-system of S_p consisting of all non-regular separations of S_p for each $p \in P$. Then $(S'_p \mid p \in P)$ is an inverse system whose non-empty inverse limit consists solely of non-regular separations and is a subset of the inverse limit of $(S_p \mid p \in P)$, contrary to the assumption that τ is regular. Thus some S_p is regular and $(S_q \mid q \geqslant p)$ is an inverse system of regular finite tree sets whose inverse limit is still isomorphic to τ. □

To lift theorems for finite tree sets to profinite tree sets it is desirable to express a profinite tree set as inverse limit of finite tree sets, not just of nested separation systems. For this we obtain a profinite tree set as an inverse system of its 'finite quotients'. We define this precisely in the next section and provide a few key lemmas for the proof of the main result, a characterization of the profinite tree sets and a method of obtaining every profinite tree set as an inverse limit of finite tree sets.

4.2 Distinguishing separations

This section lays the technical foundations for the rest of the chapter. Given a tree set τ our aim is to find a way of defining finite quotients of τ that form an inverse system of finite tree sets whose inverse limit is isomorphic to τ. The latter part of this will be done in the next two sections, while in this section we define these 'finite quotients' and analyse their properties.

To this end for any finite set of stars in τ we define an equivalence relation on τ which essentially breaks up τ into finitely many chunks. After proving a few basic facts about this equivalence relation we find certain conditions that ensure that the equivalence classes of τ form a finite tree set, as needed for the proof of Theorem 5(ii).

Following this main part of the section we analyse these equivalence relations a bit more and prove a few key lemmas for the next two sections.

Let τ be a tree set. A *selection* of τ is a non-empty finite set $D \subseteq \tau$ with $|\sigma \cap D| \neq 1$ for every splitting star σ of τ.

Let us show that any selection D of τ divides τ into different sections between the stars that meet D. We make this precise by defining an equivalence relation $\sim_D$ on τ.

Recall that, for $\vec{r}, \vec{s} \in \tau$, we write $\vec{r} \lneqq \vec{s}$ as shorthand notation for $\vec{r} \leqslant \vec{s}$ and $r \neq s$.

For a separation $\vec{s} \in \tau$ and a selection D set

$$D^+(\vec{s}\,) := \left\{ \vec{d} \in D \mid \vec{d} \lneqq \vec{s} \right\}$$

and

$$D^-(\vec{s}\,) := D^+(\overleftarrow{s}).$$

Two separations $\vec{s}, \vec{r}$ are *D-equivalent* for a selection D, denoted as $\vec{s} \sim_D \vec{r}$, if $D^+(\vec{s}\,) = D^+(\vec{r}\,)$ and $D^-(\vec{s}\,) = D^-(\vec{r}\,)$. This is an equivalence relation with finitely many classes. Write $[\vec{s}\,]_D$ for the equivalence class of $\vec{s} \in \tau$ under $\sim_D$. A separation $\vec{d} \in D$ *distinguishes* $\vec{r}$ and $\vec{s}$ if

$$\vec{d} \in \big(D^+(\vec{r}\,)\big) \triangle \big(D^+(\vec{s}\,)\big) \text{ or } \vec{d} \in \big(D^-(\vec{r}\,)\big) \triangle \big(D^-(\vec{s}\,)\big).$$

Thus $\vec{r}$ and $\vec{s}$ are D-equivalent if and only if no $\vec{d} \in D$ distinguishes them.

For a selection D of τ and separations $\vec{r} \leqslant \vec{s}$ it follows from the definitions that $D^+(\vec{r}\,) \subseteq D^+(\vec{s}\,)$ and $D^-(\vec{r}\,) \supseteq D^-(\vec{s}\,)$. Furthermore $D^+(\vec{s}\,) \cap D^-(\vec{s}\,) = \varnothing$ for all $\vec{s} \in \tau$ as any element of this intersection would be trivial with witness $\vec{s}$. This implies that $\vec{s} \sim_D \overleftarrow{s}$ if and only if $D^+(\vec{s}\,) = D^-(\vec{s}\,) = \varnothing$. But $D^+(\vec{s}\,) \cup D^-(\vec{s}\,)$ is never empty and in fact contains an element of every splitting star that meets D, so $\vec{s} \nsim_D \overleftarrow{s}$ for every $\vec{s} \in \tau$.

The next lemma shows a few basic properties of $\sim_D$. The first of these is especially important, as it will enable us to turn the equivalence classes of $\sim_D$ on τ into a separation system.

Lemma 4.2. *Let τ be a tree set, D a selection and $\vec{r}, \vec{s}, \vec{t} \in \tau$.*

(i) *If $\vec{r} \sim_D \vec{s}$ then $\overleftarrow{r} \sim_D \overleftarrow{s}$.*

(ii) *If* $\vec{r} \leqslant \vec{s} \leqslant \vec{t}$ *and* $\vec{r} \sim_D \vec{t}$ *then* $\vec{r} \sim_D \vec{s} \sim_D \vec{t}$*.*

(iii) *If there is no* $\vec{d} \in D$ *with* $\vec{d} \lneq \vec{s}$*, and* $\vec{r} \leqslant \vec{s}$*, then* $\vec{r} \sim_D \vec{s}$*.*

(iv) *If* $\vec{r} \leqslant \vec{s} \leqslant \overleftarrow{t}$ *and* $\vec{s} \sim_D \vec{t}$ *then* $\vec{r} \sim_D \vec{s}$*.*

(v) *If* $\vec{r} \geqslant \vec{s} \geqslant \overleftarrow{t}$ *and* $\vec{s} \sim_D \vec{t}$ *then* $\vec{r} \sim_D \vec{s}$*.*

Proof. (i) This follows from

$$D^+(\overleftarrow{s}) = D^-(\vec{s}) = D^-(\vec{r}) = D^+(\overleftarrow{r})$$

and

$$D^-(\overleftarrow{s}) = D^+(\vec{s}) = D^+(\vec{r}) = D^-(\overleftarrow{r}).$$

(ii) By the observation above

$$D^+(\vec{r}) \subseteq D^+(\vec{s}) \subseteq D^+(\vec{t}) = D^+(\vec{r})$$

and similarly $D^-(\vec{r}) = D^-(\vec{s})$, hence $\vec{r} \sim_D \vec{s}$.

(iii) By assumption $D^+(\vec{r}) = D^+(\vec{s}) = \varnothing$. Furthermore $D^-(\vec{s}) \subseteq D^-(\vec{r})$. Suppose there is a $\vec{d} \in D$ with $\vec{d} \lneq \overleftarrow{r}$ but not $\vec{d} \lneq \overleftarrow{s}$. Then $\overleftarrow{d} \lneq \vec{s}$ as $\vec{d} \not\lneq \vec{s}$ by assumption. Let σ be the splitting star containing $\vec{d}$ and $\vec{e} \in \sigma \cap D$ with $\vec{d} \neq \vec{e}$. Then $\vec{e} \leqslant \overleftarrow{d} \lneq \vec{s}$ by the star property, contradicting the assumption that there is no such $\vec{e} \in D$.

(iv) As $D^+(\vec{s}) \subseteq D^+(\overleftarrow{t}) = D^+(\overleftarrow{s})$ there can be no $\vec{d} \in D$ with $\vec{d} \lneq \vec{s}$ as it would be trivial with witness s. Thus $\vec{s} \sim_D \vec{r}$ by (iii).

(v) This follows from (i) and (iv). □

We now define precisely in which way we want to turn the equivalence classes of τ into a separation system.

Let D be a selection of a tree set τ. Then write

$$\tau/D := (\{[\vec{s}]_D \mid \vec{s} \in \tau\}, \leqslant, *)$$

with $([\vec{s}]_D)^* := [\overleftarrow{s}]_D$ and $[\vec{s}]_D \leqslant [\vec{r}]_D$ if there are $\vec{s'} \in [\vec{s}]_D$ and $\vec{r'} \in [\vec{r}]_D$ with $\vec{s'} \leqslant \vec{r'}$.

If for some selection D the relation $\leqslant$ of τ/D is a partial order then τ/D is a separation system by Lemma 4.2(i). In that case τ/D would even be nested because τ is.

Our aim is to ensure that τ/D is a tree set. For this we first find sufficient conditions for $\leqslant$ to be a partial order, and then show that these conditions are strong enough to ensure that τ/D does not contain any trivial elements.

The relation $\leqslant$ on τ/D is reflexive by definition, thus we need to show that it is transitive and anti-symmetric. For the latter no further assumptions are needed, so we begin by proving the anti-symmetry.

Lemma 4.3. *Let* τ *be a tree set and* D *a selection. If* $\vec{r} \leqslant \vec{x}$ *and* $\vec{s} \geqslant \vec{y}$ *for* $\vec{r}, \vec{s}, \vec{x}, \vec{y} \in \tau$ *with* $\vec{r} \sim_D \vec{s}$ *and* $\vec{x} \sim_D \vec{y}$ *then also* $\vec{r} \sim_D \vec{x}$*.*

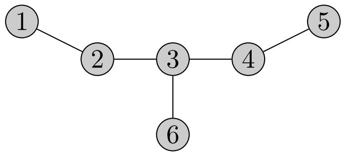
Proof. We have

$$D^+(\vec{r}) \subseteq D^+(\vec{x}) = D^+(\vec{x}) \subseteq D^+(\vec{s}) = D^+(\vec{r})$$

and similarly $D^-(\vec{r}) = D^-(\vec{x})$. □

This shows that $\leqslant$ on τ/D is antisymmetric. To prove transitivity we need further assumptions, as the following example demonstrates.

Example 4.4. Let T be the following graph.



The tree set $\tau(T)$ is regular and we have $(6,3) \leqslant (3,2)$ and $(4,3) \leqslant (3,6)$. For the selection $D = \{(1,2),(3,2),(3,4),(5,4)\}$ the edges $(2,3)$ and $(3,4)$ get identified in $\tau(T)/D$, implying

$$[(6,3)]_D \leqslant [(3,2)]_D = [(4,3)]_D \leqslant [(3,6)]_D$$

in $\tau(T)/D$. But $(6,3) \nleqslant (3,6)$, so $\leqslant$ is not transitive on $\tau(T)/D$.

This example exploits the fact that there is a branching star that does not meet D between two splitting stars that do meet D. In order to prevent this counterexample to transitivity one could ask that D meets every branching star that lies between two elements of D. But this alone is not enough to ensure that $\leqslant$ on τ/D is transitive: if we replace the separation $(6,3)$ in Example 4.4 above with a chain of order type ω the resulting tree set would not have any branching stars, but the transitivity of $\leqslant$ would still fail for the same reason. Therefore we also need an assumption on τ that ensures that whenever there is a three-star as in the example above we can also find a branching star, which would then be subject to the condition on D.

Recall that $\vec{b} \in \tau$ is a branching point of τ if $\vec{b}$ lies in a splitting star of size greater than two.

Call a selection D of a tree set τ *branch-closed* if $\vec{b} \in D$ for every branching point $\vec{b}$ of τ for which there are $\vec{d}_1, \vec{d}_2 \in D$ with $\vec{d}_1 \leqslant \vec{b} \leqslant \vec{d}_2$. Furthermore τ is *chain-complete* if every non-empty[2] chain $C \subseteq \tau$ has a supremum in τ.

Example 4.5. For every non-empty compact subset $X \subseteq \mathbb{R}$ the natural tree set $\tau(X)$ on X is a chain-complete tree set.

Conversely, for every unbounded subset $X \subseteq R$ the natural tree set $\tau(X)$ on X is not a chain-complete tree set.

We claim that the two conditions that τ is chain-complete and D branch-closed are enough to ensure that $\leqslant$ on τ/D is transitive and hence a partial order. Before we prove this claim we need to establish some basic properties of chain-complete tree sets, beginning with the fact that every chain has not only a supremum but an infimum too:

[2] This differs from the common definition of a chain-complete poset as it does not imply that τ has a smallest element.

Lemma 4.6. *Let τ be a chain-complete tree set and C a chain. Then C has an infimum in τ.*

Proof. Consider the chain

$$C' := \left\{ \overleftarrow{t} \mid \vec{t} \in C \right\}$$

and let $\overleftarrow{s}$ be its supremum in τ. Then $\vec{s}$ is the infimum of C. □

Recall that an orientation O of τ is splitting if every element of O lies below some maximal element of O.

The usual way to find a branching star in a tree set τ is to define a consistent orientation with three or more maximal elements and then show that it is splitting. The first part can be done with the Extension Lemma. For the latter part the following lemma provides a sufficient condition for a consistent orientation to be splitting. It turns out that having *two* maximal elements is already enough, if τ is chain-complete:

Lemma 4.7. *Let τ be a chain-complete tree set and O a consistent orientation of τ with two or more maximal elements. Then O is splitting.*

Proof. Let $\vec{r}, \vec{s}$ be two maximal elements of O and let $\vec{t} \in O$ be any separation. If $\vec{t}$ lies below $\vec{r}$ or $\vec{s}$ there is nothing to show. If not consider the up-closure $C := \lfloor \vec{t} \rfloor \subseteq O$ of $\vec{t}$ in O. As O is consistent C is a chain, which has a supremum $\vec{m} \in \tau$ by assumption. $\overleftarrow{r}$ and $\overleftarrow{s}$ are upper bounds for C, so $\overleftarrow{m} \geqslant \vec{r}, \vec{s}$. But $\vec{r}$ and $\vec{s}$ are maximal in O, implying $\overleftarrow{m} \notin O$ and hence $\vec{m} \in O$. □

With Lemma 4.7 we can now show that if we have a three-star in a chain-complete tree set we can find a branching star 'in the same location':

Proposition 4.8. *Let τ be a chain-complete tree set and σ a star with exactly three elements. Then there is a unique branching star σ' of τ such that every element of σ lies below a different element of σ'.*

Proof. Let $\sigma = \{\vec{r}, \vec{s}, \vec{t}\}$ and

$$R = \left\{ \vec{x} \in \tau \mid \vec{r} \leqslant \vec{x} \leqslant \overleftarrow{s},\ \vec{r} \leqslant \vec{x} \leqslant \overleftarrow{t} \right\}.$$

Then R is a chain and by assumption $\vec{r'} = \sup R$ exists. As $\overleftarrow{s}$ and $\overleftarrow{t}$ are lower bounds for R we have $\vec{r'} \in R$. Similarly define

$$S = \left\{ \vec{x} \in \tau \mid \vec{s} \leqslant \vec{x} \leqslant \overleftarrow{r},\ \vec{s} \leqslant \vec{x} \leqslant \overleftarrow{t} \right\}$$

and

$$T = \left\{ \vec{x} \in \tau \mid \vec{t} \leqslant \vec{x} \leqslant \overleftarrow{r},\ \vec{t} \leqslant \vec{x} \leqslant \overleftarrow{s} \right\}$$

as well as $\vec{s'} = \sup S$ and $\vec{t'} = \sup T$. As R, S and T are disjoint $\{\vec{r'}, \vec{s'}, \vec{t'}\}$ forms a three-star. Additionally there is no $\vec{x} \in \tau$ that lies between $\vec{r'}$ and $\overleftarrow{s'}$, $\vec{s'}$ and $\overleftarrow{t'}$ or $\vec{t'}$ and $\overleftarrow{r'}$: if, say, $\vec{r'} < \vec{x} < \overleftarrow{s'}$ then either $\vec{x} \in R$ or $\overleftarrow{x} \in S$, contradicting the choice of either $\vec{r'}$ or $\vec{s'}$.

Applying the Extension Lemma to $\{\vec{r'}, \vec{s'}, \vec{t'}\}$ yields a consistent orientation O in which $\vec{r'}$ is maximal. Then $\vec{s'}$ and $\vec{t'}$ are maximal in O too. It follows from

Lemma 4.7 that O is splitting, so the set σ' of its maximal elements is the desired branching star.

The uniqueness follows from the fact that if σ_1 and σ_2 are two distinct splitting stars, there is a $\vec{s} \in \sigma_2$ which is an upper bound for all elements of σ_1 but one, as we already showed in Lemma 3.1. Hence if three separations lie below different elements of σ_1, at least two of them will lie below the same element of σ_2. □

This proposition is useful as it often allows us to work with splitting stars without loss of generality in the context of selections. We will also use it in the next section, especially the uniqueness part which is not important in this section.

We now show in three steps that $\leqslant$ on τ/D is transitive for branch-closed D and chain-complete τ. First we show that if a counterexample to the transitivity exists it must be a three-star with one element equivalent to the inverse of the second, as in Example 4.4. Then we apply Proposition 4.8 to this three star to obtain a branching star, of which we show that it is still a counterexample to the transitivity. Finally we derive a contradiction to the assumption that D is branch-closed.

Lemma 4.9. *Let τ be a tree set and D a selection. If $\leqslant$ on τ/D is not transitive then there is a three-star $\{\vec{r}, \vec{s}_1, \overleftarrow{s}_2\}$ such that $\vec{s}_1 \sim_D \vec{s}_2$ but neither $\vec{r} \sim_D \vec{s}_1$ nor $\vec{r} \sim_D \overleftarrow{s}_1$.*

Proof. Suppose there are $[\vec{x}]_D, [\vec{y}]_D, [\vec{z}]_D \in \tau/D$ such that $[\vec{x}]_D \leqslant [\vec{y}]_D$ and $[\vec{y}]_D \leqslant [\vec{z}]_D$ but $[\vec{x}]_D \nleqslant [\vec{z}]_D$. Pick $\vec{r} \in [\vec{x}]_D$, $\vec{s}_1, \vec{s}_2 \in [\vec{y}]_D$ and $\vec{t} \in [\vec{z}]_D$ with $\vec{r} \leqslant \vec{s}_1$ and $\vec{s}_2 \leqslant \vec{t}$.

At most one of $\vec{r}$ and $\vec{t}$ can be D-equivalent to $\overleftarrow{s}_1$ by assumption. Suppose that $\vec{r} \nsim_D \overleftarrow{s}_1$ (the case $\vec{t} \nsim_D \overleftarrow{s}_1$ is symmetrical).

Because τ is a tree set s_1 and s_2 have comparable orientations. If $\vec{s}_1 \leqslant \vec{s}_2$ then $\vec{r} \leqslant \vec{t}$, contradicting $[\vec{x}]_D \nleqslant [\vec{z}]_D$. By Lemma 4.2(iv) and (v) $\vec{s}_1 \nleqslant \overleftarrow{s}_2$ and $\vec{s}_1 \ngeqslant \overleftarrow{s}_2$ as $\vec{r} \nsim_D \vec{s}_1, \overleftarrow{s}_1$. Hence $\vec{s}_1 \geqslant \vec{s}_2$. Furthermore $\vec{r} \nleqslant \vec{s}_2$ as $\vec{r} \nleqslant \vec{t}$, and $\vec{r} \ngeqslant \vec{s}_2$ by Lemma 4.2(i), so $\{\vec{r}, \overleftarrow{s}_1, \vec{s}_2\}$ must be a three-star. □

This was the first of the three steps. In the next step we show that if a counterexample to the transitivity of $\leqslant$ exists there is a counterexample which is a branching star.

Lemma 4.10. *Let τ be a chain-complete tree set and D a selection. If $\leqslant$ on τ/D is not transitive then there is a three-star $\{\vec{r}, \vec{s}_1, \overleftarrow{s}_2\}$ of branching points such that $\vec{s}_1 \sim_D \vec{s}_2$ but neither $\vec{r} \sim_D \vec{s}_1$ nor $\vec{r} \sim_D \overleftarrow{s}_1$.*

Proof. By Lemma 4.9 there is a three-star $\{\vec{x}, \vec{y}_1, \overleftarrow{y}_2\}$ with $\vec{y}_1 \sim_D \vec{y}_2$ and $\vec{x} \nsim_D \vec{y}_1, \overleftarrow{y}_1$. An application of Proposition 4.8 yields a branching star σ with a three-star $\{\vec{r}, \vec{s}_1, \overleftarrow{s}_2\} \subseteq \sigma$ for which $\vec{x} \leqslant \vec{r}$ and $\vec{y}_1 \leqslant \vec{s}_1 \leqslant \vec{s}_2 \leqslant \vec{y}_2$. From Lemma 4.2(ii) it follows that $\vec{y}_1 \sim_D \vec{s}_1 \sim_D \vec{s}_2 \sim_D \vec{y}_2$. Furthermore Lemma 4.2(i) and (iv) imply that $\vec{r} \nsim_D \vec{s}_1$ and $\vec{r} \nsim_D \overleftarrow{s}_1$, as otherwise $\vec{x} \sim_D \vec{y}_1$ or $\vec{x} \sim_D \overleftarrow{y}_1$ contrary to assumption. Thus $\{\vec{r}, \vec{s}_1, \overleftarrow{s}_2\}$ is the desired three-star. □

For the third step we need to show that there are elements of D that allow us to apply the branch-closedness of D to derive a contradiction.

Lemma 4.11. *Let τ be a chain-complete tree set and D a selection. Let $\{\vec{r}, \vec{s}_1, \overleftarrow{s}_2\}$ be a three-star in τ with $\vec{s}_1 \sim_D \vec{s}_2$ but neither $\vec{r} \sim_D \vec{s}_1$ nor $\vec{r} \sim_D \overleftarrow{s}_1$. Then there are $\vec{d}_1, \vec{d}_2 \in D$ with $\vec{d}_1 \lneq \vec{s}_1$ and $\overleftarrow{d}_2 \lneq \overleftarrow{s}_2$.*

Proof. Let $\vec{d} \in D$ distinguish $\vec{r}$ and $\vec{s}_1$; we will show that $\vec{d} \lneq \vec{s}_1$. This $\vec{d}$ cannot lie in $D^+(\vec{r})\backslash D^+(\vec{s}_1)$ as then it would also distinguish $\vec{s}_1$ and $\vec{s}_2$. Furthermore $D^-(\vec{s}_1) = D^-(\vec{s}_2) \subseteq D^-(\vec{r})$ by the star property, so $\vec{d}$ cannot lie in $D^-(\vec{s}_1)\backslash D^-(\vec{r})$ either. If $\vec{d} \in D^-(\vec{r})\backslash D^-(\vec{s}_1)$ then any $\vec{e} \in \sigma \cap D$ with $\vec{e} \neq \vec{d}$ would distinguish $\vec{s}_1$ and $\vec{s}_2$, where σ is the splitting star containing $\vec{d}$. Therefore $\vec{d} \in D^+(\vec{s}_1)\backslash D^+(\vec{r})$, so in particular $\vec{d} \lneq \vec{s}_1$.

Repeating this argument for a $\vec{d}_2 \in D$ that distinguishes $\vec{r}$ and $\overleftarrow{s}_2$ shows $\overleftarrow{d}_2 \lneq \overleftarrow{s}_2$ and hence the claim. □

Finally we put the above lemmas together to prove that $\leqslant$ in τ/D is transitive.

Lemma 4.12. *Let τ be a chain-complete tree set and D a branch-closed selection. Then $\leqslant$ on τ/D is transitive.*

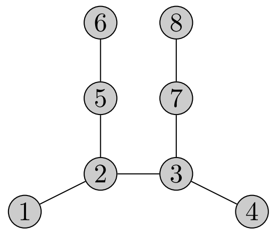
Proof. Suppose $\leqslant$ is not transitive. Then by Lemma 4.10 there is a three-star $\{\vec{r}, \vec{s}_1, \overleftarrow{s}_2\}$ of branching points such that $\vec{s}_1 \sim_D \vec{s}_2$ but neither $\vec{r} \sim_D \vec{s}_1$ nor $\vec{r} \sim_D \overleftarrow{s}_1$. Applying Lemma 4.11 to this star yields $\vec{d}_1, \vec{d}_2 \in D$ with $\vec{d}_1 \lneq \vec{s}_1 \leqslant \vec{s}_2 \lneq \vec{d}_2$. As D is branch-closed and $\vec{s}_1$ a branching point this implies $\vec{s}_1 \in D$; but then $\vec{s}_1 \in D^+(\vec{s}_2)\backslash D^+(\vec{s}_1)$, contrary to the assumption that $\vec{s}_1 \sim_D \vec{s}_2$.

Hence $\leqslant$ is transitive as claimed. □

Therefore $\leqslant$ on τ/D is a partial order, so τ/D is a nested separation system for chain-complete τ and branch-closed D. To prove that τ/D is a tree set it is thus left to show that it does not contain any trivial elements.

The next example shows that τ/D may well contain a trivial element even in cases where $\leqslant$ is a partial order. However, this too exploits that D is not branch-closed, an we will subsequently prove that τ/D is indeed a tree set for branch-closed D.

Example 4.13. Let T be the following graph.



The tree set $\tau(T)$ is regular. For the selection $D = \{(2,5),(6,5),(3,7),(8,7)\}$ we have $(1,2) \sim_D (4,3)$ and thus $[(1,2)]_D \leqslant [(2,3)]_D, [(3,2)]_D$. As D distinguishes $(1,2)$ from $(2,3)$ and from $(3,2)$ this means that $[(1,2)]_D$ is trivial in $\tau(T)/D$.

The proof that τ/D has no trivial elements if τ is chain-complete and D is branch-closed is again be carried out in multiple steps. First we show that the configuration from Example 4.13 is the only possible type of counterexample. Following that we prove that if this counterexample occurs there are elements of D we can use to apply the branch-closedness of D with.

Lemma 4.14. *Let τ be a chain-complete tree set and D a branch-closed selection. If τ/D contains a trivial element then there are $\vec{r}, \vec{s}, \vec{x} \in \tau$ with $\vec{r} \leqslant \vec{x} \leqslant \overleftarrow{s}$ and $\vec{r} \sim_D \vec{s}$ but neither $\vec{r} \sim_D \vec{x}$ nor $\vec{r} \sim_D \overleftarrow{x}$.*

Proof. If τ/D contains a trivial element then there are $\vec{r}, \vec{x} \in \tau$ with $\vec{r} \leqslant \vec{x}$ and $[\vec{r}]_D < [\vec{x}]_D, [\overleftarrow{x}]_D$ in τ/D. Then there are $\vec{s} \in [\vec{r}]_D$, $\vec{y} \in [\vec{x}]_D$ with $\vec{s} \leqslant \overleftarrow{y}$.

As neither $\vec{r} \sim_D \vec{x}$ nor $\vec{r} \sim_D \overleftarrow{x}$ by assumption Lemma 4.2 (iv) and (v) imply $\vec{x} \ngeqslant \overleftarrow{y}$ and $\vec{x} \nleqslant \overleftarrow{y}$. Furthermore if $\vec{x} \leqslant \vec{y}$ then $\vec{r} \leqslant \vec{x} \leqslant \vec{y} \leqslant \overleftarrow{s}$, so we are done.

This leaves the case $\vec{x} \geqslant \vec{y}$. If $\vec{r} \leqslant \overleftarrow{y}$ then $\{\vec{r}, \vec{y}, \overleftarrow{x}\}$ is a three-star as in Lemma 4.10 and 4.11, which we know is impossible as shown in the proof of Lemma 4.12 if D is branch-closed. By Lemma 4.2(ii) $\vec{r} \geqslant \vec{y}$ would imply $\vec{r} \sim_D \vec{x}$, so this is also impossible. Hence the only relation r and y can have is $\vec{r} \leqslant \vec{y}$, and then $\vec{r} \leqslant \vec{y} \leqslant \overleftarrow{s}$ as desired. □

The next step is to show that we can use the assumption that D is branch-closed, that is to find $\vec{d}_1, \vec{d}_2 \in D$ with the proper relations to the separations from Lemma 4.14.

Lemma 4.15. *Let τ be a tree set and D a selection. Let $\vec{r}, \vec{s}, \vec{x} \in \tau$ with $\vec{r} \leqslant \vec{x} \leqslant \overleftarrow{s}$ and $\vec{r} \sim_D \vec{s}$ but neither $\vec{r} \sim_D \vec{x}$ nor $\vec{r} \sim_D \overleftarrow{x}$. Then there are $\vec{d}_1, \vec{d}_2 \in D$ with*

$$\vec{d}_1 \lneq \overleftarrow{r}, \overleftarrow{s}, \vec{x}, \qquad \vec{d}_2 \lneq \overleftarrow{r}, \overleftarrow{s}, \overleftarrow{x}.$$

Proof. By $\vec{r} \leqslant \overleftarrow{s}$ and $\vec{r} \sim_D \vec{s}$ we have $D^+(\vec{r}) = D^+(\vec{s}) \subseteq D^-(\vec{r})$ and hence $D^+(\vec{r}) = D^+(\vec{s}) = \emptyset$. Therefore either $\vec{d} \lneq \overleftarrow{r}, \overleftarrow{s}$ or $\overleftarrow{d} \lneq \overleftarrow{r}, \overleftarrow{s}$ for all $\vec{d} \in D$. Thus if $\vec{d}_1 \in D$ distinguishes $\vec{r}$ and $\vec{x}$ then $\vec{d}_1 \lneq \overleftarrow{r}, \vec{x}$ is the only possibility, and if $\vec{d}_2 \in D$ distinguishes $\vec{s}$ and $\overleftarrow{x}$ then $\vec{d}_2 \lneq \overleftarrow{s}, \overleftarrow{x}$ is the only possibility. The claim now follows from the assumption that $\vec{r} \sim_D \vec{s}$ but neither $\vec{r} \sim_D \vec{x}$ nor $\vec{s} \sim_D \overleftarrow{x}$. □

Finally we combine the above lemmas and use Proposition 4.8 to prove that τ/D has no trivial elements.

Lemma 4.16. *Let τ be a chain-complete tree set and D a branch-closed selection. Then τ/D contains no trivial element.*

Proof. Suppose τ/D contains a trivial element. From Lemma 4.14 and 4.15 it follows that there are $\vec{r}, \vec{s}, \vec{x} \in \tau$ with $\vec{r} \leqslant \vec{x} \leqslant \overleftarrow{s}$ and $\vec{r} \sim_D \vec{s}$ but neither $\vec{r} \sim_D \vec{x}$ nor $\vec{r} \sim_D \overleftarrow{x}$, as well as $\vec{d}_1, \vec{d}_2 \in D$ with

$$\vec{d}_1 \lneq \overleftarrow{r}, \overleftarrow{s}, \vec{x}, \qquad \vec{d}_2 \lneq \overleftarrow{r}, \overleftarrow{s}, \overleftarrow{x}.$$

Proposition 4.8 applied to the three-star $\{\vec{x}, \vec{d}_2, \overleftarrow{s}\}$ then yields a branching star σ and some $\vec{b} \in \sigma$ with $\vec{x} \leqslant \vec{b}$. Then $\vec{d}_1 \leqslant \vec{b} \leqslant \overleftarrow{d}_2$ by $\vec{d}_1 \leqslant \vec{x}$ and the star property and hence $\vec{b} \in D$ as D is branch-closed. But $\vec{r} \leqslant \vec{b} \lneq \overleftarrow{s}$ by the star property, so $\vec{b}$ distinguishes $\vec{r}$ and $\vec{s}$, contradicting $\vec{r} \sim_D \vec{s}$.

Therefore τ/D cannot contain a trivial element. □

We have assembled all the parts necessary to show that τ/D is a finite tree set:

Proposition 4.17. *Let τ be a chain-complete tree set and D a branch-closed selection. Then τ/D is a finite tree set.*

Proof. As D is finite there are only finitely many subsets of D and hence only finitely many equivalence classes of $\sim_D$, so τ/D is finite. The relation $\leqslant$ on τ/D is reflexive by definition, anti-symmetric by Lemma 4.3 and transitive by Lemma 4.12 and thus a partial order. The involution $([\vec{s}]_D)^* = [\overleftarrow{s}]_D$ is order-reversing: if $[\vec{s}]_D \leqslant [\vec{r}]_D$ with $\vec{s} \leqslant \vec{r}$ then $\overleftarrow{s} \geqslant \overleftarrow{r}$ and thus $[\overleftarrow{s}]_D \geqslant [\overleftarrow{r}]_D$. Therefore τ/D is a separation system. Any two unoriented separations $\{[\vec{s}]_D, [\overleftarrow{s}]_D\}, \{[\vec{r}]_D, [\overleftarrow{r}]_D\}$ in τ/D have comparable orientations, because their representatives s and r are nested. Finally Lemma 4.16 shows that τ/D has no trivial elements and is thus a finite tree set. □

With this we have accomplished the main goal of this section. In the next two sections we will define a suitable directed set $\mathcal{D}$ of selections of τ and show $\tau \cong \varprojlim (\tau/D \mid D \in \mathcal{D})$. To help with this in the remainder of this section we establish a few independent facts about the behaviour of τ/D for later use. We show that the relation of r and s in τ can sometimes be recovered from the relation of $[\vec{r}]_D$ and $[\vec{s}]_D$ in τ/D, and that the equivalence classes of $\sim_D$ in τ are chain-complete. The latter will be crucial in the surjectivity proof in the next sections. Furthermore we study the behaviour of $\sim_D$ on infinite stars and find a sufficient condition for τ/D to be regular.

That τ/D is a tree set implies that r and s in τ have to have the same relation as $[\vec{r}]_D$ and $[\vec{s}]_D$ in τ/D, at least if those are different classes:

Lemma 4.18. *Let τ be a tree set, $\vec{r}, \vec{s} \in \tau$ and D a selection for which τ/D is a tree set. If $[\vec{r}]_D \lneqq [\vec{s}]_D$ then $\vec{r} \lneqq \vec{s}$.*

Proof. Any other relation between r and s implies either $[\vec{r}]_D = [\vec{s}]_D$ or that one of $[\vec{r}]_D$ and $[\overleftarrow{s}]_D$ would be trivial in τ/D. □

For the study of τ/D it is essential to know the behaviour of chains of τ with regard to $\sim_D$. It turns out that the equivalence classes of τ are chain-complete themselves if τ is; we don't even need the assumption that D is branch-closed for this:

Proposition 4.19. *Let τ be a chain-complete tree set, D a selection and $\vec{t} \in \tau$. Then $[\vec{t}]_D$ is chain-complete.*

In particular $[\vec{t}]_D$ has a maximal (and a minimal) element.

Proof. Let C be a chain in the equivalence class $[\vec{t}]_D$ with supremum $\vec{s}$ in τ. The claim is trivial if $\vec{s} \in C$. Thus we may assume that $\vec{s} \notin C$. Then $\overleftarrow{s}$ cannot lie in a splitting star σ of τ, as in that case some other element $\vec{s}'$ of σ would be an upper bound of C with $\vec{s}' < \vec{s}$.

Pick some $\vec{r} \in C$; we will verify that $\vec{r} \sim_D \vec{s}$. As $\vec{r} \leqslant \vec{s}$ we have $D^+(\vec{r}) \subseteq D^+(\vec{s})$ and $D^-(\vec{s}) \subseteq D^-(\vec{r})$.

Consider $\vec{d} \in D^-(\vec{r})$. As all elements of C are D-equivalent $\overleftarrow{d}$ is an upper bound for C and hence $\vec{s} \leqslant \overleftarrow{d}$. On the one hand $\vec{d} \neq \overleftarrow{s}$ as $\overleftarrow{s}$ does not lie in a splitting star, on the other hand $\vec{d} \neq \vec{s}$ as then $\vec{r}$ would be trivial. Therefore $\vec{s} \lneq \overleftarrow{d}$ and thus $D^-(\vec{r}) \subseteq D^-(\vec{s})$.

Now consider $\vec{d} \in D^+(\vec{s})$. This $\vec{d}$ cannot be an upper bound for C, hence either $\vec{d} \in D^+(\vec{r})$ or $\vec{d} \leqslant \overleftarrow{r}$. In the latter case $\overleftarrow{d}$ is an upper bound for C implying $\vec{d} \lneq \vec{s} \leqslant \overleftarrow{d}$ and thus that $\vec{d}$ would be trivial. Therefore $D^+(\vec{s}) \subseteq D^+(\vec{r})$ and hence $\vec{r} \sim_D \vec{s}$. □

For a subset $B \subseteq \tau$ and a selection D write $[B]_D := \{[\vec{b}]_D \mid \vec{b} \in B\} \subseteq \tau/D$. A direct consequence of Lemma 4.2(ii) and Proposition 4.19 is that for a chain $C \subseteq \tau$ the supremum of $[C]_D$ is the class of the supremum of C in τ:

Corollary 4.20. *Let τ be a chain-complete tree set, D a branch-closed selection, C a chain and $\vec{s}$ the supremum of C in τ. Then $[\vec{s}]_D = \max\,[C]_D$ in τ/D.*

Proof. The relation $\sim_D$ has finitely many equivalence classes, so Lemma 4.2(ii) implies that some final segment is completely contained in some class $[\vec{t}]_D$ of τ. The set $[C]_D$ in τ/D is again a chain, and as $[\vec{t}]_D$ contains a final segment of C it is the maximum of $[C]_D$ in τ/D. Proposition 4.19 now implies $\vec{s} \in [\vec{t}]_D = \max[C]_D$. □

Infinite splitting stars of τ play an important role in the upcoming section 5.4. We now analyze their behaviour with regard to $\sim_D$. This turns out to be quite simple: if a splitting star σ meets D, then all elements of $\sigma \cap D$ are pairwise non-equivalent, and all elements of $\sigma \backslash D$ get identified:

Lemma 4.21. *Let τ be a chain-complete tree set, D a branch-closed selection and σ a splitting star that meets D. Then $\vec{r} \sim_D \vec{s}$ for distinct $\vec{r}, \vec{s} \in \sigma$ if and only if $\vec{r}, \vec{s} \notin D$.*

In particular if σ is infinite there is exactly one equivalence class of $\sim_D$ containing infinitely many elements of σ, and every other equivalence class contains at most one element of σ.

Proof. Let $\vec{r}, \vec{s} \in \sigma$ be two distinct separations. For the forward direction suppose that $\vec{r} \in D$. Then $\vec{r} \in D^-(\vec{s})$ but $\vec{r} \notin D^-(\vec{r})$, so $\vec{r} \not\sim_D \vec{s}$.

For the backward direction assume that $\vec{r}, \vec{s} \notin D$. Then $D^+(\vec{s}) = \emptyset$ as otherwise $\vec{s} \in D$ by the assumptions that D is branch-closed and σ meets D. Similarly $D^+(\vec{r}) = \emptyset$. Moreover $D^-(\vec{r}) \backslash \{\vec{s}\} = D^-(\vec{s}) \cup D^+(\vec{s})$ as $\vec{r}, \vec{s}$ lie in a splitting star, so $D^-(\vec{r}) = D^-(\vec{s})$ by $\vec{s} \notin D$. □

To apply Lemma 4.21 in practice it is useful to have a sufficient condition for σ to meet D. The following lemma accomplishes this by showing that a splitting star σ of τ must meet D as soon as it meets at least three equivalence classes of $\sim_D$:

Lemma 4.22. *Let τ be a chain-complete tree set, D a branch-closed selection and σ a splitting star which meets at least three equivalence classes of $\sim_D$. Then σ meets D.*

Proof. Suppose that $\sigma \cap D = \emptyset$. Then there is $\vec{t} \in \sigma$ with $D^+(\vec{t}) \neq \emptyset$. Consider $\vec{r}, \vec{s} \in \sigma$ with $\vec{r} \not\sim_D \vec{t}$ and $\vec{s} \not\sim_D \vec{t}$. We will show that $\vec{r} \sim_D \vec{s}$, contradicting the assumption that σ meets three equivalence classes. First note that $D^+(\vec{r}) = D^+(\vec{s}) = \emptyset$ by the assumptions that D is branch-closed and $\vec{t} \notin D$. Moreover $D^-(\vec{r})\setminus\{\vec{s}\} = D^-(\vec{s}) \cup D^+(\vec{s})$ as $\vec{r}, \vec{s}$ lie in a splitting star, so $D^-(\vec{r}) = D^-(\vec{s})$ by $\vec{s} \notin D$. Hence $\vec{r} \sim_D \vec{s}$. □

For regular tree sets τ the tree set τ/D need not be regular in general. For example if τ is a regular four-star and D consists of two separations of this star then the other two separations in it get identified and form a small separation in τ/D. This is essentially the only way that τ/D can contain a small element if τ is regular: two separations that point towards each other which get identified. However if D contains all branching points of τ and τ is chain-complete then any two separations that point towards each other get distinguished:

Lemma 4.23. *Let τ be a regular chain-complete tree set, D a selection and $\vec{r}, \vec{s} \in \tau$ with $\vec{r} \sim_D \vec{s}$ and $\vec{r} \leqslant \overleftarrow{s}$. Then there is a branching point $\vec{b}$ of τ with $\vec{b} \notin D$.*

Proof. From the assumptions it follows that $\vec{d} \lneq \overleftarrow{r}, \overleftarrow{s}$ or $\overleftarrow{d} \lneq \overleftarrow{r}, \overleftarrow{s}$ for every $\vec{d} \in D$. As $D^+(\vec{r}) \cup D^-(\vec{r})$ is non-empty there is some $\vec{d} \in D$ with $\vec{d} \lneq \overleftarrow{r}, \overleftarrow{s}$. Applying Proposition 4.8 to the three-star $\{\vec{r}, \vec{s}, \vec{d}\}$ results in a branching star at least two of whose elements do not lie in D as they would distinguish $\vec{r}$ and $\vec{s}$. □

Lemma 4.23 implies that τ/D is regular if τ is chain-complete and D contains all branching points of τ:

Corollary 4.24. *Let τ be a regular chain-complete tree set and D a selection containing all branching points of τ. Then τ/D is a finite regular tree set.*

Proof. The selection D is branch-closed, so τ/D is a finite tree set by Proposition 4.17. Suppose some $[\vec{t}]_D$ in τ/D is small, i.e. $[\vec{t}]_D \leqslant [\overleftarrow{t}]_D$. Then there are $\vec{r} \in [\vec{t}]_D$ and $\overleftarrow{s} \in [\overleftarrow{t}]_D$ with $\vec{r} \leqslant \overleftarrow{s}$. By Lemma 4.2(i) $\vec{r} \sim_D \vec{s}$, so Lemma 4.23 implies that D does not contain all branching points of τ contrary to assumption. □

4.3 Regular tree sets

With the technical groundwork being done we can now set our eyes on the main goal of this chapter: to prove Theorem 5. In this section we begin by studying a much simpler case, the regular tree sets, and give a proof of Theorem 6 that does not use Theorem 5. In doing so we also obtain a characterization of the regular profinite tree sets. The main mechanics of the proof of Theorem 5 from the next section are already present here.

The general strategy is as follows. For a regular tree set τ we define a suitable directed set $\mathcal{D}$ of selections such that τ/D is a regular finite tree set for each $D \in \mathcal{D}$. For $\vec{s} \in \tau$ and selections $D \subseteq D'$ we have $[\vec{s}]_{D'} \subseteq [\vec{s}]_D$, so by taking these inclusions as the bonding maps $(\tau/D \mid D \in \mathcal{D})$ is an inverse system of regular finite tree sets. It then remains to prove that τ and $\varprojlim\, (\tau/D \mid D \in \mathcal{D})$ are isomorphic; for this we define the map $\varphi\colon \tau \to \varprojlim\, (\tau/D \mid D \in \mathcal{D})$ as

$$\varphi(\vec{s}) = ([\vec{s}]_D \mid D \in \mathcal{D}).$$

By the observation above $\varphi(\vec{s})$ is indeed always a compatible choice, and φ is a homomorphism of separation systems by Lemma 4.2(i) and the definition of $\leqslant$ in τ/D. Furthermore as every τ/D is regular so is their inverse limit, so all we need to show in order to meet the assumptions of Lemma 2.6 is that φ is a bijection.

We use Corollary 4.24 to ensure that the τ/D are finite regular tree sets. For this we assume that τ is chain-complete and let $\mathcal{D}$ be the set of all selections that contain all branching points of τ. Then $(\tau/D \mid D \in \mathcal{D})$ is an inverse system, provided that $\mathcal{D}$ is non-empty. A tree set τ contains a selection if and only if it contains a non-singleton splitting star, and if a selection contains all branching points of τ as Corollary 4.24 demands then there can only be finitely many branching points in τ. Hence we need to assume that τ has only finitely many branching points.

For the surjectivity of φ we rely on Corollary 4.20.

The map φ is injective if and only if for all distinct $\vec{r}, \vec{s} \in \tau$ the set $\mathcal{D}$ contains a selection D with $[\vec{r}]_D \neq [\vec{s}]_D$. A sufficient condition for this is that there is a splitting star of τ between any two given separations. Formally, a tree set τ is *splittable* if for every $\vec{r}, \vec{s} \in \tau$ with $\vec{r} \lneq \vec{s}$ there is a splitting star σ of τ with $\vec{r'}, \overleftarrow{s'} \in \sigma$ such that $\vec{r} \leqslant \vec{r'} \lneq \vec{s'} \leqslant \vec{s}$. As a side effect of the assumption that τ is splittable we do not need to separately assume that τ contains a non-singleton splitting star, as this follows directly from τ being splittable.

Example 4.25. For $X = \mathbb{Z}$ the natural tree set $\tau(X)$ on X is splittable. However, not every countable tree set is splittable: for $Y = \mathbb{Q}$ the natural tree set $\tau(Y)$ on Y is not splittable, because $\tau(Y)$ has no splitting orientations at all.

Let us now prove that the above assumptions are sufficient.

Proposition 4.26. *Let τ be a regular chain-complete splittable tree set with finitely many branching points and $\mathcal{D}$ the set of all selections that contain all branching stars of τ. Then $(\tau/D \mid D \in \mathcal{D})$ is an inverse system of regular finite tree sets and the map $\varphi\colon \tau \to \varprojlim\, (\tau/D \mid D \in \mathcal{D})$ with*

$$\varphi(\vec{s}) = ([\vec{s}]_D \mid D \in \mathcal{D})$$

is an isomorphism of tree sets.

Proof. From the assumptions it follows that $\mathcal{D}$ ordered by inclusion is a directed set as it is non-empty and closed under taking finite unions. Therefore $(\tau/D \mid D \in \mathcal{D})$ is an inverse system with the surjective bonding maps $f_{DE} \colon \tau/D \to \tau/E$ defined as

$$f([\vec{s}\,]_D) = [\vec{s}\,]_E$$

for $E \subset D$. Note that these bonding maps are well-defined by the definition of $\sim_D$, and are homomorphisms of tree sets by Lemma 4.2(i). Therefore $\varprojlim\, (\tau/D \mid D \in \mathcal{D})$ is a regular tree set.

The map φ is a homomorphism of tree sets by Lemma 4.2(i) and the definition of $\leqslant$ in τ/D: if $\vec{r} \leqslant \vec{s}$ for $\vec{r}, \vec{s} \in \tau$ then $[\vec{r}\,]_D \leqslant [\vec{s}\,]_D$ for all $D \in \mathcal{D}$ and hence $\varphi(\vec{r}\,) \leqslant \varphi(\vec{s}\,)$. The claim thus follows from Lemma 2.6 if we can show that φ is a bijection.

To see that φ is injective let r and s be two distinct unoriented separations in τ with, say, $\vec{r} \lneq \vec{s}$. As τ is splittable there is a splitting star σ of τ with $\vec{r'}, \overleftarrow{s'} \in \sigma$ such that $\vec{r} \leqslant \vec{r'} \lneq \vec{s'} \leqslant \vec{s}$, and for every $D \in \mathcal{D}$ with $\vec{r'}, \overleftarrow{s'} \in D$ the classes $[\vec{r}\,]_D, [\overleftarrow{r}\,]_D, [\vec{s}\,]_D$ and $[\overleftarrow{s}\,]_D$ are all distinct.

For the surjectivity let $\vec{t^*} = (\vec{t}^{\,*}_D \mid D \in \mathcal{D}) \in \varprojlim\, (\tau/D \mid D \in \mathcal{D})$ and assume for a contradiction that there is no $\vec{s} \in \tau$ with $\varphi(\vec{s}\,) = \vec{t^*}$. Set

$$X := \left\{ \vec{s} \in \tau \mid \varphi(\vec{s}\,) \leqslant \vec{t^*} \right\}, \qquad Y := \left\{ \vec{s} \in \tau \mid \varphi(\vec{s}\,) \geqslant \vec{t^*} \right\}.$$

For every $s \in \tau$ exactly one of $\vec{s}, \overleftarrow{s}$ lies in $X \cup Y$. Therefore at most one of X and Y is empty. We will show that both are non-empty. Suppose that $X \neq \emptyset$ and let C be a maximal chain in X with supremum $\vec{s}$ in τ. Then $\varphi(C)$ is also a chain and from Corollary 4.20 it follows that $\varphi(\vec{s}\,)$ is the supremum of $\varphi(C)$ in $\varprojlim\, (\tau/D \mid D \in \mathcal{D})$ and hence $\varphi(\vec{s}\,) \leqslant \vec{t^*}$. Moreover $\varphi(\vec{s}\,) \neq \vec{t^*}$ and therefore $\varphi(\vec{s}\,) \lneq \vec{t^*}$, so there is some $D \in \mathcal{D}$ such that $\varphi(\vec{s}\,) \lneq \vec{t^*_D}$. Pick $\vec{r} \in \tau$ with $[\vec{r}\,]_D = \vec{t^*_D}$. Then $\vec{s} \lneq \vec{r}$ by Lemma 4.18 and the regularity of τ/D implies that either $\varphi(\vec{r}\,) < \vec{t^*}$ or $\varphi(\vec{r}\,) > \vec{t^*}$. The first contradicts the maximality of C, therefore $\varphi(\vec{r}\,) > \vec{t^*}$ and thus $\vec{r} \in Y$.

Now let C' be a maximal chain in Y with infimum $\vec{s'}$. Again Corollary 4.20 implies that $\varphi(\vec{s'})$ is the infimum of $\varphi(C')$, and therefore $\varphi(\vec{s}\,) \leqslant \vec{t^*} \leqslant \varphi(\vec{s'})$. Pick $D' \geqslant D$ in $\mathcal{D}$ such that $\vec{t^*}_{D'} < [\vec{s'}]_{D'}$. As τ/D' is a tree set and $[\vec{s}\,]_{D'} \lneq \vec{t^*}_{D'}$ also $\vec{t^*}_{D'} \lneq [\vec{s'}]_{D'}$. Pick $\vec{t} \in \tau$ with $[\vec{t}\,]_{D'} = \vec{t^*}_{D'}$; then $\vec{s} < \vec{t} < \vec{s'}$ and thus $\overleftarrow{t} \in X \cup Y$ would imply that one of $\vec{s}, \vec{s'}$ is co-trivial. Hence $\vec{t} \in X \cup Y$, contradicting the maximality of either C or C'. $\square$

The property that a tree set contains only finitely many branching points is a bit exotic. For chain-complete tree sets it is equivalent to the much simpler condition that the tree set contains no infinite star:

Lemma 4.27. *Let τ be a chain-complete tree set. Then τ has infinitely many branching points if and only if τ contains an infinite star.*

Proof. For the backward direction suppose that the set B of all branching points of τ is finite and σ is an infinite star in τ. For a three-star σ_3 in τ let $B(\sigma_3)$ be the unique branching star of τ from Proposition 4.8 for which each element of σ_3 lies below a different element of $B(\sigma_3)$. Set

$$B' = \left\{ \vec{b} \in B \mid \vec{b} \in B(\sigma_3) \text{ for a three-star } \sigma_3 \right\}.$$

Then $B' \subseteq B$ is finite. Pick a $\vec{b} \in \bigcup B'$ that is minimal with the property that there are infinitely many $\vec{s} \in \sigma$ with $\vec{s} \leqslant \vec{b}$. Consider distinct $\vec{r}, \vec{s}, \vec{t} \in \sigma$ with $\vec{r}, \vec{s}, \vec{t} \leqslant \vec{b}$. Then $\vec{b} \notin B(\{\vec{r}, \vec{s}, \vec{t}\})$; in particular $\vec{u} \lneq \vec{b}$ for each $\vec{u} \in B\{\vec{r}, \vec{s}, \vec{t}\}$. As there are infinitely many triples $\vec{r}, \vec{s}, \vec{t} \in \sigma$ with $\vec{r}, \vec{s}, \vec{t} \leqslant \vec{b}$ and only finitely many elements in B' some $\vec{b}' \in B'$ must lie in infinitely many of the $B(\{\vec{r}, \vec{s}, \vec{t}\})$. But then $\vec{b}' \lneq \vec{b}$, contradicting the minimality of $\vec{b}$.

For the forward direction let B be the set of all branching points of τ. By Ramsey's Theorem B as a poset contains either an infinite antichain or an infinite chain. If $A \subseteq B$ is an infinite antichain either A itself or $\{\overleftarrow{a} \mid \vec{a} \in A\}$ is an infinite star. If $C \subseteq B$ is an infinite chain then for every $\vec{s} \in C$ let $\sigma(\vec{s})$ be the branching star containing $\vec{s}$ and $f(\vec{s})$ be some element of $\sigma(\vec{s})$ such that $f(\vec{s}) \lneq \vec{t}$ for some $\vec{t} \in C$. The set $\{f(\vec{s}) \mid \vec{s} \in C\}$ is an infinite star. □

Note that only the backward implication in Lemma 4.27 used the assumption that τ is chain-complete. The forward implication holds for all tree sets.

This Lemma allows us to formulate a cleaner version of Proposition 4.26:

Corollary 4.28. *A regular tree set τ that is chain-complete, splittable and contains no infinite star is profinite.*

Proof. By Lemma 4.27 the tree set τ contains only finitely many branching points. The claim then follows from Proposition 4.26. □

To obtain a characterization of the regular profinite tree sets we need to show that the assumptions we made in Corollary 4.28 are not only sufficient but also necessary.

We begin by showing that all profinite tree sets are chain-complete. This holds for all profinite tree sets, not just for regular ones.

As a first step we establish an analogue of Corollary 4.20 for arbitrary profinite tree sets. Note that for a chain in a profinite tree set all its projections are finite chains.

Lemma 4.29. *Let $\tau = \varprojlim (S_p \mid p \in P)$ be a profinite tree set and C a chain in τ. Then $(\max C_p \mid p \in P)$ is a compatible choice, where $C_p := C \upharpoonright p$.*

Proof. Let $p < q$ be given. Pick $\vec{r}, \vec{s} \in C$ with $\vec{r}_q = \max C_q$ and $\vec{s}_p = \max C_p$. Then $\vec{s}_q \leqslant \vec{r}_q$ and hence $\max C_p = \vec{s}_p \leqslant \vec{r}_p = f_{qp}(\max C_q)$. □

Now we just need to check that this coordinate-wise maximum is indeed the supremum of the given chain.

Proposition 4.30. *Profinite tree sets are chain-complete.*

Proof. Let $\tau = \varprojlim (S_p \mid p \in P)$ be a profinite tree set, C a non-empty chain in τ and $C_p := C \upharpoonright p$ for $p \in P$. By Lemma 4.29 $\vec{s} := (\max C_p \mid p \in P)$ is a compatible choice, so $\vec{s} \in \tau$. By definition $\vec{s}$ is an upper bound for C. Let $\vec{r} \in \tau$ be another upper bound for C. Then $\vec{s}_p \leqslant \vec{r}_p$ for each $p \in P$ and hence $\vec{s} \leqslant \vec{r}$ as $\vec{s}_p$ is the projection of some element in C for which $\vec{r}$ is an upper bound. This shows that $\vec{s}$ is the least upper bound for C. □

As with chain-completeness all profinite tree sets are splittable, not just the regular ones. The proof of this uses the same technique as the proof of chain-completeness.

Proposition 4.31. *Profinite tree sets are splittable.*

Proof. Let $\tau = \varprojlim (S_p \mid p \in P)$ be a profinite tree set and $\vec{r}, \vec{s} \in \tau$ with $\vec{r} \lneq \vec{s}$. Fix $p \in P$ such that $\vec{r}_p \lneq \vec{s}_p$ and set

$$X := \left\{ \vec{x} \in \tau \mid \vec{r} \leqslant \vec{x} \leqslant \vec{s} \text{ and } \vec{x}_p = \vec{r}_p \right\}.$$

Then X is a chain with $\vec{r} \in X$ which by Lemma 4.29 and Proposition 4.30 has a supremum $\vec{r'} = (\max X_q \mid q \in P)$. Now set

$$Y := \left\{ \vec{y} \in \tau \mid \vec{r} \leqslant \vec{y} \leqslant \vec{s} \text{ and } \vec{r}_p < \vec{y}_p \right\}.$$

This is a chain with $\vec{s} \in Y$ and infimum $\vec{s'} = (\min Y_q \mid q \in P)$. By definition we have $\vec{r} \leqslant \vec{r'} \lneq \vec{s'} \leqslant \vec{s}$. Furthermore there is no $\vec{t} \in \tau$ with $\vec{r'} < \vec{t} < \vec{s'}$ as this $\vec{t}$ would lie in $X \cup Y$ and thus contradict the definition of either $\vec{r'}$ or $\vec{s'}$. By the Extension Lemma there is a consistent orientation O of τ extending $\{\vec{r'}, \overleftarrow{s'}\}$ in which $\vec{r'}$ and therefore $\overleftarrow{s'}$ is maximal. Lemma 4.7 says that O is splitting, so $\vec{r'}$ and $\overleftarrow{s'}$ lie in a common splitting star. □

We have established that all profinite tree sets are chain-complete and splittable. However it is not true that all profinite tree sets contain only finite stars, and for this we really do need the assumption that the tree set is regular.

The next proposition says that every maximal infinite star in a profinite tree set contains a small separation, and its proof shows the process in which this small separation is generated as a limit of the infinite star. Both the assertion and this process will be used in the next section.

Proposition 4.32. *Let τ be a profinite tree set. Then every infinite star which is maximal by inclusion contains a small separation.*

Proof. Suppose $\sigma \subseteq \tau$ is an infinite maximal star, and $\tau = \varprojlim (S_p \mid p \in P)$. Let σ_p be the projection of σ to S_p. For every $p \in P$ there must be some $\vec{s}_p \in S_p$ which is the image of infinitely many elements of σ. As σ is a star such a $\vec{s}_p$ has to be small. For $p \in P$ let σ'_p be the set of all $\vec{s}_p \in S_p$ which are the image of infinitely many elements of σ. Then $(\sigma'_p \mid p \in P)$ is an inverse system of finite sets with a non-empty inverse limit, and its elements are also elements of τ. Let $\vec{s} \in \varprojlim (\sigma'_p \mid p \in P)$ be such an element. As every $\vec{s}_p$ is small so is $\vec{s}$. Moreover $\vec{s}_p \leqslant \overleftarrow{r}_p$ for all $p \in P$ and $\vec{r} \in \sigma$, so $\vec{s} \in \sigma$ by maximality. □

Call a tree set τ *star-finite* if it contains no infinite star. Then Proposition 4.32 implies that all regular profinite tree sets are star-finite:

Corollary 4.33. *Regular profinite tree sets are star-finite.*

Proof. If a profinite tree set contains an infinite star by Proposition 4.32 it also contains a small separation. Hence regular profinite tree sets do not contain infinite stars. □

Together Lemma 4.27, Corollary 4.28 and the Propositions 4.30, 4.31 and 4.32 imply the following characterization of the regular profinite tree sets:

Theorem 7. *A regular tree set τ is profinite if and only if it is chain-complete, splittable and star-finite.* □

Using Proposition 4.26 and Theorem 7 we can prove Theorem 6(ii) without using Theorem 5(ii):

Proof of Theorem 6(ii). By Theorem 7 and Lemma 4.27 every regular profinite tree set meets the assumptions of Proposition 4.26 and is thus the inverse limit of an inverse system of regular finite tree sets, as claimed. □

Therefore the regular profinite tree sets are indeed precisely those tree sets that can be obtained as an inverse limit of regular finite tree sets.

The following three examples show that the three conditions in Theorem 7 do not imply each other.

Example 4.34. Let R be a one-way infinite ray. Then $\vec{E}(R)$ is a regular tree set which is splittable and star-finite but not chain-complete.

Example 4.35. Let $X = [0, 1]$ and $\tau = \tau(X)$ the natural tree set on X. Then τ is chain-complete and star-finite but not splittable as the only splitting stars are $\{\vec{1}\}$ and $\{\overleftarrow{0}\}$.

Example 4.36. Let S be an infinite star. Then $\vec{E}(S)$ is a regular tree set which is chain-complete and splittable but not star-finite.

4.4 Irregular tree sets

In this section we prove Theorem 5:

Theorem 5.

(i) *Every inverse limit of finite tree sets is a tree set.*

(ii) *Every profinite tree set is an inverse limit of finite tree sets.*

Our approach is essentially the same as for the proof of Theorem 6 in the previous section: find a suitable directed set $\mathcal{D}$ of selections such that $(\tau/D \mid D \in \mathcal{D})$ is an inverse system of finite tree sets with inverse limit τ.

As in the previous chapter we assume that τ is chain-complete and splittable.

We again define the map $\varphi\colon \tau \to (\tau/D \mid D \in \mathcal{D})$ as

$$\varphi(\vec{s}) = ([\vec{s}]_D \mid D \in \mathcal{D}),$$

but this time we need to verify the conditions of Lemma 2.7 instead of Lemma 2.6 as $\varprojlim (\tau/D \mid D \in \mathcal{D})$ need not be regular.

In order to use Proposition 4.17 every $D \in \mathcal{D}$ needs to be branch-closed. However, to infer the injectivity of φ from the assumption that τ is splittable we need that every two-element subset of every splitting star of τ is contained in some $D \in \mathcal{D}$. This creates a problem for ensuring $\mathcal{D}$ is a directed set: if τ contains two non-singleton splitting stars with infinitely many branching points between them then no branch-closed selection can meet both of them. We thus need to assume that there are only finitely many branching points between any two non-singleton splitting stars.

To make this formal, for unoriented separations $s, s' \in \tau$ let $C(s, s')$ denote the set of all branching points $\vec{b}$ of τ with $\vec{s} \leqslant \vec{b} \leqslant \vec{s}'$ for some orientation of s and s'. Then $C(s, s')$ is always the disjoint union of two chains, and if $C(s, s')$ meets a star σ in an element other than s or s' then it meets that star in exactly two elements. Furthermore a selection D of τ is branch-closed if and only if $C(s, s') \subseteq D$ for all $\vec{s}, \vec{s}' \in D$.

If we assume that $C(s, s')$ is finite for all regular s, s' in τ then no two non-singleton splitting stars can have infinitely many branching points between them as each of these stars must contain a regular separation. Under this assumption the set of all branch-closed selections is a directed set.

With these assumptions we would be able to prove that $(\tau/D \mid D \in \mathcal{D})$ is an inverse system of finite tree sets, where $\mathcal{D}$ is the set of all branch-closed selections, and that φ is an injective homomorphism of tree sets. To apply Lemma 2.7 we still need to show that the pre-images of small separations are small and that φ is surjective. The first can be done with a little bit of case-checking. The latter needs an additional assumption: in the proof of Proposition 4.32 we have seen that every infinite star of τ has a small separation as a limit when dis- and reassembled as in $\varprojlim (\tau/D \mid D \in \mathcal{D})$. Concretely, by Lemma 4.21, if σ is an infinite splitting star of τ that meets a $D \in \mathcal{D}$, the elements of $\sigma \backslash D$ all lie in the same equivalence class, and the family of these $[\sigma \backslash D]_D$ over all $D \in \mathcal{D}$ that meet σ is a compatible choice and thus defines a separation in $\varprojlim (\tau/D \mid D \in \mathcal{D})$. This separation is not the image $\varphi(\vec{s})$ for any $\vec{s} \in \sigma$, as $\vec{s} \in D$ for some $D \in \mathcal{D}$ and then $[\vec{s}]_D \neq [\sigma \backslash D]_D$.

To make φ surjective we thus need to 'reserve' a $\vec{s} \in \sigma$ to be the pre-image of this limit separation. We can do this by forbidding all $D \in \mathcal{D}$ to contain $\vec{s}$.

Then $[\vec{s}]_D = [\sigma\backslash D]_D$ for every D that meets σ and the set $\mathcal{D}$ is still a directed set. Proposition 4.32 tells us that the limit separation of σ will be small, so this $\vec{s}$ needs to be small too. The assumption needed is thus that every infinite splitting star of τ contains a small separation.

Let us now prove that these assumptions are sufficient.

Proposition 4.37. *Let τ be a chain-complete splittable tree set with no infinite regular splitting star, in which $C(s, s')$ is finite for all regular s, s' in τ. For every infinite splitting star σ let $\nu(\sigma)$ be a small separation in σ, and let $\mathcal{D}$ be the set of all branch-closed selections of τ with $\nu(\sigma) \notin D$ for every infinite splitting star σ of τ.*

Then $(\tau/D \mid D \in \mathcal{D})$ is an inverse system of finite tree sets and the map $\varphi\colon \tau \to \varprojlim\,(\tau/D \mid D \in \mathcal{D})$ with

$$\varphi(\vec{s}) = ([\vec{s}]_D \mid D \in \mathcal{D})$$

is an isomorphism of tree sets.

Proof. By Proposition 4.17 each τ/D is a finite tree set. The set $\mathcal{D}$ ordered by inclusion is a directed set: let $D, D' \in \mathcal{D}$ and set

$$E := D \cup D' \cup \bigcup_{\vec{s},\vec{s'} \in D \cup D'} C(s, s').$$

Then $E \in \mathcal{D}$ with $D, D' \subseteq E$: E is finite by the assumption that $C(s, s')$ is finite for all regular s, s', and E is branch-closed by construction. It contains no $\nu(\sigma)$ of any infinite splitting star σ, as neither D nor D' does and thus $\nu(\sigma) \notin C(s, s')$ for all $\vec{s}, \vec{s'} \in D \cup D'$. Furthermore E meets every splitting star at least twice or not at all, as both D and D' do and each $C(s, s')$ does for all splitting stars that do not already meet D or D'. Hence $\mathcal{D}$ is a directed set.

Therefore $(\tau/D \mid D \in \mathcal{D})$ is an inverse system with the surjective bonding maps $f_{DE}\colon \tau/D \to \tau/E$ defined as

$$f([\vec{s}]_D) = [\vec{s}]_E$$

for $E \subset D$. Note that these bonding maps are well-defined by the definition of $\sim_D$, and are homomorphisms of tree sets by Lemma 4.2(i). Thus $\varprojlim\,(\tau/D \mid D \in \mathcal{D})$ is a tree set by Theorem 5(i).

The map φ is a homomorphism of tree sets by Lemma 4.2(i) and the definition of $\leqslant$ in τ/D: if $\vec{r} \leqslant \vec{s}$ then $[\vec{r}]_D \leqslant [\vec{s}]_D$ for all $D \in \mathcal{D}$ and hence $\varphi(\vec{r}) \leqslant \varphi(\vec{s})$. The claim thus follows from Lemma 2.7 if we can show that φ is a bijection, and that pre-images of small separations are small.

For the injectivity let $\vec{r}, \vec{s} \in \tau$ be two distinct separations. Then $\varphi(\vec{r}) \neq \varphi(\vec{s})$ follows from the assumption that τ is splittable, unless one of $\vec{r}$ and $\vec{s}$ is $\nu(\sigma)$ for some infinite splitting star σ. Suppose $\vec{r} = \nu(\sigma)$. If s does not meet σ the injectivity again follows from τ being splittable; if on the other hand s meets σ then any $D \in \mathcal{D}$ that meets s witnesses $\varphi(\vec{r}) \neq \varphi(\vec{s})$ by Proposition 4.21.

To show that pre-images of small separations are small consider a regular $\vec{s} \in \tau$; we will find a $D \in \mathcal{D}$ for which $[\vec{s}]_D$ is regular. This implies the claim as if $\varphi(\vec{s})$ and thus every $[\vec{s}]_D$ is small we know that $\vec{s}$ is not regular and thus either small or co-small, and it cannot be co-small as φ is a homomorphism. If there are $\vec{r}, \vec{t} \in \tau$ with $\vec{r} \lneq \vec{s} \lneq \vec{t}$ we can obtain a suitable selection $D \in \mathcal{D}$ by applying the splittability of τ to $\vec{r}, \vec{s}$ and $\vec{s}, \vec{t}$, and then taking as D those two-stars and all branching points between them. We may therefore assume that there is no $\vec{r} \in \tau$ with $\vec{r} \lneq \vec{s}$ (the other case is symmetrical). If $\vec{s}$ lies in a splitting star σ of τ then any non-singleton finite subset of σ containing $\vec{s}$ but not $\nu(\sigma)$ is a selection $D \in \mathcal{D}$ for which $[\vec{s}]_D$ is regular. If $\vec{s}$ does not lie in a splitting star there is a separation $\vec{t} \in \tau$ with $\vec{s} \lneq \vec{t}$ and t regular. Consider $C(s,t)$, which is finite by assumption. If it is empty then by applying the splittability of τ to $\vec{s}$, $\vec{t}$ we obtain a two-element subset of a splitting star which is a selection $D \in \mathcal{D}$ with $[\vec{s}]_D$ regular. If $C(s,t)$ is non-empty let $\vec{d} \in C(s,t)$ be minimal with $\vec{s} \lneq \vec{d}$, and $\overleftarrow{d'} \in C(s,t)$ maximal with $\vec{s} \lneq \overleftarrow{d'}$. Then $\vec{d}, \overleftarrow{d'}$ lie in a common branching star σ and $D = \{\vec{d}, \overleftarrow{d'}\}$ is a selection in $\mathcal{D}$ for which $[\vec{s}]_D$ is regular.

For the surjectivity of φ let $\vec{t^*} = (\vec{t^*_D} \mid D \in \mathcal{D}) \in \varprojlim (\tau/D \mid D \in \mathcal{D})$ and assume for a contradiction that there is no $\vec{s} \in \tau$ with $\varphi(\vec{s}) = \vec{t^*}$. This implies that $\vec{t^*_D}$ is an infinite equivalence class for each $D \in \mathcal{D}$. Set

$$X := \left\{ \vec{s} \in \tau \mid \varphi(\vec{s}) \leqslant \vec{t^*} \right\}, \qquad Y := \left\{ \vec{s} \in \tau \mid \varphi(\vec{s}) \geqslant \vec{t^*} \right\}.$$

For every $s \in \tau$ exactly one of $\vec{s}$, $\overleftarrow{s}$ lies in $X \cup Y$. Therefore at most one of X and Y is empty. If both are non-empty we may repeat the proof of Proposition 4.26, so we may assume that $Y = \varnothing$ (the other case is symmetrical).

Let C be a maximal chain in X with supremum $\vec{s}$. By Corollary 4.20 $\varphi(\vec{s})$ is the supremum of C in $\varprojlim (\tau/D \mid D \in \mathcal{D})$ and hence $\varphi(\vec{s}) \leqslant \vec{t^*}$. Moreover $\varphi(\overleftarrow{s}) \neq \vec{t^*}$ and therefore $\varphi(\vec{s}) \lneq \vec{t^*}$. Let $\mathcal{D}'$ be the set of all $D \in \mathcal{D}$ with $[\vec{s}]_D \lneq \vec{t^*_D}$. This is a cofinal set in $\mathcal{D}$. For each $D \in \mathcal{D}'$ let $\vec{M}(D)$ be the set of minimal elements of the equivalence class $\vec{t^*_D}$, which is non-empty by Proposition 4.19. Consider a $D \in \mathcal{D}'$ and $\vec{r} \in \vec{M}(D)$. Then $\vec{s} \lneq \vec{r}$, and by the maximality of C and the minimality of $\vec{r}$ there can be no $\vec{t} \in \tau$ with $\vec{s} \lneq \vec{t} \lneq \vec{r}$. Lemma 4.7 thus implies that $\vec{s}$ and $\overleftarrow{r}$ lie in a common splitting star σ of τ. As σ is the unique splitting star of τ with $\vec{s} \in \sigma$ and both $D \in \mathcal{D}'$ and $\vec{r} \in \vec{M}(D)$ were arbitrary this shows $\overleftarrow{M}(D) := \{\overleftarrow{r} \mid \vec{r} \in \vec{M}(D)\} \subseteq \sigma$ for every $D \in \mathcal{D}'$. We will show that $\vec{M}(D)$ and thus σ is infinite for every $D \in \mathcal{D}'$ and deduce $\overleftarrow{t^*} = \varphi(\nu(\sigma))$.

Suppose for a contradiction that there is a $D \in \mathcal{D}'$ with $\vec{M}(D)$ finite and let $D' \in \mathcal{D}'$ with $D' \geqslant D$ be such that $[\vec{r}]_{D'} \neq \vec{t^*}_{D'}$ for each $\vec{r} \in \vec{M}$. Pick a $\vec{u} \in \vec{M}(D')$. Then $\overleftarrow{u} \in \overleftarrow{M}(D') \subseteq \sigma$. By compatibility $[\vec{u}]_D = \vec{t^*_D}$ and hence $\vec{r} < \vec{u}$ for some $\vec{r} \in \vec{M}(D)$, contradicting $\overleftarrow{r}, \overleftarrow{u} \in \sigma$. Therefore $\overleftarrow{M}(D) \subseteq \sigma$ is infinite for every $D \in \mathcal{D}'$.

Pick a $\overleftarrow{r} \in \sigma$ with $\overleftarrow{r} \neq \vec{s}$ and fix some $D \in \mathcal{D}'$ with $[\overleftarrow{r}]_D \neq [\vec{s}]_D$ and $[\overleftarrow{r}]_D \neq \vec{t^*_D}$. Then Lemma 4.21 and Lemma 4.22 imply that $\overleftarrow{t^*}_{D'} = [\nu(\sigma)]_{D'}$ for every $D' \in \mathcal{D}'$ with $D' \geqslant D$ as $\nu(\sigma) \notin g(D')$ by assumption. As the set of all $D' \in \mathcal{D}'$ with $D' \geqslant D$ is cofinal in $\mathcal{D}$ this shows $\overleftarrow{t^*} = \varphi(\nu(\sigma))$.

Thus φ is a bijection and the claim follows from Lemma 2.7. □

To obtain a characterization of the profinite tree sets we now prove that the assumptions of Proposition 4.37 are necessary, that is, that every profinite tree set meets these assumptions. The Propositions 4.30, 4.31 and 4.32 from the previous section already established that all profinite tree sets are chain-complete and splittable and contain no regular infinite splitting star.

We show that $C(s, s')$ is finite for all regular s, s' in a profinite tree set τ in three steps. First we show that every infinite chain has some limit element. Then we show that if $\vec{m} \in \tau$ is the limit of a chain of branching points it must be co-small; and finally we infer that $C(s, s')$ can only be finite if one of s and s' is non-regular.

The first step is more about posets and chain-complete tree sets than about profinite tree sets:

Lemma 4.38. *Let τ be a chain-complete tree set and C an infinite chain in τ. Then there is a sub-chain $C' \subseteq C$ that does not contain both its infimum and its supremum in τ.*

Proof. We may assume that C contains its infimum and supremum in τ as otherwise $C' := C$ is as desired.

For $\vec{s} \in C$ let

$$C_{\leqslant \vec{s}} := \left\{ \vec{r} \in C \mid \vec{r} \leqslant \vec{s} \right\}$$

and

$$L = \left\{ \vec{s} \in C \mid \left|C_{\leqslant \vec{s}}\right| < \infty \right\}.$$

Then L is a non-empty sub-chain of C. Let $\vec{l}$ be the supremum of L in τ; if $\vec{l} \notin L$ then L is as desired. If on the other hand $\vec{l} \in L$ then L is finite, so $R := C \backslash L$ is infinite. Let $\vec{r}$ be the infimum of R in τ. If $\vec{r} \in R$ then $C_{\leqslant \vec{r}}$ is infinite, so there is a $\vec{t} \in C_{\leqslant \vec{r}} \backslash (L \cup \{\vec{r}\})$. But this contradicts the fact that $\vec{r}$ is the infimum of R. Therefore $\vec{r} \notin R$ and R is the desired sub-chain. □

Now we prove that the limit of a chain of branching points is co-small. The proof of this is somewhat analogous to the proof of Proposition 4.32:

Lemma 4.39. *Let $\tau = \varprojlim (S_p \mid p \in P)$ be a profinite tree set, C a chain of infinitely many branching points and $\vec{m}$ the supremum of C in τ. If $\vec{m} \notin C$ then $\vec{m}$ is co-small.*

Proof. As co-small separations are maximal in τ we may assume without loss of generality that C is a chain of order type ω.

Let $C = \{\vec{s}^{\,n} \mid n \in \mathbb{N}\}$ with $\vec{s}^{\,n} \lneq \vec{s}^{\,n+1}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ pick an element $\vec{t}^{\,n}$ of the branching star containing $\vec{s}^{\,n}$ with $\vec{t}^{\,n} \lneq \vec{s}^{\,n+1}$. Let $\vec{m}$ be the supremum of C. For any fixed $p \in P$ Lemma 4.29 implies that there is a $n \in \mathbb{N}$ with $\vec{s}^{\,n}_p = \vec{m}_p$; let $k(p) \in \mathbb{N}$ be the minimal such index and write

$$T_p := \left\{ \vec{t}^{\,n}_p \mid n \geqslant k(p) \right\}.$$

Observe that if $n \geqslant k(p)$ for some $p \in P$ then $\vec{t}^{\,n} \leqslant \vec{m}, \overleftarrow{s}^{\,n}$ by definition and hence $\vec{t}^{\,n}_p \leqslant \vec{m}_p$ as well as $\vec{t}^{\,n}_p \leqslant \overleftarrow{s}^{\,n}_p = \overleftarrow{m}_p$, and as a consequence also $\vec{t}^{\,n}_p \leqslant \overleftarrow{t}^{\,n}_p$.

Moreover $k(p) \leqslant k(q)$ for $p < q$, so $(T_p \mid p \in P)$ is an inverse system whose inverse limit is a subset of τ. Let $\vec{t} \in \varprojlim (T_p \mid p \in P)$. By the above observation $\vec{t} \leqslant \vec{m}$, $\vec{t} \leqslant \overleftarrow{m}$ and $\vec{t} \leqslant \overleftarrow{t}$. Thus $t = m$, and accordingly either $\vec{t} = \vec{m}$ or $\vec{t} = \overleftarrow{m}$. But $\vec{m}$ cannot be small as then every $\vec{s}^{\,n}$ would be trvial. Therefore $\vec{t} = \overleftarrow{m}$ and $\vec{m}$ is co-small. □

With these two lemmas we can now show that $C(s, s')$ is finite for all regular s, s' in profinite tree sets:

Proposition 4.40. *Let τ be a profinite tree set and $s, s' \in \tau$ two regular unoriented separations. Then $C(s, s')$ is finite.*

Proof. Suppose that $C(s, s')$ is infinite. Then $C(s, s')$ is the disjoint union of two infinite chains. Let C be one of them. By Lemma 4.38 there is a sub-chain C' of C that does not contain both its infimum and supremum in τ; suppose that C' does not contain its supremum (the other case is symmetrical). Let $\vec{m}$ be the supremum of C'. Lemma 4.39 implies that $\vec{m}$ is co-small. But one of $\vec{s}$, $\overleftarrow{s}$, $\vec{s'}$ or $\overleftarrow{s'}$ is an upper bound for C'. As co-small elements are maximal in τ it follows that $m = s$ or $m = s'$, a contradiction. □

Proposition 4.37, 4.30, 4.31, 4.32 and 4.40 combine into the following Theorem characterizing the profinite tree sets:

Theorem 8. *A tree set τ is profinite if and only if it is chain-complete and splittable, contains no regular infinite splitting star, and has the property that $C(s, s')$ is finite for all regular s, s'.* □

Moreover we can now prove Theorem 5(ii), that is, that every profinite tree set is an inverse limit of finite tree sets:

Proof of Theorem 5(ii). Let τ be a profinite tree set. From Theorem 8 it follows that τ meets the assumptions of Proposition 4.37, which together with Proposition 4.17 implies that τ is an inverse limit of finite tree sets. □

Therefore the profinite tree sets are indeed precisely those tree sets that are an inverse limit of finite tree sets.

Finally we show by example that the four properties in Theorem 8 do not imply each other. Example 4.34, 4.35 and 4.36 from the previous section already showed this for three of the four properties. The following example shows a tree set that is chain-complete and splittable with no infinite splitting star, but has an infinite $C(s, s')$.

Example 4.41. Let B be the tree set with ground set $\vec{m}, \overleftarrow{m}$ and $\vec{s}_n$, $\overleftarrow{s}_n$, $\vec{t}_n$ and $\overleftarrow{t}_n$ for every $n \in \mathbb{N}$, with the following relations:

1. $\vec{m} \geqslant \vec{s}_n, \vec{t}_n$ and $\overleftarrow{m} \leqslant \overleftarrow{s}_n, \overleftarrow{t}_n$ for all $n \in \mathbb{N}$,
2. $\vec{s}_i \leqslant \vec{s}_j$ and $\overleftarrow{s}_i \geqslant \overleftarrow{s}_j$ if and only if $i \leqslant j$,
3. $\vec{t}_i \leqslant \overleftarrow{t}_j$ if and only if $i \neq j$,
4. $\vec{s}_i \leqslant \overleftarrow{t}_j$ and $\overleftarrow{s}_i \geqslant \vec{t}_j$ if and only if $i \leqslant j$,
5. $\vec{s}_i \geqslant \vec{t}_j$ and $\overleftarrow{s}_i \leqslant \overleftarrow{t}_j$ if and only if $i > j$.

Then B is a tree set which is chain-complete and splittable with no infinite splitting star, but $C(s_1, m)$ is infinite despite s_1 and m being regular.

4.5 Orientations of profinite tree sets

In our study of tree sets, splitting stars and orientations of tree sets have played an important role in formulating and proving the various characterization theorems of the previous chapters and sections. Profinite tree sets naturally carry a topology: the inverse limit topology. We may therefore try to describe splitting orientations in topological terms. A first guess is that the splitting orientations of a profinite tree set are precisely those orientations that are topologically closed. And indeed, every closed orientation is splitting, and most splitting orientations are closed. In fact these two properties are equivalent in star-finite profinite tree sets, and thus in particular in regular profinite tree sets. However, in profinite tree sets that do contain an infinite star there might be splitting orientations that are not topologically closed. We show that this is indeed the case for every profinite tree set with an infinite star.

Recall that if $\tau = \varprojlim\, (S_p \,|\, p \in P)$ is a profinite tree set, the inverse limit topology on τ is the subspace topology of the product space

$$\varprojlim\, (S_p \,|\, p \in P) \subseteq \prod_{p \in P} S_p,$$

where each S_p carries the discrete topology. Note that this topology depends on the inverse system used.

The following lemma characterizes the closed sets in a profinite separations system $\vec{S}$, and we use it in the rest of this section to determine whether an orientation is closed or not.

Lemma 4.42. [2] *Let $\vec{S} = \varprojlim\, (\vec{S}_p \,|\, p \in P)$ be a profinite separation system. A set $O \subseteq \vec{S}$ is closed in $\vec{S}$ if and only if there are sets $O_p \subseteq \vec{S}_p$ such that $O = \varprojlim\, (O_p \,|\, p \in P)$ with the maps $O_q \to O_p$ induced from $(\vec{S}_p \,|\, p \in P)$.*

It is straightforward to show that every closed set in τ is bounded from above in the same sense as splitting orientations:

Lemma 4.43. [2] *Let $\vec{S}$ be a profinite separation system. If $O \subseteq \vec{S}$ is closed in $\vec{S}$, then for every $\vec{r} \in O$ there exists in O some $\vec{s} \geqslant \vec{r}$ that is maximal in O.*

Lemma 4.43 implies that every closed consistent orientation is splitting:

Corollary 4.44. *Let τ be a profinite tree set and O a consistent orientation of τ. If O is closed in τ it is also splitting.*

Proof. This follows immediately from Lemma 4.43. □

We now show that all splitting orientations are closed, with one possible exception.

Lemma 4.45. *Let τ be a profinite tree set and O an orientation of τ with a greatest element $\vec{m}$. Then O is a consistent orientation, and if $\vec{m}$ is not co-small then O is closed in τ.*

Proof. If O contains $\vec{r}, \vec{s}$ with $\overleftarrow{r} \leqslant \vec{s}$ then $\vec{m} \geqslant \vec{r} \geqslant \overleftarrow{s} \geqslant \overleftarrow{m}$, a contradiction. Thus O is consistent.

Now suppose that $O \neq O' := \varprojlim\, (O_p \,|\, p \in P)$, where $O_p = O \uparrow p$. Then O' contains some $\overleftarrow{s}$ with $\vec{s} \in O$. If $\vec{s} < \vec{m}$ then for every $p \in P$ there is some

$\vec{r} \in O$ with $\vec{r}_p = \overleftarrow{s}_p$. As $\vec{r} \leqslant \vec{m}$ also $\overleftarrow{s}_p \leqslant \vec{m}_p$. But this holds for every $p \in P$, hence $\vec{s}, \overleftarrow{s} < \vec{m}$, a contradiction.

If on the other hand $\overleftarrow{m} \in O'$ then for every $p \in P$ there is some $\vec{r} \in O$ with $\vec{r}_p = \overleftarrow{m}_p$. Due to $\vec{r} \leqslant \vec{m}$ also $\overleftarrow{m}_p \leqslant \vec{m}_p$ for every $p \in P$, hence $\vec{m}$ is co-small. □

If a consistent orientation O of a profinite tree set has two or more maximal elements none of them can be co-small. Thus if O is splitting one would expect O to be closed. This is indeed the case.

Lemma 4.46. *Let τ be a profinite tree set and O a splitting orientation of τ with two or more maximal elements. Then O is closed in τ.*

Proof. Suppose that $O \neq O' := \varprojlim (O_p \mid p \in P)$, where $O_p = O \uparrow p$. Then O' contains some $\overleftarrow{s}$ with $\vec{s} \in O$. This $\vec{s}$ lies below some maximal element $\vec{m} \in O$. Let $\vec{n} \neq \vec{m}$ be another maximal element of O and fix a $q \in P$ such that $m_q \neq n_q$ and thus $\vec{n}_q < \overleftarrow{m}_q$.

Consider any $p \geqslant q$. There is some $\vec{r} \in O$ with $\vec{r}_p = \overleftarrow{s}_p$. This $\vec{r}$ lies below a maximal element of O; suppose first that $\vec{r} \leqslant \vec{n}$. Then $\vec{n}_p \geqslant \vec{r}_p = \overleftarrow{s}_p \geqslant \overleftarrow{m}_p$, contradicting $\vec{n}_p < \overleftarrow{m}_p$ as $p \geqslant q$.

Thus $\vec{r}$ lies below some other maximal element of O and hence points towards $\vec{n}$, that is $\vec{r} \leqslant \overleftarrow{n}$. Then $\overleftarrow{n}_p \geqslant \vec{r}_p = \overleftarrow{s}_p \geqslant \overleftarrow{m}_p$ and also $\overleftarrow{n}_p \geqslant \vec{m}_p$ as $\vec{m}$ and $\vec{n}$ point towards each other. As this holds for every $p \geqslant q$ it implies $\overleftarrow{n} > \vec{m}, \overleftarrow{m}$, a contradiction. Hence O is closed by Lemma 4.42. □

Combining Lemma 4.7, Lemma 4.30 and Lemma 4.46 gives the following Lemma.

Lemma 4.47. *Let τ be a profinite tree set and O a consistent orientation of τ with two or more maximal elements. Then O is both splitting and closed.* □

As we have seen in the proof of Lemma 4.45 the only way that a splitting consistent orientation O of a profinite tree set τ can fail to be closed is if O has a co-small greatest element $\vec{m}$ such that $\overleftarrow{m}$ is in the closure of O. A slight modification of Example 4.41 from the previous section shows that this can indeed occur.

Example 4.48. Let B be the tree set with ground set $\vec{m}, \overleftarrow{m}$ and $\vec{s}_n, \overleftarrow{s}_n, \vec{t}_n$ and $\overleftarrow{t}_n$ for every $n \in \mathbb{N}$, with the following relations:

1. $\vec{m} \geqslant \vec{s}_n, \vec{t}_n$ and $\overleftarrow{m} \leqslant \overleftarrow{s}_n, \overleftarrow{t}_n$ for all $n \in \mathbb{N}$,
2. $\vec{m} \geqslant \overleftarrow{m}$,
3. $\vec{s}_i \leqslant \vec{s}_j$ and $\overleftarrow{s}_i \geqslant \overleftarrow{s}_j$ if and only if $i \leqslant j$,
4. $\vec{t}_i \leqslant \overleftarrow{t}_j$ if and only if $i \neq j$,
5. $\vec{s}_i \leqslant \overleftarrow{t}_j$ and $\overleftarrow{s}_i \geqslant \vec{t}_j$ if and only if $i \leqslant j$,
6. $\vec{s}_i \geqslant \vec{t}_j$ and $\overleftarrow{s}_i \leqslant \overleftarrow{t}_j$ if and only if $i > j$.

Then B is profinite by Theorem 8. Consider the consistent orientation

$$O := \{\,\vec{s}_n,\, \vec{t}_n \mid n \in \mathbb{N}\} \cup \{\vec{m}\}$$

of B. This O is splitting but not closed because $\overleftarrow{m}$ lies in the closure of $\{\,\vec{t}_n \mid n \in \mathbb{N}\} \subset O$, as the proof of Lemma 4.39 demonstrates.

Another example arises from the proof of Proposition 4.32, which together with Lemma 4.42 tells us which elements of an infinite star can be used to generate splitting non-closed orientations.

Example 4.49. Let S be an infinite graph-theoretical star and σ the infinite splitting star of the edge tree set $\tau(S)$. Let τ be the tree set obtained from $\tau(S)$ by making one separation $\vec{s} \in \sigma$ co-small. Then τ is profinite by Theorem 8. By Proposition 4.32 $\vec{s}$ is in the closure of $\sigma\backslash\{\,\vec{s}\,\}$, so the orientation $O = \{\,\overleftarrow{s}\,\} \cup (\sigma\backslash\{\,\vec{s}\,\})$ is splitting but not closed.

In both Example 4.48 and Example 4.49 the separation in the closure of O was the limit of an infinite star: in Example 4.48 of $\{\,\vec{t}_n \mid n \in \mathbb{N}\}$ and in Example 4.49 of $\sigma\backslash\{\,\vec{s}\,\}$. In fact the existence of any infinite star is enough to guarantee a consistent orientation that is splitting but not closed:

Lemma 4.50. *Let $\tau = \varprojlim\,(S_p \mid p \in P)$ be a profinite tree set and σ an infinite star of τ. Then τ has a consistent orientation O that is splitting but not closed.*

Proof. We will show that σ contains a small separation $\vec{s}$ that lies in the closure of $\sigma\backslash\{\,\vec{s}\,\}$. Then we can apply the Extension Lemma to $\{\,\overleftarrow{s}\,\}$ to obtain an orientation O with greatest element $\overleftarrow{s}$ that is not closed as the closure of O contains $\vec{s}$. The proof is similar to that of Proposition 4.32.

For $p \in P$ let σ_p be the projection $\{\,\vec{t}_p \mid \vec{t} \in \sigma\}$ into S_p and σ'_p the set of all $\vec{t}_p \in S_p$ for which there are infinitely many $\vec{r} \in \sigma$ with $\vec{t}_p = \vec{r}_p$. As σ is infinite and S_p is finite σ'_p is non-empty and finite, and as σ is a star all elements of σ'_p are small. Thus $(\sigma'_p \mid p \in P)$ is an inverse system of finite sets. Pick $\vec{s} \in \varprojlim\,(\sigma'_p \mid p \in P)$. As every $\vec{s}_p$ is small so is $\vec{s}$, and for every $p \in P$ there is $\vec{r} \in \sigma\backslash\{\,\vec{s}\,\}$ with $\vec{r}_p = \vec{s}_p$. Hence $\vec{s}$ lies in the closure of $\sigma\backslash\{\,\vec{s}\,\}$. $\square$

The converse holds too: whenever a splitting orientation fails to be closed a limit separation of an infinite star is at fault.

Lemma 4.51. *Let $\tau = \varprojlim\,(S_p \mid p \in P)$ be a profinite tree set and O a consistent orientation that is splitting but not closed. Then τ contains an infinite star.*

Proof. From Lemma 4.45 and 4.47 it follows that O must have a greatest element $\vec{m}$ with $\overleftarrow{m}$ in the closure of O. Suppose that τ does not contain an infinite star. We will find a $p \in P$ for which $\vec{s}_p \neq \overleftarrow{m}_p$ for all $\vec{s} \in O$, showing that O is closed.

As the minimal elements of τ form a star it follows that τ has only finitely many minimal elements. Furthermore $\vec{s} \in O$ for all minimal elements of τ except for $\overleftarrow{m}$ by consistency. For any $\vec{s} \in O$ with $\vec{s} \neq \overleftarrow{m}$ there is a $p \in P$ such that $\vec{s}_p \lneqq \vec{m}_p$ and $\vec{s}_p$ is not trivial with witness m_p, as $\vec{s}$ is not trivial with witness m. As there are only finitely many minimal elements and P is a directed set there is a $p \in P$ such that $\vec{s}_p \lneqq \vec{m}_p$ and $\vec{s}_p$ is not trivial with witness m_p for all minimal elements $\vec{s}$ of τ simultaneously.

Now suppose that there is a $\vec{t} \in O$ with $\vec{t}_p = \overleftarrow{m}_p$. Then $\vec{s} \leqslant \vec{t}$ for some minimal element of τ with $\vec{s} \in O$ (that is, $\vec{s} \neq \overleftarrow{m}$). But then both $\vec{s}_p \lneqq \vec{m}_p$ and $\vec{s}_p \leqslant \vec{t}_p = \overleftarrow{m}_p$ as $\vec{s} \leqslant \vec{t}$, contradicting the choice of p. $\square$

The combination of Lemma 4.50 and 4.51 gives the following characterization of profinite tree sets for which 'splitting' and 'closed' are equivalent for consistent orientations.

Proposition 4.52. *Let τ be a profinite tree set. The splitting orientations of τ are precisely the closed consistent orientations if and only of τ is star-finite.*

In particular, for regular profinite tree sets a consistent orientation is splitting if and only if it is closed.

Proof. The forward direction follows from Lemma 4.50 and the backward direction from Lemma 4.51. The additional claim about regular profinite tree sets is a consequence of Theorem 7. □

4.6 Application: representing profinite tree sets as bipartitions of a set

As an application of this study of profinite tree sets we extend Theorem 3 to profinite tree sets by using the characterization of regular profinite tree sets from Theorem 7.

Recall that $\mathcal{O} = \mathcal{O}(\tau)$ denotes the set of all consistent orientations of τ and $\mathcal{O}' = \mathcal{O}'(\tau)$ that of all $O \in \mathcal{O}$ that have a greatest element. Furthermore $\mathcal{O}(\vec{s})$ is the set of all $O \in \mathcal{O}$ that contain $\vec{s}$, and similarly $\mathcal{O}'(\vec{s})$ is the set of all $O \in \mathcal{O}'$ with $\vec{s} \in O$.

Theorem 9. *Let τ be a regular profinite tree set. The map $\varphi\colon \tau \to \mathcal{B}(\mathcal{O}')$ with $\varphi(\vec{s}) = (\mathcal{O}'(\overleftarrow{s}), \mathcal{O}'(\vec{s}))$ is an isomorphism of tree sets between τ and its image in $\mathcal{B}(\mathcal{O}')$ if and only if it is injective, which it is if and only if τ has no splitting star of size two.*

Proof. For the backward direction verify the assumptions of Lemma 2.6. By Theorem 7 τ is chain-complete and splittable. First we show that the image of φ in $\mathcal{B}(\mathcal{O}')$ is regular. Let $\vec{s} \in \tau$ and C a maximal chain in τ containing $\vec{s}$ with supremum $\vec{m}$. Applying the Extension Lemma to C yields a consistent orientation O with $C \subseteq O$ which has $\vec{m}$ as a maximal element. As O is consistent and C was chosen maximally $\vec{m}$ is even the greatest element of O. Hence $O \in \mathcal{O}'(\vec{s})$ and the image of φ in $\mathcal{B}(\mathcal{O}')$ is regular.

Moreover φ clearly commutes with the involution, and it is order-preserving by consistency. Thus it is left to show that φ is injective. For this let $\vec{r}, \vec{s} \in \tau$ with $\vec{r} \neq \vec{s}$. If $\vec{r} \leqslant \overleftarrow{s}$ then any $O \in \mathcal{O}'(\overleftarrow{s})$ contains $\vec{r}$ but not $\vec{s}$ and hence witnesses that $\mathcal{O}'(\vec{r}) \neq \mathcal{O}'(\vec{s})$. If $\overleftarrow{r} \leqslant \vec{s}$ then any $O \in \mathcal{O}'(\vec{r})$ again contains $\vec{r}$ but not $\vec{s}$ and hence witnesses that $\mathcal{O}'(\vec{r}) \neq \mathcal{O}'(\vec{s})$. Finally if $\vec{r}$ and $\vec{s}$ are comparable with, say, $\vec{r} \lneq \vec{s}$, due to τ being splittable there are $\vec{r'}, \vec{s'} \in \tau$ with $\vec{r} \leqslant \vec{r'} \lneq \vec{s'} \leqslant \vec{s}$ and a splitting star σ of τ containing $\vec{r'}$ and $\overleftarrow{s'}$. As $|\sigma| \neq 2$ there is a $\vec{t} \in \sigma$ with $\vec{t} \neq \vec{r'}, \overleftarrow{s'}$. Then $\vec{t} \leqslant \overleftarrow{r'} \leqslant \overleftarrow{r}$ and $\vec{t} \leqslant \overleftarrow{s'} \leqslant \overleftarrow{s}$. Pick any $O \in \mathcal{O}'(\overleftarrow{t})$. By consistency $\vec{r}, \overleftarrow{s} \in O$, so O witnesses that $\mathcal{O}'(\vec{r}) \neq \mathcal{O}'(\vec{s})$.

Hence φ is injective and the claim follows from Lemma 2.6.

For the forward direction let $\{\vec{r}, \vec{s}\}$ be a splitting star of τ with $\vec{r} \neq \vec{s}$. We will show $\mathcal{O}'(\vec{r}) = \mathcal{O}'(\overleftarrow{s})$. By consistency $\mathcal{O}'(\vec{r}) \subseteq \mathcal{O}'(\overleftarrow{s})$, so suppose there is an $O \in \mathcal{O}'(\vec{r})$ with $\overleftarrow{s} \notin O$. Let $\vec{t}$ be the greatest element of O. Then $t \neq r$ and $t \neq s$, so $\vec{r}, \vec{s} \lneq \vec{t}$ as $\vec{r}, \vec{s} \in O$. But this contradicts the assumption that $\{\vec{r}, \vec{s}\}$ is a splitting star of τ. Therefore $\mathcal{O}'(\vec{r}) = \mathcal{O}'(\overleftarrow{s})$ and hence $\varphi(\vec{r}) = \varphi(\overleftarrow{s})$ despite $\vec{r} \neq \overleftarrow{s}$. $\square$

As every finite tree set is profinite Theorem 9 immediately implies Theorem 3.

4.7 Further study

Our study of profinite tree sets was motivated by the prospect of extending the tangle-tree theorem from [4] to profinite separation systems. However, even if the theorem is applied to a profinite separation system, not all tree sets that occur in the tangle-tree theorem are necessarily profinite themselves.

Example 4.53. Let V be an infinite ground set and $\tau \subset \mathcal{B}(V)$ the tree set consisting of all bipartitions of the form $\{V - x\,,\,x\}$ for $x \in V$. Then τ is an infinite regular star and hence not profinite. Its closure $\overline{\tau}$ in $\mathcal{B}(V)$ is profinite but not a tree set: in addition to τ it contains the trivial separation $\{\emptyset\,,\,V\}$.

Furthermore for every $x \in V$ the set

$$P_x := \{(A, B) \in \mathcal{B}(V) \mid x \in B\}$$

is a tangle, that is a consistent orientation of $\mathcal{B}(V)$. Write $\mathcal{P}$ for the set of all such P_x with $x \in V$. Then τ is a tree set which distinguishes every pair $P_x, P_y \in \mathcal{P}$: the tangles P_x and P_y orient the bipartition $\{V - x\,,\,x\} \in \tau$ differently. Therefore a tree set like τ is indeed a possible result of an application of an infinite tangle-tree theorem.

In view of Example 4.53 it seems natural to extend the study of profinite tree sets to a broader class of nested separation systems. Call a separation system τ a *rooted tree set* with *root* $\vec{r} \in \tau$ if $\vec{r}$ is the least element of τ and $\tau - r$ is a tree set, and call τ *regular* if $\vec{r}$ is the only small separation in τ. Note that the root of a rooted tree set τ is always trivial in τ by definition.

With this definition the closure of τ in $\mathcal{B}(V)$ from Example 4.53 is a regular profinite rooted tree set with root $(\emptyset\,,\,V)$.

Proposition 4.30 and 4.31 still hold for profinite rooted tree sets with the same proofs, so every profinite rooted tree set is chain-complete and splittable. Proposition 4.32 trivially holds for rooted tree sets: every star in a rooted tree set that is maximal by inclusion contains the root and hence a small separation. However, profinite rooted tree sets may contain infinite regular splitting stars as seen in Example 4.53 above. Finally Proposition 4.40 should hold for profinite rooted tree sets as well.

Therefore a careful translation of Section 5.2 and 5.4 to rooted tree sets might resolve the following open problems:

Open Problem 1. *Is every profinite rooted tree set an inverse limit of finite rooted tree sets?*

Open Problem 2. *Let τ be a chain-complete splittable rooted tree set with finite $C(s, s')$ for all regular $s, s' \in \tau$. Is τ profinite?*

For tree sets our study revealed that regular profinite tree sets are a very sparse subclass of the class of profinite tree sets: regular profinite tree sets do not contain infinite stars and can thus be thought of as certain subdivisions of finite tree sets. However Example 4.53 suggests that regularity is a much less restrictive property for profinite rooted tree sets, as they may contain infinite regular splitting stars. An answer to the second open problem is likely to also clarify how powerful the regular profinite rooted tree sets are compared to the irregular profinite rooted tree sets.

5 Tree-like spaces

5.1 Graph-like spaces

As we have seen in Chapter 3, not every tree set, even regular, can be represented as the edge tree set of a tree. In this last chapter we find a (topological) relaxation of the notion of a (graph-theoretical) tree, to be called *tree-like spaces*, which, like trees, have regular edge tree sets, but which are just general enough that, conversely, every regular tree set can be represented as the edge tree set of a tree-like space.

The concept of graph-like spaces was first introduced in [10] by Thomassen and Vella, and further studied in [1] by Bowler, Carmesin and Christian. In [1] the authors discuss the connections between graph-like spaces and graphic matroids, which are of no interest to us here. Instead we determine when a graph-like space is tree-like, and then show that every regular tree set can be represented as the edge tree set of a tree-like space.

Graph-like spaces are limit objects of graphs that are not themselves graphs. In short they consist of the usual vertices and edges, together with a topology that allows the vertices and edges to be limits of each other. The formal definition is as follows.

Definition 5.1. [1] A *graph-like space* G is a topological space (also denoted G) together with a *vertex set* $V = V(G)$, an *edge set* $E = E(G)$ and for each $e \in E$ a continuous map $\iota_e^G : [0,1] \to G$ (the superscript may be omitted if G is clear from the context) such that:

- The underlying set of G is $V \dot\cup [(0,1) \times E]$.
- For any $x \in (0,1)$ and $e \in E$ we have $\iota_e(x) = (x, e)$.
- $\iota_e(0)$ and $\iota_e(1)$ are vertices (called the *end-vertices* of e).
- $\iota_e \restriction_{(0,1)}$ is an open map.
- For any two distinct $v, v' \in V$, there are disjoint open subsets U, U' of G partitioning $V(G)$ and with $v \in U$ and $v' \in U'$.

The *inner points* of the edge e are the elements of $(0,1) \times \{e\}$, and we abbreviate the subspace $G \backslash \{e\}$ as $G - e$.

Note that G is always Hausdorff. For an edge $e \in E(G)$ the definition of graph-like space allows $\iota_e(0) = \iota_e(1)$. We call such an edge a *loop*. In our discussions of graph-like spaces loops are irrelevant, so the reader may imagine all graph-like spaces to be loop-free.

If U, U' are disjoint open subsets of G partitioning $V(G)$ we call the set of edges with end-vertices in both U and U' a *topological cut* of G and say that the pair (U, U') *induces* that cut. The last property of graph-like spaces then says that any two vertices can be separated by a topological cut.

For reasons of cardinality arc-connectedness is not a very useful notion in graph-like spaces. Instead we work with an adapted concept of arcs. A subspace $P \subseteq G$ is a *pseudo-arc* if P is a compact connected graph-like space with a start-vertex a and an end-vertex b satisfying the following:

- For each $e \in E(P)$ the vertices a and b are separated in $P - e$.
- For any two $x, y \in V(P)$ there is an edge $e \in E$ such that x and y are separated in $P - e$.

If P contains an edge then $a \neq b$; otherwise we call P *trivial*. The graph-like space G is *pseudo-arc connected* if for all vertices $a, b \in V(G)$ there is a pseudo-arc $P \subseteq G$ with start-vertex a and end-vertex b.

The adapted notion of circles is analogous. A subspace $C \subseteq G$ of a graph-like space G is a *pseudo-circle* if it is a compact connected graph-like space with at least one edge satisfying the following:

- Removing any edge from C does not disconnect C but removing any pair does.
- Any two vertices of C can be separated in C by removing a pair of edges.

Pseudo-arcs and pseudo-circles are related as follows:

Lemma 5.2. [1] *Let G be a graph-like space, C a pseudo-circle in G and $e \in E(C)$. Then $C - e$ is a pseudo-arc in G joining the end-vertices of e.*

Conversely, let P and Q be non-trivial non-loop pseudo-arcs in G that meet precisely in their end-vertices Then $P \cup Q$ is a pseudo-circle in G.

5.2 Tree-like spaces

There are many different equivalent ways of defining the graph-theoretical trees:

Proposition 5.3. *For a graph $T = (V, E)$ the following are equivalent.*

(i) *For any two vertices $a, b \in V(T)$ there is a unique path in T from a to b;*

(ii) *T is connected but $T - e$ is not for any edge $e \in E(T)$;*

(iii) *T is connected and contains no circle.*

A graph T is a tree if it has one (and thus all) of the above properties. In some situations one of these properties is easier to work with than the others, and their equivalence is used implicitly in many places in graph theory.

The above properties can be translated into the setting of graph-like spaces to say when a graph-like space is tree-like as follows:

Definition 5.4. A compact loop-free graph-like space G is a *tree-like space* if one of the following conditions holds:

(i) For any two vertices $a, b \in V(G)$ there is a unique pseudo-arc in T from a to b;

(ii) G is connected but $G - e$ is not for any edge $e \in E(G)$;

(iii) G is connected and contains no pseudo-circle.

For graph-theoretical trees the proof of the equivalence of these definitions is easy:

Proof of Proposition 5.3. (i) $\Rightarrow$ (iii): Clearly T is connected. Suppose C is a circle in T. Pick some edge $e \in E(C)$ with end-vertices a and b. Then both aeb and $C - e$ are a–b-paths in T, contrary to the assumption that there exists only one such path.

(iii) $\Rightarrow$ (ii): Suppose $T - e$ is still connected for some $e \in E(T)$ with end-vertices a and b. Then $T - e$ contains an a–b-path P, which together with e forms a circle in T.

(ii) $\Rightarrow$ (i): Given two vertices $a, b \in T$ there is at least one a–b-path P in T as T is connected. Suppose there is another such path Q. Then some edge $e \in E(T)$ lies on exactly one of the two paths P and Q; for this edge $T - e$ is still connected, contrary to assumption. □

However, proving the analogous equivalence for tree-like spaces is much tougher:

Proposition 5.5. *For compact loop-free graph-like spaces the conditions in Definition 5.4 are equivalent.*

In the proof of Proposition 5.3 the implications (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) both used the fact that a connected graph contains a path between any two given vertices. For graphs this is simply the definition of connectedness. However for graph-like spaces it is not obvious that every topologically connected graph-like space is pseudo-arc connected.

To this end we establish the following theorem:

Theorem 10. *A compact graph-like space is connected if and only if it is pseudo-arc connected.*

Before we prove Theorem 10 we show that it indeed implies Proposition 5.5. The argument is very similar to the proof of Proposition 5.3, but one additional technical lemma is needed: if two vertices a and b of a graph G are joined by two different paths it is obvious that some edge $e \in E(G)$ lies on exactly one of the two paths, as used in the implication (ii) $\Rightarrow$ (i). However for graph-like spaces and pseudo-arcs this intuitive fact requires a surprising amount of set-up to prove (see [1]).

We forego this technical set-up and simply use the following lemma:

Lemma 5.6. [1] *Any non-trivial pseudo-arc in a graph-like space is the closure of the inner points of its edges.*

Lemma 5.6 immediately implies that if two vertices a and b of a graph-like space G are joined by two distinct pseudo-arcs P and Q then there is an edge $e \in E(G)$ which lies on exactly one of the two pseudo-arcs. In fact slightly more is true: both P and Q contain an edge that does not lie on the other pseudo-arc. For if the edge set of Q was a proper subset of the edge set of P then Q would be disconnected as the removal of any edge from P separates a and b in P.

Proof that Theorem 10 implies Proposition 5.5. (i) $\Rightarrow$ (iii): Let G be a compact loop-free graph-like space with property (i). Then G is connected. Suppose C is a pseudo-circle in G; then for any $e \in E(C)$ both e and $C - e$ define pseudo-arcs in G joining the end-vertices of e, contradicting (i).

(iii) $\Rightarrow$ (ii): Let G be a compact loop-free graph-like space with property (iii). Suppose $G - e$ is still connected for some $e \in E(G)$ with end-vertices a and b. Then by Theorem 10 $G - e$ contains a pseudo-arc P between a and b, which together with e forms a pseudo-circle by Lemma 5.2.

(ii) $\Rightarrow$ (i): Let G be a compact loop-free graph-like space with property (ii). Theorem 10 implies that G is pseudo-arc connected. For the uniqueness suppose G contains two different pseudo-arcs P and Q between two vertices a and b. Lemma 5.6 implies that there is an edge $e \in E(G)$ which lies on exactly one of the two pseudo-arcs. But then $G - e$ is still pseudo-arc connected[3] and therefore connected, a contradiction. □

Now we turn to the proof of Theorem 10. The backwards implication is clear as pseudo-arcs are connected.

For the remainder of this section let G be a compact connected graph-like space and a and b two vertices of G.

The strategy of the proof of the forward implication is as follows. Given vertices a and b which we want to connect with a pseudo-arc, first we find a minimal set L of edges which meets every a–b-cut (that is, every cut of G that separates a and b). We then want to show that the closure of these edges in G is the desired pseudo-arc. By minimality for every edge $e \in L$ there is a *signature cut*, that is, an a–b-cut for which e is the only cross-edge of L. This allows us to define a linear order on L: to compare two edges $e, f \in L$ check on which side of e's signature cut f lies. By extending this order to the points in L's closure in G we

[3]See Lemma 4.16 in[1].

can perform finite-intersection-arguments for suitable initial segments in order to prove connectedness.

We start off with a technical lemma that allows us to work with 'tidy' versions of our a–b-cuts. It also establishes that all topological cuts are finite if G is a compact graph-like space, which is important for the application of Zorn's Lemma.

Lemma 5.7. *Let C be a topological cut in G. Then there are disjoint open sets X, Y partitioning the vertices of G such that the edges in C are precisely those edges that are not completely in X or completely in Y. Furthermore, C is finite.*

Proof. Let X', Y' be two disjoint open sets inducing the topological cut C. Without loss of generality we may assume that every edge that meets exactly one of X', Y' is completely contained in that set. An edge that meets both X' and Y' cannot be partitioned by those two sets as it is connected. Consider the open covering F of G consisting of X', Y' and for each edge $e \in E(G)$ that meets both X' and Y' the set of inner points of e. No subsystem of F covers G, so by compactness F is a finite covering. Thus there are only finitely many edges meeting both X' and Y', which also implies that C is finite. For every such edge e with both end-vertices in X' we can add the inner points of e to X' and delete the entire edge from Y', and we can do the same thing for all such edges with both end-vertices in Y'. The resulting sets X, Y are still open and are as desired. □

This lemma justifies the following formal definition of an a–b-cut .
A pair (A, B) of disjoint open sets in G is an *a–b-cut* if:

(i) $a \in A$ and $b \in B$;

(ii) $V(G) \subseteq A \cup B$;

(iii) for every edge $e \in E(G)$ with both end-vertices in A we have $\mathring{e} \in A$;

(iv) for every edge $e \in E(G)$ with both end-vertices in B we have $\mathring{e} \in B$.

That is, (A, B) is a cut separating a and b which is 'clean' in the sense of Lemma 5.7. In this case the set C of edges with end-vertices in both A and B is also called an a–b-cut , and we say that C is *induced* by (A, B). The set of all a–b-cuts is denoted by $\mathcal{C}_{a,b}$. This set is non-empty: by the axioms of graph-like spaces there are open disjoint sets X, Y partitioning $V(G)$ and separating a and b, so the existence of an a–b-cut follows from Lemma 5.7.

Now we set up the application of Zorn's Lemma to obtain a minimal set of edges that meets every a–b-cut . Let

$$X := \{e \in E(G) : e \in C \text{ for some } C \in \mathcal{C}_{a,b}\}.$$

This is non-empty as there is a $C \in \mathcal{C}_{a,b}$ which is non-empty by the connectedness of G. Now let

$$\mathcal{L} := \{L \subseteq X : L \cap C \neq \emptyset \text{ for all } C \in \mathcal{C}_{a,b}\}.$$

As $X \in \mathcal{L}$, this set is non-empty as well. Order the elements of $\mathcal{L}$ by inclusion. For any descending chain $M_i \in \mathcal{L}$, $i \in I$ the set $M := \bigcap_{i \in I} M_i$ is a lower bound in $\mathcal{L}$: for each $C \in \mathcal{C}_{a,b}$ every M_i contains at least one edge of C, but as C is finite, so does M. Therefore Zorn's Lemma implies the existence of a minimal element $L \in \mathcal{L}$. We show that L is the set of edges of a pseudo-arc joining a and b.

For an edge $e \in L$ a $C \in \mathcal{C}_{a,b}$ is a *signature cut* of e if $L \cap C = \{e\}$. In that case we also call open disjoint sets (A, B) inducing C a signature cut of e. Such a cut exists for every $e \in L$ by the minimality of L.

Note that if (A, B) is a signature cut of an edge $e \in L$, then for any other $f \in L$ either $\mathring{f} \subseteq A$ or $\mathring{f} \subseteq B$.

For an edge $e \in L$ with end-vertices $x \neq y$ and a signature cut (A, B) of e we say that e *runs from* x to y if $x \in A$ and $y \in B$.

For two edges $e, f \in L$ we set $e < f$ if there is a signature cut (A, B) of e with $\mathring{f} \subseteq B$. Furthermore we set $e \leqslant e$ for all edges $e \in L$.

Before proceeding we need to check that neither the orientation of an edge $e \in L$ nor the definition of $e < f$ depends on the signature cut at hand, and that $\leqslant$ is a linear order on L. The general strategy in the following proofs is this: assume a counterexample to the claim exists. Consider the signature cuts of all edges involved, then for a contradiction find a suitable corner or union of corners of these cuts that is still an a–b-cut but contains no edge of L.

Lemma 5.8. *If $e \in L$ runs from x to y then $x \in A$ and $y \in B$ for all signature cuts (A, B) of e. Furthermore if $e < f$ for $e, f \in L$ then $\mathring{f} \subseteq B$ for all signature cuts (A, B) of e.*

Proof. Suppose there is an edge $e \in L$ with end-vertices x, y and signature cuts $(A_1, B_1), (A_2, B_2)$, for which $x \in A_1, B_2$ and $y \in A_2, B_1$. But then $(A_1 \cap A_2, B_1 \cup B_2)$ would induce an a–b-cut containing no edge of L: all edges of L apart from e have both their end-vertices either in $B_1 \cup B_2$ or in $A_1 \cap A_2$, and e has no end-vertex in $A_1 \cap A_2$. This contradicts the definition of L. Hence $x \in A$ and $y \in B$ for all signature cuts (A, B) of e.

Now suppose there are edges $e, f \in L$ and signature cuts $(A_1, B_1), (A_2, B_2)$ of e such that $f \in B_1, A_2$. Let (A_3, B_3) be a signature cut of f. If $\mathring{e} \subseteq A_3$ then $(A_1 \cup A_2 \cup A_3, B_1 \cap B_2 \cap B_3)$ induces an a–b-cut containing no edge of L. But if $\mathring{e} \subseteq B_3$ then $(A_1 \cap A_2 \cap A_3, B_1 \cup B_2 \cup B_3)$ induces an a–b-cut containing no edge of L, a contradiction. Hence if $e < f$ then $\mathring{f} \subseteq B$ for all signature cuts (A, B) of e. □

Lemma 5.9. *The relation $\leqslant$ on L is a linear order.*

Proof. It is reflexive: this is true by definition.

Every two edges of L are comparable: suppose there are two distinct edges $e, f \in L$ with respective signature cuts (A_1, B_1) and (A_2, B_2), for which $\mathring{e} \subseteq A_2$ and $\mathring{f} \subseteq A_1$. Then $(A_1 \cap A_2, B_1 \cup B_2)$ induces an a–b-cut containing no edge of L, a contradiction.

It is antisymmetric: suppose there are two distinct edges $e, f \in L$ with respective signature cuts (A_1, B_1) and (A_2, B_2), for which $\mathring{e} \subseteq B_2$ and $\mathring{f} \subseteq B_1$. Then $(A_1 \cup A_2, B_1 \cap B_2)$ induces an a–b-cut containing no edge of L, a contradiction.

It is transitive: suppose there are three distinct edges $e, f, g \in L$, $e < f$ and $f < g$, with signature cuts (A_1, B_1) of e and (A_2, B_2) of f for which $\mathring{f} \subseteq B_1$

and $\mathring{g} \subseteq B_2$ but $\mathring{g} \subseteq A_1$. Then $(A_1 \cup A_2, B_1 \cap B_2)$ is a signature cut of f (as $\mathring{e} \subseteq A_2$) with $\mathring{g} \subseteq A_1 \cup A_2$, which contradicts $f < g$. □

Finally we define the pseudo-arc that shall join a and b. Write $\overline{L}$ for

$$\overline{L} := \overline{\bigcup \{\mathring{e} \mid e \in L\}}.$$

As G is compact $\overline{L}$ is a compact subspace of G. Furthermore the removal of any edge $e \in L$ from $\overline{L}$ (that is, removal of $\mathring{e}$) separates a and b in $\overline{L}$ as any signature cut of e witnesses.

To prove that $\overline{L}$ is connected we perform finite-intersection arguments on suitable initial segments of $\overline{L}$. In order for this to be possible we first need to extend the order $\leqslant$ on L to an order $\prec$ on $\overline{L}$.

Let (A, B) be a signature cut of some $e \in L$ and $x \in \overline{L} \backslash \mathring{e}$. Then we write $x \prec e$ if $x \in A$, and $x \succ e$ if $x \in B$. For $x, y \in \overline{L}$ we write $x \preceq y$ if any of the following holds:

(i) there are edges $e, f \in L$ with $x \in \mathring{e}$, $y \in \mathring{f}$ and $e < f$;

(ii) there is an edge $e \in L$ with $x \prec e \prec y$;

(iii) there is an edge $e \in L$ with end-vertices v, w, running from v to w, such that $x, y \in \mathring{e}$ and $\iota^{-1}(x) < \iota^{-1}(y)$ in the parametrization ι of e with $\iota(0) = v$ and $\iota(1) = w$.

In addition we set $x \preceq x$ for all $x \in \overline{L}$.

As for $\leqslant$ we prove in the following lemma that $\prec$ is well-defined in the sense that $x \prec e$ implies $x \in A$ for all signature cuts (A, B) of e.

Lemma 5.10. *If $x \prec e$ for $x \in \overline{L} \backslash \mathring{e}$ and $e \in L$ then $x \in A$ for all signature cuts (A, B) of e.*

Proof. Suppose there are two signature cuts $(A_1, B_1), (A_2, B_2)$ of e with $x \in A_1, B_2$. If x is an end-vertex of e this is an immediate contradiction to Lemma 5.8. If x is not an end-vertex of e consider $D := \overline{(A_1 \cap B_2) \backslash \mathring{e}}$. This is an open set containing x, so due to $x \in \overline{L} = \overline{\{\mathring{e} : e \in L\}}$ there is an edge $f \neq e \in L$ with $D \cap \mathring{f} \neq \varnothing$. But then $\mathring{f} \subseteq D$, contradicting Lemma 5.8 as well. □

As one readily checks $\preceq$ is a partial order on $\overline{L}$. If $x, y \in \overline{L}$ are incomparable then x and y are both vertices that are not the end-vertex of any edge in L. To show that $\overline{L}$ is a pseudo-arc from a to b we need to show that any two vertices $x, y \in \overline{L}$ are separated in $\overline{L} - \mathring{e}$ for some $e \in L$. That is, we need to show that $\preceq$ is a linear order on $\overline{L}$. We shall achieve this with a finite intersection property argument for initial segments of $\overline{L}$.

Let $C \in \mathcal{C}_{a,b}$ be some a–b-cut and $L(C) := L \cap C = \{e_1, \ldots, e_n\}$ with $e_1 < \cdots < e_n$. For $k \in [n+1]$ the k-th *segment* of $\overline{L}$ with regard to C is

$$S_C(k) := \left\{x \in \overline{L} : e_{k-1} \prec x \prec e_k\right\}$$

for $k \neq 1, n+1$, and $S_C(1) := \left\{x \in \overline{L} : x \prec e_1\right\}$ as well as $S_C(n+1) := \left\{x \in \overline{L} : x \succ e_n\right\}$.

As in the analogous scenario with paths and cuts in graphs one would expect the segments of $\overline{L}$ with regard to an a–b-cut (A, B) to alternate between being contained in A or in B. The next lemma shows that this is the case, and helps locate an edge which separates two given vertices in $\overline{L}$.

Lemma 5.11. *Let $C \in \mathcal{C}_{a,b}$ be induced by (A, B) with $L(C) = \{e_1, \ldots, e_n\}$ and $e_1 < \cdots < e_n$. For $k \in [n+1]$ the following holds.*

(i) *If k is odd then $S_C(k) \subseteq A$;*

(ii) *If k is even then $S_C(k) \subseteq B$.*

In particular, if an edge $e_k \in L(C)$ has end-vertices x, y with $x \preceq y$ then e_k runs from x to y if k is odd and from y to x if k is even.

Proof. For clarity we only consider the case where k is odd; the other case follows analogously.

$k = 1$: Suppose there is an $x \in S_C(1)$ with $x \in B$. Let (A_1, B_1) be a signature cut of e_1. Then $x \in B \cap A_1$ as $x \prec e_1$. Due to $x \in \overline{L}$ there has to be an edge $f \in L$ with $\mathring{f} \cap (B \cap A_1) \neq \emptyset$. This implies $\mathring{f} \subseteq B \cap A_1$ and in particular $e_1 \neq f$. Let (A_f, B_f) be a signature cut of f. Then $(A \cap A_1 \cap A_f, B \cup B_1 \cup B_f)$ is an a–b-cut not containing any edge of L: suppose $g \in L$ is an edge with end-vertices v, w such that $v \in A \cap A_1 \cap A_f$ and $w \in B \cup B_1 \cup B_f$. Then $w \in A_1 \cap A_f$ implying $w \in B$ and thus $g \in L(C)$, but also $g < e_1$, a contradiction.

$k > 1$: Suppose there is an $x \in S_C(k)$ with $x \in B$. Let $(A_{k-1}, B_{k-1}), (A_k, B_k)$ be signature cuts of e_{k-1} and e_k respectively. Then $x \in B \cap B_{k-1} \cap A_k$ as $e_{k-1} \prec x \prec e_k$. Due to $x \in \overline{L}$ there has to be an edge $f \in L$ with $\mathring{f} \cap (B \cap B_{k-1} \cap A_k) \neq \emptyset$. This implies $\mathring{f} \subseteq B \cap B_{k-1} \cap A_k$ and in particular $f \neq e_{k-1}, e_k$. Let (A_f, B_f) be a signature cut of f. Then

$$((B_{k-1} \cap B_f) \cap (A \cup (B \cap B_k)), A_{k-1} \cup A_f \cup (B \cap A_k))$$

is an a–b-cut not containing any edge of L: suppose $g \in L$ is an edge with end-vertices v, w such that $v \in (B_{k-1} \cap B_f) \cap (A \cup (B \cap B_k))$ and $w \in A_{k-1} \cup A_f \cup (B \cap A_k)$. Then $w \in B_{k-1} \cap B_f$ and therefore $w \in B \cap A_k$, implying $v \in A_k$ and thus $v \in A$. This means $g \in L(C)$ but $e_{k-1} < g < e_k$, a contradiction. □

Lemma 5.11 indeed implies that any two vertices of $\overline{L}$ can be separated by some $e \in L$.

Lemma 5.12. *Let $v \neq w$ be two vertices in $\overline{L}$. Then there is an edge $e \in L$ which separates v and w in $\overline{L}$.*

Proof. If C is an a–b-cut with v and w on different sides, then by Lemma 5.11 v and w are in different segments, $S_C(k_v)$ and $S_C(k_w)$, say. For $k := \min\{k_v, k_w\}$ the edge $e_k \in L(C)$ separates v and w in $\overline{L}$: as $x \prec e \prec y$ for any signature cut (A, B) of e we have $x \in A$ and $y \in B$, which gives a partition of $\overline{L} \setminus \mathring{e}$ into two relatively open sets.

It is thus left to show that an a–b-cut with v and w on different sides exists. Let (A, B) be any a–b-cut and (V, W) be a v-w-cut. If v and w are on different sides of (A, B) or if (V, W) is an a–b-cut we are done. If not, then $v, w \in A$ and $a, b \in V$, say. But then $(A \cap V, B \cup W)$ is the desired cut. □

From this it follows that $\preceq$ is in fact a linear order on $\overline{L}$. Next we prove that $a \in \overline{L}$ (which is surprisingly non-obvious) by finding a minimum of $\overline{L}$ and showing that this has to be a.

Note that for any vertex $c \neq a$ there is an a–b-cut with c on the b-side: let (A, B) be an a–b-cut and (A', C) be an a–c-cut. Then $(A \cap A', B \cup C)$ is the desired cut.

Lemma 5.13. *The minimum of $\overline{L}$ with regard to $\preceq$ is a and the maximum is b. In particular $a, b \in \overline{L}$.*

Proof. We only show this for a.

If L has a minimum $m \in L$, let a' be the smaller one of its end-vertices (that is, m runs from a' to its other end-vertex). Then a' is the minimum of $\overline{L}$ by Lemma 5.12. Suppose $a \neq a'$. Let C be an a–b-cut induced by (A, B) with $a' \in B$. Then $a' \notin S_C(1)$, so $e_1 \prec a'$ implying $e_1 < m$ a contradiction to the minimality of m.

If L does not have a minimum then for $e \in L$ set

$$X_e := \overline{\bigcup \left\{ \mathring{f} : f \in L,\ f < e \right\}}.$$

Then $X_e \subseteq \overline{L}$ for all $e \in L$. Because G is compact $\overline{L}$ has the finite intersection property. Therefore

$$X := \bigcap_{e \in L} X_e \neq \varnothing.$$

For any edge $e \in L$ no inner point $x \in \mathring{e}$ of e is in X, as $x \notin X_e$. Thus X contains a vertex a'. If there were another vertex $a'' \in X$, then a' and a'' could be separated by an edge $e \in L$ by Lemma 5.12 and one of them would not be in X_e. So $X = \{a'\}$. Suppose $a \neq a'$. Let C be an a–b-cut induced by (A, B) with $a' \in B$ and let $L(C) = \{e_1, \ldots, e_n\}$ with $e_1 < \cdots < e_n$. Then $a' \notin S_C(1)$ as $a' \in B$, so $e_1 \prec a'$. But this means $a' \notin X_{e_1}$, a contradiction. □

The final property needed of $\overline{L}$ to be a pseudo-arc joining a and b is that it is connected. The proof of this is similar to the proof of Lemma 5.13.

Lemma 5.14. *The subspace $\overline{L}$ of G is connected.*

Proof. Suppose $X, Y \subseteq \overline{L}$ are two non-empty disjoint sets partitioning $\overline{L}$ which are open in the subspace topology of $\overline{L}$ with $a \in X$. As edges are connected, $\mathring{e} \subseteq X$ or $\mathring{e} \subseteq Y$ for all $e \in L$. Let $S := \{e \in L \mid \mathring{e} \subseteq Y\}$ and $\overline{S} := \overline{\{\mathring{e} : e \in S\}}$. Then S is non-empty as Y contains a point of $\overline{L}$ and thus an inner point of an edge of L.

We aim to find a minimum of $Y = \overline{S}$ with regard to $\preceq$.

If S has a minimum $m \in S$ with regard to $\leqslant$ then let y be the smaller one of its end-vertices. Then $y \in Y$ and $y \preceq z$ for all $z \in \overline{S}$.

If S does not have a minimum then for $e \in S$ set

$$R_e := \overline{\bigcup \left\{ \mathring{f} : f \in S,\ f < e \right\}}.$$

Every R_e is a non-empty closed subset of $\overline{L}$. By the finite intersection property $R := \bigcap_{e \in S} R_e$ is non-empty. For any edge $e \in S$ no inner point $x \in \mathring{e}$ of e is in R, as $y \notin R_e$. Thus R contains a vertex y. If there were another vertex

$y' \in R$, then y and y' could be separated by an edge $e \in L$ by Lemma 5.12, with $y \prec e \prec y'$, say. This edge e cannot be in S as in that case y would not be in R_e. Thus $\mathring{e} \subseteq X$. Let (A, B) be a signature cut of e. As $e < f$ for all $f \in S$ due to $e \prec y' \prec f$ we have $y \in A$ and

$$\bigcup \left\{ \mathring{f} : f \in S \right\} \subseteq B.$$

But then $A \cap \overline{L}$ witnesses that $y \notin \overline{S}$, a contradiction.

Therefore $R = \{y\}$ and y is the minimum of $\overline{S}$.

Now set

$$X' := \{x \in X : x \prec e \text{ for all } e \in S\}$$

and let $U := \{e \in L \,|\, \mathring{e} \subseteq X'\}$. By a similar argument as above X' has a maximum x. Let y be the minimum of $Y = \overline{S}$ and $e \in L$ an edge separating x and y. If $y \prec e \prec x$ then either $e \in S$ and $x \notin X'$ or $e \in U$ and $y \notin Y$. So $x \prec e \prec y$, which implies $e \in U$. But this contradicts the fact that x is the maximum of X'. □

We have succeeded in proving that $\overline{L}$ is a pseudo-arc containing a and b. This concludes our proof of Theorem 10. □

5.3 Tree-sets and tree-like spaces

Similarly to graph-theoretical trees every tree-like space gives rise to a regular tree set. In this section we show that the tree-like spaces are rich enough that one can obtain every regular tree set from them. This is in contrast to Chapter 3 where we showed that the regular tree sets coming from trees are precisely those with no chain of order type $\omega + 1$. This restriction was owed to the fact that graph-theoretical trees cannot have edges that are the limit of other edges. But tree-like spaces *can* have limit edges, so this is no longer a restriction.

For a tree-like space T we can define the *edge tree set* $\tau(T)$ in a way that is very similar to the definition of $\tau(T)$ in Chapter 3. Let

$$\vec{E}(T) := \{(\iota_e(0), \iota_e(1)) \mid e \in E(T)\} \cup \{(\iota_e(1), \iota_e(0)) \mid e \in E(T)\}$$

be the set of *oriented edges* of T. As tree-like spaces cannot contain loops every element of $\vec{E}(T)$ is a pair of two distinct vertices of T. For vertices $u, v \in V(T)$ let $P(u, v)$ be the unique pseudo-arc in T with end-vertices u and v. Then $\tau(T) := (\vec{E}(T), \leqslant, *)$ becomes a separation system by setting $(x, y)^* := (y, x)$ and $(x, y) < (v, w)$ for $(x, y), (v, w) \in \vec{E}(T)$ with $\{x, y\} \neq \{v, w\}$ whenever

$$P(y, v) \subseteq P(x, v) \subseteq P(x, w).$$

It is straightforward to check that $\tau(T)$ is a regular tree set.

For the other direction let τ be a regular tree set; we define the *tree-like space of the consistent orientations of* τ, denoted $T(\tau)$, and show that $\tau \cong \tau(T(\tau))$. Let $V = \mathcal{O}(\tau)$ be the set of consistent orientations and E the set of unoriented separations of τ. As in Chapter 3 let $O(\vec{s})$ be the unique $O \in \mathcal{O}(\vec{s})$ in which $\vec{s}$ is maximal. We define the tree-like space $T = T(\tau)$ with vertex set V and edge set E, that is with ground set $V \cup [(0, 1) \times E]$. For this we need to define the maps $\iota_e \colon [0, 1] \to T$.

Fix any orientation O' of τ. For each $\vec{e} \in O'$ let $\iota_e \colon [0, 1] \to T$ be the map

$$\iota_e(x) = \begin{cases} O(\overleftarrow{e}), & x = 0 \\ (x, e), & 0 < x < 1 \\ O(\vec{e}), & x = 1 \end{cases} .$$

So far the definition of $V(T)$ and the adjacencies in $T(\tau)$ have been analogous to the construction from Chapter 3. But to make $T(\tau)$ into a graph-like space we also need to define a topology.

For $\vec{e} \in O'$ let $E^+(\vec{e})$ be the set of all $\vec{s} \in O'$ with $\vec{e} < \vec{s}$ or $\vec{e} < \overleftarrow{s}$, and $E^-(\vec{e})$ the set of all $\vec{s} \in O'$ with $\vec{s} < \vec{e}$. For $\vec{e} \in O'$ and $r \in (0, 1)$ set

$$S(\vec{e}, r) := \mathcal{O}(\vec{e}) \cup \big((0, 1) \times E^+(\vec{e})\big) \cup ((r, 1) \times e)$$

and

$$S(\overleftarrow{e}, r) := \mathcal{O}(\overleftarrow{e}) \cup \big((0, 1) \times E^-(\vec{e})\big) \cup ((0, r) \times e)\,.$$

We define the sub-base of the topology on T as $\mathcal{S} := \big\{ S(\vec{e}, r) \mid \vec{e} \in \tau,\ r \in (0, 1) \big\}$. Note that only the notation depends on the choice of O' but the topology on T does not. It is clear that T is a graph-like space: for any two vertices $a, b \in V = \mathcal{O}(\tau)$ pick any $\vec{e}$ in the symmetric difference of a and b, viewed as orientations of τ. Then $S(\vec{e}, \frac{1}{2})$ and $S(\overleftarrow{e}, \frac{1}{2})$ are disjoint open sets partitioning V and $\{a, b\}$.

To show that T is a tree-like space first we need to show that it is compact. By the Alexander sub-base theorem from general topology it suffices to show that any open covering of sets in $\mathcal{S}$ has a finite sub-cover. Suppose that $\mathcal{C}$ is a sub-basic open cover of T with no finite sub-cover. Let $E(\mathcal{C})$ be the set of all $\vec{e} \in \tau$ such that $S(\vec{e}, x) \in \mathcal{C}$ for some $x \in (0,1)$. If $\vec{r} \leqslant \overleftarrow{s}$ for any $\vec{r}, \vec{s} \in E(\mathcal{C})$ then their corresponding sets in $\mathcal{C}$ already cover all of T, except possibly for $(0,1) \times r$ if $\vec{r} = \overleftarrow{s}$, which can be finitely covered. Thus we may assume that $\vec{r} \nleqslant \overleftarrow{s}$ for all $\vec{r}, \vec{s} \in E(\mathcal{C})$. Then the set

$$E^*(\mathcal{C}) := \{\overleftarrow{e} \mid \vec{e} \in E(\mathcal{C})\}$$

is a consistent partial orientation of τ, so by the Extension Lemma there is an $O \in \mathcal{O}(\tau) = V(T)$ with $E^*(\mathcal{C}) \subseteq O$. But $O \notin S(\vec{e}, r)$ for every $\vec{e} \in E(\mathcal{C})$ and $r \in (0,1)$, so $\mathcal{C}$ was not a cover of T. Therefore T is a compact graph-like space.

Now we verify that T is connected but $T - e$ is not for every $e \in E(T)$. The latter follows immediately from the definition of $\mathcal{S}$: for any edge $e \in E(T)$ the sets $S(\vec{e}, \frac{1}{2})$ and $S(\overleftarrow{e}, \frac{1}{2})$ define a partition of $T - e$ into non-empty disjoint open sets. To show that T is connected first note that any non-empty open set in T contains an inner point of an edge. Suppose that A, B are non-empty disjoint open sets partitioning T. For any edge $e \in E$ the image of ι_e in T is connected, hence every edge whose inner points meet A is completely contained in A, and similarly for B. Write τ_A for the set of $\vec{e} \in \tau$ with $\mathring{e} \subseteq A$, and τ_B for the set of $\vec{e} \in \tau$ with $\mathring{e} \subseteq B$. Then τ_A and τ_B partition τ and are closed under involution. Fix any $\vec{a} \in \tau_A$ and $\vec{b} \in \tau_B$ with $\vec{a} \leqslant \overleftarrow{b}$ and write $C = \{\vec{r} \in \tau \mid \vec{a} \leqslant \vec{r} \leqslant \overleftarrow{b}\}$ for the chain of elements between $\vec{a}$ and $\overleftarrow{b}$. Let C_A be a maximal initial segment of C with $C_A \subset \tau_A$ and C_B a maximal initial segment of C^* with $C_B \subset \tau_B$, where C^* is the image of C under the involution. The set $C_A \cup C_B$ is a consistent partial orientation of τ, so by the Extension Lemma there is an $O \in \mathcal{O}(\tau) = V(T)$ with $C_A \cup C_B \subseteq O$. Suppose that $O \in A$, say. Let $X \subseteq \tau$ be minimal in size with the property that

$$O \in \mathcal{X} := \bigcap_{\vec{x} \in X} S(\vec{x}, r(\vec{x})) \subseteq A$$

for suitable $r(\vec{x}) \in (0,1)$. From our assumptions it follows that such an X exists and is a finite subset of O, and the minimality implies that X is a star. Observe that $\mathring{b} \subseteq S(\vec{x}, r(\vec{x}))$ for all $\vec{x} \in X$ with $\vec{x} < \overleftarrow{b}$. As $\mathcal{X}$ does not meet B there must be a (unique) $\vec{x} \in X$ with $\vec{x} \geqslant \vec{b}$ and thus $\overleftarrow{x} \in C$. If $\vec{x} \in \tau_B$ then $\mathcal{X}$ again meets B, hence $\vec{x} \in \tau_A$. As $\vec{x} \in O$ and thus $\overleftarrow{x} \notin C_A$ there is a $\vec{t} \in \tau_B \cap O$ with $\vec{x} \leqslant \vec{t}$. But then $\mathring{t} \subset \mathcal{X}$, a contradiction. Therefore T is connected and hence a tree-like space.

Finally we prove that $\tau \cong \tau(T(\tau))$. For two vertices $u, v \in V(T) = \mathcal{O}(\tau)$ the set $C = v \backslash u$ is a chain in τ. Set

$$P(u,v) := \overline{\bigcup \{\mathring{e} \mid \vec{e} \in C\}} \subseteq T(\tau).$$

Then $P(u,v) = P(v,u)$ and $P(u,v)$ is the unique pseudo-arc in T with u and v as end-vertices[4]. Define the map $\varphi \colon \tau \to \vec{E}(T(\tau))$ as

$$\varphi(\vec{e}) := \begin{cases} (\iota_e(0), \iota_e(1)), & \vec{e} \in \iota_e(1) \\ (\iota_e(1), \iota_e(0)), & \vec{e} \in \iota_e(0) \end{cases}.$$

[4]This follows immediately if one uses the machinery established in [1], which we do not introduce here. Alternatively one can show the connectedness of $P(u,v)$ by repeating the proof that $T(\tau)$ is connected, and verifying the other properties of a pseudo-arc directly.

This is a bijection between τ and $\vec{E}(T(\tau))$ that commutes with the involution. The claim follows from Lemma 2.6 if we can show that φ is order-preserving. For this let $\vec{r}, \vec{s} \in \tau$ with $\vec{r} < \vec{s}$. Let (x, y) be the end-vertices of $r \in E(T)$ with $\vec{r} \in y$ and (v, w) the end-vertices of $s \in E(T)$ with $\vec{s} \in w$. Then

$$v \backslash y = (v \backslash x) \backslash \{\vec{r}\}$$

and

$$v \backslash x = (w \backslash x) \backslash \{\vec{s}\},$$

so $P(y, v) \subseteq P(x, v) \subseteq P(x, w)$ and hence $\varphi(\vec{r}) = (x, y) \leqslant (v, w) = \varphi(\vec{s})$.

In summary we have established the following theorem.

Theorem 11. *A tree set is isomorphic to the edge tree set of a tree-like space if and only if it is regular. Indeed, if τ is a regular tree set, then $\tau = \tau(T(\tau))$, where $T(\tau)$ is the tree-like space of the consistent orientations of τ.* □

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