HAMILTONICITY IN LOCALLY FINITE GRAPHS: TWO EXTENSIONS AND A COUNTEREXAMPLE

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Abstract. We state a sufficient condition for the square of a locally finite graph to contain a Hamilton circle, extending a result of Harary and Schwenk about finite graphs.

We also give an alternative proof of an extension to locally finite graphs of the result of Chartrand and Harary that a finite graph not containing K^4 or $K_{2,3}$ as a minor is Hamiltonian if and only if it is 2-connected. We show furthermore that, if a Hamilton circle exists in such a graph, then it is unique and spanned by the 2-contractible edges.

The third result of this paper is a construction of a graph which answers positively the question of Mohar whether regular infinite graphs with a unique Hamilton circle exist.

1. Introduction

Results about Hamilton cycles in finite graphs can be extended to locally finite graphs in the following way. For a locally finite connected graph G we consider its Freudenthal compactification $|G|$ [7, 8]. This is a topological space obtained by taking G, seen as a 1-complex, and adding the *ends* of G, which are the equivalence classes of the rays of G under the relation of being inseparable by finitely many vertices, as additional points. Extending the notion of cycles, we define *circles* [9, 10] in $|G|$ as homeomorphic images of the unit circle $S^1 \subseteq \mathbb{R}^2$ in $|G|$, and we call them *Hamilton circles* of G, if they contain all vertices of G. As a consequence of being a closed subspace of $|G|$, Hamilton circles also contain all ends of G. Following this notion we call G *Hamiltonian* if there is a Hamilton circle in $|G|$.

One of the first and probably one of the deepest results about Hamilton circles was Georgakopoulos's extension of Fleischner's theorem to locally finite graphs.

Theorem 1.1. [13] *The square of any finite 2-connected graph is Hamiltonian.*

Theorem 1.2. [14, Thm. 3] *The square of any locally finite 2-connected graph is Hamiltonian.*

Following this breakthrough, more Hamiltonicity theorems have been extended to locally finite graphs in this way [1, 4, 14, 15, 18, 19, 21].

The purpose of this paper is to extend two more Hamiltonicity results about finite graphs to locally finite ones and to construct a graph which shows that another result does not extend.

The first result we consider is a corollary of the following theorem of Harary and Schwenk. A *caterpillar* is a tree such that after deleting its leaves only a path is left. Let $S(K_{1,3})$ denote the graph obtained by taking the star with three leaves, $K_{1,3}$, and subdividing each edge once.

Theorem 1.3. [16, Thm. 1] *Let T be a finite tree with at least three vertices. Then the following statements are equivalent:*

(i) *T^2 is Hamiltonian.*
(ii) *T does not contain $S(K_{1,3})$ as a subgraph.*

(iii) *T is a caterpillar.*

Theorem 1.3 has the following obvious corollary.

Corollary 1.4. [16] *The square of any finite graph G on at least three vertices such that G contains a spanning caterpillar is Hamiltonian.*

While the proof of Corollary 1.4 is immediate, the proof of the following extension of it, which is the first result of this paper, needs more work. We call the closure $\overline{H}$ in $|G|$ of a subgraph H of G a *standard subspace* of $|G|$. Extending the notion of trees, we define *topological trees* as topologically connected standard subspaces not containing any circles. As an analogue of a path, we define an *arc* as a homeomorphic image of the unit interval $[0, 1] \subseteq \mathbb{R}$ in $|G|$. Note that for standard subspaces being topologically connected is equivalent to being arc-connected by Lemma 2.4. For our extension we adapt the notion of a caterpillar to the space $|G|$ and work with *topological caterpillars*, which are topological trees $\overline{T}$ such that $\overline{T - L}$ is an arc, where T is a forest in G and L denotes the set of vertices of degree 1 in T.

Theorem 1.5. *The square of any locally finite connected graph G on at least three vertices such that $|G|$ contains a spanning topological caterpillar is Hamiltonian.*

The other two results of this paper concern the uniqueness of Hamilton circles. The first is about finite *outerplanar graphs*. These are finite graphs that can be embedded in the plane so that all vertices lie on the boundary of a common face. Clearly, finite outerplanar graphs have a Hamilton cycle if and only if they are 2-connected. It is also easy to see that any finite 2-connected outerplanar graph has a unique Hamilton cycle, which consists precisely of the 2-*contractible edges*, i.e., those edges each of whose contraction leaves the graph 2-connected (except for the case where the graph is a K^3), as pointed out by Sysło. We summarise this with the following proposition.

Proposition 1.6. (i) *A finite outerplanar graph is Hamiltonian if and only if it is 2-connected.*

(ii) [26, Thm. 6] *Finite 2-connected outerplanar graphs have a unique Hamilton cycle, which consists precisely of the 2-contractible edges unless the graph is isomorphic to a K^3.*

Finite outerplanar graphs can also be characterised by forbidden minors, which was done by Chartrand and Harary.

Theorem 1.7. [6, Thm. 1] *A finite graph is outerplanar if and only if it contains neither a K^4 nor a $K_{2,3}$ as a minor.*[1]

In the light of Theorem 1.7 we first prove the following extension of statement (i) of Proposition 1.6 to locally finite graphs.

Theorem 1.8. *Let G be a locally finite connected graph. Then the following statements are equivalent:*

(i) *G is 2-connected and contains neither K^4 nor $K_{2,3}$ as a minor.*[1]

(ii) *$|G|$ has a Hamilton circle C and there exists an embedding of $|G|$ into a closed disk such that C is mapped onto the boundary of the disk.*

Furthermore, if statements (i) and (ii) hold, then $|G|$ has a unique Hamilton circle.

From this we then obtain the following corollary, which extends statement (ii) of Proposition 1.6.

[1] Actually these statements can be strengthened a little bit by replacing the part about not containing a K^4 as a minor by not containing it as a subgraph. This follows from Lemma 4.1.

Corollary 1.9. *The edges contained in the Hamilton circle of a locally finite 2-connected graph not containing K^4 or $K_{2,3}$ as a minor are precisely the 2-contractible edges of the graph unless the graph is isomorphic to a K^3.*

We should note here that parts of Theorem 1.8 and Corollary 1.9 are already known. Chan [5, Thm. 20 with Thm. 27] proved that a locally finite 2-connected graph not containing K^4 or $K_{2,3}$ as a minor has a Hamilton circle that contains the 2-contractible edges of the graph, but no further ones. He deduces this from other general results about 2-contractible edges in locally finite 2-connected graphs. In our proof, however, we directly construct the Hamilton circle and show its uniqueness without working with 2-contractible edges. Afterwards, we deduce Corollary 1.9.

Our third result is related to the following conjecture Sheehan made for finite graphs.

Conjecture 1.10. [25] *For every $r > 2$ there is no finite r-regular graph with a unique Hamilton cycle.*

This conjecture is still open, but some partial results have been proved [17, 28, 29]. For $r = 3$ the statement of the conjecture was verified by Smith first. This was noted in an article of Tutte [30] where the statement for $r = 3$ was published for the first time.

For infinite graphs Conjecture 1.10 is not true in this formulation. It fails already with $r = 3$. To see this consider the graph depicted in Figure 1, called the *double ladder.*

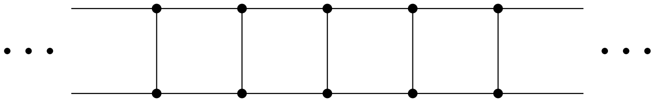

FIGURE 1. The double ladder

It is easy to check that the double ladder has a unique Hamilton circle, but all vertices have degree 3. Mohar has modified the statement of the conjecture and raised the following question. To state them we need to define two terms. For a graph G we call the equivalence classes of rays under the relation of being inseparable by finitely many vertices the *ends* of G. We define the *vertex-* or *edge-degree* of an end ω to be the supremum of the number of vertex- or edge-disjoint rays in ω, respectively. In particular, ends of a graph G can have infinite degree even if G is locally finite.

Question 1. [22] *Does an infinite graph exist that has a unique Hamilton circle and degree $r > 2$ at every vertex as well as vertex-degree r at every end?*

Our result shows in contrast to Conjecture 1.10 and its known cases that there are infinite graphs having the same degree at every vertex and end while being Hamiltonian in a unique way.

Theorem 1.11. *There exists an infinite connected graph G with a unique Hamilton circle that has degree 3 at every vertex and vertex- as well as edge-degree 3 at every end.*

So with Theorem 1.11 we answer Question 1 positively and therefore disprove the modified version of Conjecture 1.10 for infinite graphs in the way Mohar suggested by considering degrees of both vertices and ends.

The rest of this paper is structured as follows. In Section 2 we establish the notation and terminology we need for the rest of the paper. We also list some lemmas that will serve as auxiliary tools for the proofs of the main theorems. Section 3 is dedicated to Theorem 1.5 where at the beginning of that section we discuss how one can sensibly extend Corollary 1.4 and which problems arise when we try to extend Theorem 1.3 in a similar way. In Section 4 we present a proof of Theorem 1.8 and describe afterwards how a different proof of this theorem works that is copying the ideas of a proof of statement (i) of Proposition 1.6. The last section, Section 5, contains the construction of a graph witnessing Theorem 1.11.

2. Preliminaries

When we mention a graph in this paper we always mean an undirected and simple graph. For basic facts and notation about finite as well as infinite graphs we refer the reader to [7]. For a broader survey about locally finite graphs and a topological approach to them see [8].

Now we list important notions and concepts that we shall need in this paper followed by useful statements about them. In a graph G with a vertex v we denote by $\delta(v)$ the set of edges incident with v in G. Similar for a subgraph H of G or just its vertex set we denote by $\delta(H)$ the set of edges that have only one endvertex in H. Although formally different, we will not always distinguish between a cut $\delta(H)$ and the partition $(V(H), V(G) \setminus V(H))$ it is induced by. For two vertices $v, w \in V(G)$ let $d_G(v, w)$ denote the distance between v and w in G.

We call a finite graph *outerplanar* if it can be embedded in the plane such that all vertices lie on the boundary of a common face.

For a graph G and an integer $k \geq 2$ we define the *k-th power* of G as the graph obtained by taking G and adding additional edges vw for any two vertices $v, w \in V(G)$ such that $1 < d_G(v, w) \leq k$.

A tree is called a *caterpillar* if after the deletion of its leaves only a path is left.

We denote by $S(K_{1,3})$ the graph obtained by taking the star with three leaves $K_{1,3}$ and subdividing each edge once.

We call a graph *locally finite* if each vertex has finite degree.

A one-way infinite path in a graph G is called a *ray* of G, while we call a two-way infinite path in G a *double ray* of G. An equivalence relation can be defined on the set of rays of a graph G by saying that two rays are equivalent if and only if they cannot be separated by finitely many vertices in G. The equivalence classes of this relation are called the *ends* of G.

For a locally finite and connected graph G we can endow G together with its ends with a topology that yields the space $|G|$. A precise definition of $|G|$ can be found in [7, Ch. 8.5]. Let us point out here that a ray of G converges in $|G|$ to the end of G it is contained in. Another way of describing $|G|$ is to endow G with the topology of a 1-complex and then forming the Freudenthal compactification [11].

For a point set X in $|G|$, we denote its closure in $|G|$ by $\overline{X}$. We shall often write $\overline{M}$ for some M that is a set of edges or a subgraph of G. In this case we implicitly assume to first identify M with the set of points in $|G|$ which corresponds to the edges and vertices that are contained in M.

We call a subspace Z of $|G|$ *standard* if $Z = \overline{H}$ for a subgraph H of G.

A *circle* in $|G|$ is the image of a homeomorphism having the unit circle S^1 in $\mathbb{R}^2$ as domain and mapping into $|G|$. Note that all finite cycles of a locally finite graph G correspond to circles in $|G|$, but there might also be infinite subgraphs H of G such that $\overline{H}$ is a circle in $|G|$. Similar to finite graphs we call a locally finite graph G *Hamiltonian* if there exists a circle in $|G|$ which contains all vertices of G. Such circles are called *Hamilton circles* of G.

We call the image of a homeomorphism with the closed real unit interval $[0,1]$ as domain and mapping into $|G|$ an *arc* in $|G|$. Given an arc α in $|G|$, we call a point x of $|G|$ an *endpoint* of α if 0 or 1 is mapped to x by the homeomorphism defining α. Similar as for paths, we call an arc an x–y arc if x and y are the endpoints of the arc. The possibly simplest example of a nontrivial arc is a ray together with the end it converges to. However, the structure of arcs is more complicated in general and they might contain up to $2^{\aleph_0}$ many ends. We call a subspace X of $|G|$ *arc-connected* if for any two points x and y of X there is an x–y arc in X.

Using the notions of circles and arc-connectedness we now extend trees in a similar topological way. We call an arc-connected standard subspace of $|G|$ a *topological tree* if it does not contain any circle. Generalizing the definition of caterpillars, we call a topological tree $\overline{T}$ in $|G|$ a *topological caterpillar* if $\overline{T - L}$ is an arc, where T is a forest in G and L denotes the set of all leaves of T, i.e., vertices of degree 1 in T.

Now let ω be an end of a locally finite graph G. We define the *vertex-* or *edge-degree of ω in G* as the supremum of the number of vertex- or edge-disjoint rays in G, respectively, which are contained in ω. By this definition ends may have infinite vertex- or edge-degree. Similar we define the *vertex-* or *edge-degree of ω in a standard subspace X of G* as the supremum of vertex- or edge-disjoint arcs in X, respectively, that have ω as an endpoint. We should mention here that the supremum is actually an attained maximum in both definitions. Furthermore, these definitions coincide when we take $X = |G|$. The proofs of these statements are nontrivial and since it is enough for us to work with the supremum, we will not go into detail here.

We make one last definition with respect to end degrees which allows us to distinguish the parity of degrees of ends when they are infinite. The idea of this definition is due to Bruhn and Stein [3]. We call the vertex- or edge-degree of an end ω of G in a standard subspace X of $|G|$ *even* if there is a finite set $S \subseteq V(G)$ such that for every finite set $S' \subseteq V(G)$ with $S \subseteq S'$ the maximum number of vertex- or edge-disjoint arcs in X, respectively, whose endpoints are ω and some $s \in S'$ is even. Otherwise, we call the vertex- or edge-degree of ω in X, respectively, *odd*.

Next we collect some useful statements about the space $|G|$ for a locally finite graph G.

Proposition 2.1. [7, Prop. 8.5.1] *If G is a locally finite connected graph, then $|G|$ is a compact Hausdorff space.*

Having Proposition 2.1 in mind the following basic lemma helps us to work with continuous maps and verify homeomorphisms, for example when considering circles or arcs.

Lemma 2.2. *Let X be a compact space, Y be a Hausdorff space and $f : X \longrightarrow Y$ be a continuous injection. Then f^{-1} is continuous too.*

The following lemma tells us an important combinatorial property of arcs. To state the lemma more easily, let $\mathring{F}$ denote the set of inner points of edges $e \in F$ in $|G|$ for an edge set $F \subseteq E(G)$.

Lemma 2.3. [7, Lemma 8.5.3] *Let G be a locally finite connected graph and $F \subseteq E(G)$ be a cut with sides V_1 and V_2.*

(i) *If F is finite, then $\overline{V_1} \cap \overline{V_2} = \emptyset$, and there is no arc in $|G| \setminus \mathring{F}$ with one endpoint in V_1 and the other in V_2.*
(ii) *If F is infinite, then $\overline{V_1} \cap \overline{V_2} \neq \emptyset$, and there may be such an arc.*

The next lemma ensures that connectedness and arc-connectedness are equivalent for the spaces we are mostly interested in, namely standard subspaces, which are closed by definition.

Lemma 2.4. [12, Thm. 2.6] *If G is a locally finite connected graph, then every closed topologically connected subset of $|G|$ is arc-connected.*

Continuing with the idea of Lemma 2.3 of characterising important topological properties of the space $|G|$ in terms of combinatorial ones, the following lemma about arc-connected subspaces was obtained, which will be convenient for us to use in a proof later on.

Lemma 2.5. [7, Lemma 8.5.5] *If G is a locally finite connected graph, then a standard subspace of $|G|$ is topologically connected (equivalently: arc-connected) if and only if it contains an edge from every finite cut of G of which it meets both sides.*

The next theorem is actually part of a bigger one containing more equivalent statements. Since we shall need only one equivalence, we reduced it to the following formulation. For us it will be helpful to check or at least bound the degree of an end in a standard subspace just by looking at finite cuts instead of dealing with the homeomorphisms that actually define the relevant arcs.

Theorem 2.6. [8, Thm. 2.5] *Let G be a locally finite connected graph. Then the following are equivalent for $D \subseteq E(G)$:*

(i) *D meets every finite cut in an even number of edges.*
(ii) *Every vertex and every end of G has even degree or edge-degree in $\overline{D}$, respectively.*

The following lemma gives us a nice combinatorial description of circles and will be useful especially in combination with Theorem2.6 and Lemma 2.5.

Lemma 2.7. [3, Prop. 3] *Let C be a subgraph of a locally finite connected graph G. Then $\overline{C}$ is a circle if and only if $\overline{C}$ is topologically connected and every vertex or end x of G with $x \in \overline{C}$ has degree or edge-degree 2 in $\overline{C}$, respectively.*

We obtain the following corollary, which is a basic fact for finite graphs.

Corollary 2.8. *Every locally finite connected Hamiltonian graph is 2-connected.*

Proof. Let G be a locally finite connected Hamiltonian graph and suppose for a contradiction that it is not 2-connected. Fix a subgraph C of G whose closure $\overline{C}$ is a Hamilton circle of G and a cut vertex v of G. Let K_1 and K_2 be two different components of $G - v$. By Theorem 2.6 the circle $\overline{C}$ uses evenly many edges of each of the finite cuts $\delta(K_1)$ and $\delta(K_2)$. Since $\overline{C}$ is a Hamilton circle and therefore topologically connected, we get also that it uses at least two edges of each of these cuts by Lemma 2.5. This implies that v has degree at least 4 in C which contradicts Lemma 2.7. □

3. Topological caterpillars

In this section we close a gap with respect to the general question when the k-th power of a graph has a Hamilton circle. Let us begin by summarizing the results in this field. We start with finite graphs. The first result to mention is the famous theorem of Fleischner, Theorem 1.1, which deals with 2-connected graphs.

For higher powers of graphs the following theorem captures the whole situation.

Theorem 3.1. [20, 24] *The cube of any finite connected graph on at least three vertices is Hamiltonian.*

These theorems leave the question whether and when one can weaken the assumption of being 2-connected and still maintain the property of being Hamiltonian. Theorem 1.3 gives an answer to this question.

Now let us turn our attention towards locally finite infinite graphs. As mentioned in the introduction, Georgakopoulos has completely generalized Theorem 1.1 to locally finite graphs by proving Theorem 1.2. Furthermore, he also gave a complete generalization of Theorem 3.1 to locally finite graphs with the following theorem.

Theorem 3.2. [14, Thm. 5] *The cube of any locally finite connected graph on at least three vertices is Hamiltonian.*

What is left and what we do in the rest of this section is to prove lemmas about locally finite graphs covering implications similar to those in Theorem 1.3, and mainly Theorem 1.5, which extends Corollary 1.4 to locally finite graphs. Note first that Theorem 1.3 remains true if we consider locally finite infinite trees T and Hamilton circles in $|T^2|$ where the definition of a caterpillar should now include rays and double rays. Actually the same proof can be used to show this.

Corollary 1.4 is also true for locally finite graphs, but its proof is not trivial anymore. The problem is that for a spanning tree T of a locally finite connected graph G the topological spaces $|T^2|$ and $|G^2|$ might differ not only in inner points of edges but also in ends. More precisely, there might be two equivalent rays in G^2 that belong to different ends of T^2. So the Hamiltonicity of T^2 does not directly imply the one of G^2. However, for T being a spanning caterpillar of G, this problem can only occur when T contains a double ray such that all subrays belong to the same end of G. Then the same construction as in the proof for the implication from (iii) to (i) of Theorem 1.3 can be used to build a spanning double ray in T^2 which ends up being a Hamilton circle in $|G^2|$. The idea for the construction which is used for this implication is covered in Lemma 3.4.

For an infinite graph the assumption of having a spanning caterpillar is quite restrictive. Such graphs can especially have at most two ends since having three ends would imply that the spanning caterpillar must contain three disjoint rays, which is impossible because it would force the caterpillar to contain a $S(K_{1,3})$. For this reason we have defined a topological version of a caterpillar, which allows graphs with arbitrary many ends to have a spanning one and yields with Theorem 1.5 an extension of Corollary 1.4 for locally finite graphs. We recall the definition of a topological caterpillar $\overline{T}$ being a topological tree such that $\overline{T - L}$ is an arc, where T is a forest in G and L denotes the set of all leaves of T, i.e., vertices of degree 1 in T.

The following basic lemma about topological caterpillars is easy to show and so we omit its proof. It is an analogue of the equivalence of the statements (ii) and (iii) of Theorem 1.3 for topological caterpillars.

Lemma 3.3. *A topological tree $\overline{T}$ is a caterpillar if and only if T does neither contain $S(K_{1,3})$ as a subgraph nor an end of vertex-degree at least 3 in it.*

Note that we do not get a full extension of Theorem 1.3 to locally finite graphs because $|T^2|$ has a Hamilton circle if and only if T is a topological caterpillar with at most two ends, as noted above.

We continue with another basic lemma, which covers the idea of the proof that statement (iii) of Theorem 1.3 implies statement (i) of Theorem 1.3. We shall also need this in the proof of Theorem 1.5.

Lemma 3.4. *Let $\overline{T}$ be a topological caterpillar in $|G|$ for a locally finite connected graph G. Then there exists a partition of the vertices of T and a linear order $<_T$ of the partition classes such that:*

(i) *Any two different vertices belonging to the same partition class have distance 2 from each other in T.*
(ii) *For consecutive partition classes Q, R and S with $Q <_T R <_T S$ there is a unique vertex in Q that is not a leaf of T and has distance 2 to every vertex of S.*

Proof. If T has only two vertices, the statement is obvious. So we may assume that T has at least three vertices. Let L be the set of leaves of T. We know by definition that $\overline{T - L}$ is an arc A. This arc induces a linear order $<_A$ of the vertices of $V(T) - L$. Using this linear order we define the desired partition of $V(T)$. For consecutive vertices $v, w \in V(T) - L$ with $v <_A w$ we define the set $P_w = \{w\} \cup (N_T(v) \cap L)$ (cf. Figure 2). If A has a maximal element m with respect to $<_A$, we define an additional set $P^+ = N_T(m) \cap L$. Should A have a minimal element s with respect to $<_A$, we define another additional set $P^- = \{s\}$. By definition of topological caterpillars, the sets P_w together with P^+ and P^- form a partition of $V(T)$ where all vertices in a partition class have distance 2 in T. This proves part (i).

Note for statement (ii) that the linear order $<_A$ induces a linear order $<_T$ on the partition classes in the following way. For vertices $v, w \in V(T) - L$ with $v <_A w$ set $P_v <_T P_w$. If P^+ (resp. P^-) exists, set $P_v <_T P^+$ (resp. $P^- <_T P_v$) for every $v \in V(T) - L$. Now the definition of the partition classes ensures that for consecutive partition classes P_u, P_v and P_w with $P_u <_T P_v <_T P_w$ the vertex u has distance 2 in T to every vertex of P_w. For $P_u = P^-$ the same is true with the unique vertex $s \in P^-$ by definition. □

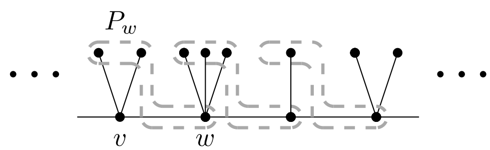

FIGURE 2. Partition classes as in Lemma 3.4.

Referring to statement (ii) of Lemma 3.4 let us call the vertex in a partition class Q that is not a leaf of T the *jumping vertex* of Q.

We still need a bit of notation and preparation work before we can prove the main theorem of this section.

Let $\overline{T}$ be a topological spanning caterpillar of a locally finite graph G. Next take a partition and a linear order $<_T$ on its classes as in Lemma 3.4. For a vertex $v \in V(G)$ let V_v be the partition class containing v. For two vertices $v, w \in V(G)$ with $V_v \leq_T V_w$ let $I_{vw} = \bigcup\{V_u \,;\, V_v \leq_T V_u \leq_T V_w\}$.

Now let $\overline{T}$ denote a topological caterpillar with only one graph-theoretical component. Let $(\mathcal{X}_1, \mathcal{X}_2)$ be a bipartition of the partition classes V_v such that consecutive classes with respect to $\leq_T$ lie not both in $\mathcal{X}_1$ or $\mathcal{X}_2$. Furthermore, let $v, w \in V(T)$ be two vertices, say with $V_v \leq_T V_w$, whose distance is even in T. We define a (v, w) *square string* in T^2 as a path in T^2 which uses only vertices of partitions that lie in the bipartition class $\mathcal{X}_i$ in which V_v and V_w lie and which contains all vertices of partition classes $V_u \in \mathcal{X}_i$ for $V_v <_T V_u <_T V_w$, but only v and w from V_v and V_w, respectively. Similarly, we define $(v, w]$, $[v, w)$ and $[v, w]$ square strings in T^2, but with the difference that they should also contain all vertices of V_w, V_v and $V_v \cup V_w$, respectively. We call the first two types of square strings that were defined *left open* and the latter ones *left closed.* The notion of being *right open* and *right closed* is analogously defined. Lemma 3.4 contains the idea of how to construct square strings.

The next lemma gives us two possibilities to decompose a graph-theoretical component of a topological caterpillar $\overline{T}$ that contains a double ray into two, possibly infinite, paths of T^2. Later on we will use these decompositions to connect the parts of all graph-theoretical components of $\overline{T}$ in a certain way such that a Hamilton circle of G^2 is formed in the end.

Lemma 3.5. *For a locally finite connected graph G, let $\overline{T}$ be a topological caterpillar in $|G|$ with only one graph-theoretical component and which contains a double ray. Furthermore, let v and w be vertices of T with $V_v \leq_T V_w$.*

- (i) *If $d_T(v, w)$ is even, then $V(T)$ can disjointly be decomposed into a v–w path and a double ray of T^2 as well as into two rays R_v and R_w of T^2 with endvertices v and w, respectively, such that $R_v \cap V_x = \emptyset$ for every $V_x >_T V_w$ and $R_w \cap V_y = \emptyset$ for every $V_y <_T V_v$.*
- (ii) *If $d_T(v, w)$ is odd, then $V(T)$ can disjointly be decomposed into two rays R_v and R_w of T^2 with endvertices v and w, respectively, such that $R_v \cap V_x = \emptyset$ for every $V_x >_T V_w$ and $R_w \cap V_y = \emptyset$ for every $V_y <_T V_v$ as well as into two rays R_v and R_w of T^2 with endvertices v and w, respectively, such that $R_v \cap V_x = \emptyset$ for every $V_x <_T V_v$ and $R_w \cap V_y = \emptyset$ for every $V_y >_T V_w$.*

Proof. We sketch the proof of statement (i). As v–w path for the first decomposition, we take a square string S_{vw} in T^2 with v and w as endvertices. Depending whether v is a jumping vertex or not we take a left open or closed square string, respectively. Depending on w we take a right closed or open square string if w is a jumping vertex or not, respectively. Since $d_T(v, w)$ is even, we can find such square strings. To construct a corresponding double ray start with a $(v^-, w^-]$ square string in T^2 where v^- and w^- denote the jumping vertices in the partition classes proceeding V_v and V_w, respectively. Using Lemma 3.4 the $(v^-, w^-]$ square string can be extend to a desired double ray containing all vertices of T that do not lie in S_{vw}.

For the second decomposition, we start for the definition of R_v with a square string S_v having v as one endvertex. For the definition of S_v we distinguish four cases. If v and w are jumping vertices, we set S_v as a path obtained by taking a $(v, w]$ square string and deleting w from it. If v is not a jumping vertex, but w is one, take a $[v, w]$ square string, delete w from it and set the remaining path as S_v. In the case that v is a jumping vertex, but w is none, S_v is defined as a path obtained from a deleting w from a (v, w) square string. In the case that neither v nor w is a jumping vertex, we take a $[v, w)$ square string, delete w from it and set the remaining path as S_v. Next we extend S_v using a square string to a path with v as one endvertex containing all vertices in partition classes V_u with $V_v <_T V_u <_T V_w$. We extend the remaining path to a ray that contains also all vertices in partition classes V_u with $V_u \leq_T V_v$, but none from partition classes V_x for $V_x >_T V_w$. The desired second ray R_w can now easily be build in $T^2 - R_v$.

The decompositions for statement (ii) are defined in a very similar way (cf. Figure 3). Therefore, we omit their definitions here. □

The following lemma is essential for connecting parts of decomposed graph-theoretical components of $\overline{T}$. Especially, here we make use of the structure of $|G|$ instead of arguing only inside of $\overline{T}$. This allows us basically to build a Hamilton circle using square strings and to "jump over" an end to avoid producing an edge-degree bigger than 2 at that end.

Lemma 3.6. *Let $\overline{T}$ be a topological spanning caterpillar of a locally finite connected graph G and $v, w \in V(G)$ where $V_v \leq_T V_w$. Then for any two vertices x, y with $V_v <_T V_x <_T V_w$ and $V_v <_T V_y <_T V_w$ there exists an x–y path in $G[I_{vw}]$.*

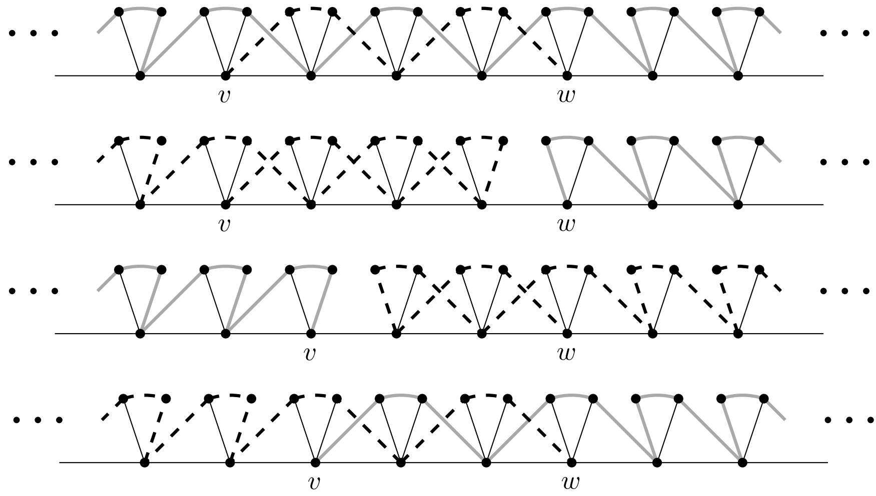

FIGURE 3. Examples for decompositions as in Lemma 3.5.

Proof. Let the vertices v, w, x and y be as in the statement of the lemma. Now suppose for a contradiction that there is no x–y path in $G[I_{vw}]$. Then we can find an empty cut D of $G[I_{vw}]$ with sides L and R such that x and y lie on different sides of it. Since $\overline{T \cap G[I_{vw}]}$ contains an x–y arc, there must exist an end $\omega \in \overline{L} \cap \overline{R}$. By definition, $\overline{T}$ contains an arc with all vertices of G that are no leaves in T, and every vertex of T is only adjacent to finitely many leaves, because G is locally finite. Therefore, we can find an open set O in $|G|$ containing ω, but no vertex of $V(G) \setminus I_{vw}$. Inside O we can find a basic open set B around ω, which contains a graph-theoretical connected subgraph with all vertices of B. Now B contains vertices of R and L as well as a path between them, which must then also exist in $G[I_{vw}]$. Such a path would have to cross D contradicting the assumption that D is an empty cut in $G[I_{vw}]$. □

To figure out which parts of which decomposed graph-theoretical components of $\overline{T}$ we can connect such that afterwards we are still able to extend this construction to a Hamilton circle of G, we shall use the next lemma. For the formulation of the lemma, we use the notion of *splits*.

Let G be a multigraph and $v \in V(G)$. Furthermore, let $E_1, E_2 \subseteq \delta(v)$ such that $E_1 \cup E_2 = \delta(v)$ where $E_i \neq \emptyset$ for $i \in \{1, 2\}$. Now we call a multigraph G' a *v-split* of G if

$$V(G') = V(G) \setminus \{v\} \cup \{v_1, v_2\}$$

with $v_1, v_2 \notin V(G)$ and

$$E(G') = E(G - v) \cup \{v_1 w \; ; \; wv \in E_1\} \cup \{v_2 u \; ; \; uv \in E_2\}.$$

We call the vertices v_1 and v_2 *replacement vertices* of v.

Lemma 3.7. *Let G be a finite Eulerian multigraph and v be a vertex of degree 4 in G. Then there exist two v-splits G_1 and G_2 of G which are Eulerian too.*

Proof. There are $\frac{1}{2} \cdot \binom{4}{2} = 3$ possible non-isomorphic v-splits of G such that v_1 and v_2 have degree 2 in the v-split. Assume that one of them, call it G', is not Eulerian. This can only be the case if G' is not connected. Let (A, B) be an empty cut of G'. Note that $G - v$ has precisely two components C_1 and C_2 since G is Eulerian and v has degree 4 in G. So C_1 and C_2 must lie in different sides of (A, B), say $C_1 \subseteq A$. Since G was connected, we get that v_1 and v_2 lie in different sides of the cut (A, B), say $v_1 \in A$. Therefore, $A = C_1 \cup \{v_1\}$ and $B = C_2 \cup \{v_2\}$. If $\delta(v) = \{vw_1, vw_2, vw_3, vw_4\}$ and $\{v_1w_1, v_1w_2\}, \{v_2w_3, v_2w_4\} \subseteq E(G')$, set G_1 and G_2 as v-splits of G such that the inclusions $\{v_1w_1, v_1w_3\}, \{v_2w_2, v_2w_4\} \subseteq E(G_1)$ and $\{v_1w_1, v_1w_4\}, \{v_2w_2, v_2w_3\} \subseteq E(G_2)$ hold. Now G_1 and G_2 are Eulerian, because every vertex has even degree in each of those multigraphs and both multigraphs are connected. To see the latter statement, note that any empty cut (X, Y) of G_i for $i \in \{1, 2\}$ would need to have C_1 and C_2 on different sides. If also v_1 and v_2 are on different sides, we would have $(A, B) = (X, Y)$, which does not define an empty cut of G_i by definition of G_i. But having v_1 and v_2 on the same side of the cut (X, Y), this would induce an empty cut in G after identifying v_1 and v_2 in G_i and yield a contradiction to the assumption that G is Eulerian and therefore especially connected. □

Now we have all tools together to proof Theorem 1.5.

Proof of Theorem 1.5. Let G be a graph as in the statement of the theorem and let $\overline{T}$ be a topological spanning caterpillar of G. We fix a partition of $V(G)$ and an order $\leq_T$ on it as in Lemma 3.4 with respect to T where V_v shall denote the partition class containing a vertex $v \in V(G)$. We may assume by Corollary 1.4 that G has infinitely many vertices. Now let us fix an enumeration of the vertices, which is possible since every locally finite connected graph is countable. We build a Hamilton circle of G^2 inductively in at most ω many steps where we have two disjoint arcs $\overline{A^i}$ and $\overline{B^i}$ in $|G^2|$ in each step $i \in \mathbb{N}$ whose endpoints are vertices of subgraphs A^i and B^i of G^2, respectively. Let a^i_ℓ and a^i_r (resp. b^i_ℓ and b^i_r) denote the endvertices of $\overline{A^i}$ (resp. $\overline{B^i}$) such that $V_{a^i_\ell} \leq_T V_{a^i_r}$ (resp. $V_{b^i_\ell} \leq_T V_{b^i_r}$). For the construction we ensure the following properties in each step $i \in \mathbb{N}$:

(1) The vertices a^i_r and b^i_r are the jumping vertices of $V_{a^i_r}$ and $V_{b^i_r}$, respectively.
(2) The partition sets $V_{a^i_\ell}$ and $V_{b^i_\ell}$ as well as $V_{a^i_r}$ and $V_{b^i_r}$ are consecutive with respect to $\leq_T$.
(3) If $V_v \cap V(A^i \cup B^i) \neq \emptyset$ holds for any vertex $v \in V(G)$, then $V_v \subseteq V(A^i \cup B^i)$.
(4) If for any vertex $v \in V(G)$ there are vertices $u, w \in V(G)$ such that $V_u, V_w \subseteq V(A^i \cup B^i)$ and $V_u \leq_T V_v \leq_T V_w$, then $V_v \subseteq V(A^i \cup B^i)$ is true.
(5) $A^i \cap A^{i+1} = A^i$ and $B^i \cap B^{i+1} = B^i$, but $V(A^{i+1} \cup B^{i+1})$ contains the least vertex with respect to the fixed vertex enumeration that was not already contained in $V(A^i \cup B^i)$.

We start the construction by picking two adjacent vertices t and t' in T that are no leaves in T. Then V_t and V'_t are consecutive with respect to $\leq_T$. Since $G^2[V_t]$ and $G^2[V_{t'}]$ are cliques by statement (i) of Lemma 3.4, we set A^1 to be a Hamilton path of $G^2[V_t]$ with endvertex t and B^1 to be one of $G^2[V_{t'}]$ with endvertex t'. This completes the first step of the construction.

Suppose we have already constructed A^n and B^n. Let $v \in V(G)$ be the least vertex with respect to the fixed vertex enumeration that is not already contained in $V(A^n \cup B^n)$. We know by our construction that either $V_v <_T V_x$ or $V_v >_T V_x$ for every vertex $x \in V(A^n \cup B^n)$. Consider the second case, since the argument for the first works analogously. Let $v' \in V(G)$ be a vertex such that $V_{v'}$ is the predecessor of V_v with respect to $\leq_T$ and $w \in V(G)$ be a vertex such that $V_w >_T V_{a^n_r}, V_{b^n_r}$ and V_w is the successor of either $V_{a^n_r}$ or $V_{b^n_r}$, say $V_{b^n_r}$. By Lemma 3.6 there exists a v'–w path P in $G[I_{b^n_r, v}]$. We may assume that $E(P) \setminus E(T)$ does not contain an edge whose endvertices lie in the same graph-theoretical component of T and that every graph-theoretical component of T is incident with at most two edges of $E(P) \setminus E(T)$. Otherwise we could use square strings to reduce the situation to the assumptions we made.

Next we inductively define a finite sequence of finite Eulerian auxiliary multigraphs $H_1, \dots, H_k$ for some $k \in \mathbb{N}$ where every vertex has either degree 2 or 4 in each of these multigraphs and we obtain H_{i+1} from H_i as a h-split for some vertex $h \in V(H_i)$ of degree 4 until we end up with a multigraph H_k that is a cycle.

As $V(H_1)$ take the set of all graph-theoretical components $T_1, \dots, T_n$ of T that are incident with an edge of $E(P) \setminus E(T)$. Two vertices T_i and T_j are adjacent if either there is an edge in $E(P) \setminus E(T)$ whose endpoints lie in T_i and T_j or there is a t_i–t_j arc $\overline{A}$ in $\overline{T}$ for a subgraph A of T and vertices $t_i \in V(T_i)$ and $t_j \in V(T_j)$ such that no endvertex of any edge of $E(P) \setminus E(T)$ lies in $V(A) \cup N_T(A)$. Since $\overline{T}$ is a topological spanning caterpillar, the multigraph H_1 is connected and by definition of P it is also Eulerian where all vertices have either degree 2 or 4.

Now suppose we have already constructed H_i and there exists a vertex $h \in V(H_i)$ with degree 4 in H_i. Since H_i is obtained from H_1 via repeated splitting operations, we know that h is incident with two edges d, e in H_i that correspond to edges d_P, e_P of $E(P) \setminus E(T)$ and with two edges f, g that correspond to arcs $\overline{A_f}$ and $\overline{A_g}$, respectively, of $\overline{T}$ for subgraphs A_f and A_g of T such that neither $V(A_f) \cup N_T(A_f)$ nor $V(A_g) \cup N_T(A_g)$ contain an endvertex of an edge of $E(P) \setminus E(T)$. Let T_j be the graph-theoretical component of T in which each of d_P and e_P has an endvertex, say w_d and w_e, respectively. Here we consider two cases:

Case 1. *The distance in T_j between w_d and w_e is even.*

In this case we define H_{i+1} as a Eulerian h-split of H_i such that the edge in H_{i+1} corresponding to d is either adjacent to the one corresponding to e or to the one corresponding to either f or g with the property that the path in T_j connecting w_d and A_f (resp. A_g) does not contain w_e. This is possible since two of the three possible non-isomorphic v-splits of H_i are Eulerian by Lemma 3.7.

Case 2. *The distance in T_j between w_d and w_e is odd.*

Here we set H_{i+1} as a Eulerian h-split of H_i such that the edge in H_{i+1} corresponding to d is not adjacent to the one corresponding to e. As in the first case, this is possible because two of the three possible non-isomorphic h-splits of H_i are Eulerian by Lemma 3.7. This completes the definition of the sequence of auxiliary multigraphs.

Now we use the last auxiliary multigraph H_k of the sequence to define the arcs $\overline{A^{n+1}}$ and $\overline{B^{n+1}}$. Note that P is a w–v' path in $G[I_{b_r^n,v}]$ where v' and w lie in the same graph-theoretical components $T_{v'}$ and T_w of T as v and b_r^n, respectively. Since we may assume that $E(P) \setminus E(T) \neq \emptyset$ holds, let $e \in E(P) \setminus E(T)$ denote the edge which contains one endvertex w_e in T_w. Then either the distance between w_e and a_r^n or between w_e and b_r^n is even, say the latter one holds. Now we first extend B^n via a $(b_r^n, w_e]$ square string in T^2 and A^n by a $(a_r^n, w_e^+]$ square string in T^2 where $V_{w_e^+}$ is the successor of V_{w_e} with respect to $\leq_T$ and w_e^+ is the jumping vertex of $V_{w_e^+}$. Then we further extend A^n using a ray to contain all vertices of partition classes V_x with $V_x >_T V_{w_e^+}$ for $x \in T_w$. This is possible by Lemma 3.4.

Next let P_1 and P_2 be the two edge-disjoint $T_{v'}$–T_w paths in H_k. Since every edge of $E(P) \setminus E(T)$ corresponds to an edge of H_k, we get that e corresponds either to P_1 or P_2, say to the former one. Therefore, we will use P_1 to obtain arcs to extend B^n and P_2 for arcs extending A^n. The way we have defined H_k via splittings ensures that for any vertex T_j of H_1 of degree 4 we have performed a T_j-split such that the partition of the edges incident with T_j into pairs of edges incident with a replacement vertex of T_j corresponds to a decomposition of T_j as in Lemma 3.5. So for every vertex of H_1 of degree 4 we take such a decomposition. For every graph-theoretical component T_m of T such that there exist two consecutive edges T_iT_j and T_jT_ℓ of P_1 or P_2 that do not correspond to edges of $E(P)\setminus E(T)$ and $V_{t_i} <_T V_{t_m} <_T V_{t_j}$ or $V_{t_j} <_T V_{t_m} <_T V_{t_\ell}$ holds for every choice of $t_i \in T_i$, $t_j \in T_j$, $t_\ell \in T_\ell$ and $t_m \in T_m$, we take a spanning double ray of T_m^2. We can find such spanning double rays using Lemma 3.4. Since $H_k = P_1 \cup P_2$ is a cycle, we can use these decompositions and double rays to extend $\overline{A^n}$ and $\overline{B^n}$ to be disjoint arcs α^n and β^n with endvertices on $T_{v'}$. With the same construction that we have used for extending A^n and B^n on T_w, we can extend α^n and β^n to have endvertices v'_j and v_j which are the jumping vertices of $V_{v'}$ and V_v, respectively, and containing all vertices of partition classes V_y for $y \in T_{v'}$ and $V_y \leq V_v$. Then we take these arcs as $\overline{A^{n+1}}$ and $\overline{B^{n+1}}$ where A^{n+1} and B^{n+1} are the corresponding subgraphs of G^2 whose closures give the arcs. By setting a_r^{n+1} and b_r^{n+1} to be v'_j and v_j, depending on which of the two arcs $\overline{A^{n+1}}$ or $\overline{B^{n+1}}$ ends in these vertices, we have guaranteed all properties from (1) to (5) for the construction.

Now the properties (3) – (5) yield not only that $\overline{A}$ and $\overline{B}$ are disjoint arcs for $A = \bigcup_{i\in\mathbb{N}} A^i$ and $B = \bigcup_{i\in\mathbb{N}} B^i$, but also that $V(G) = V(A \cup B)$. If there exists neither a maximal nor minimal partition class with respect to $\leq_T$, the union $\overline{A \cup B}$ forms a Hamilton circle of G^2 by Lemma 2.7. Should there exist a maximal partition class, say $V_{a_r^n}$ for some $n \in \mathbb{N}$ with jumping vertex a_r^n, the vertex a_r^n will also be an endvertex of $\overline{A}$. In this case we connect the endvertices a_r^n and b_r^n of $\overline{A}$ and $\overline{B}$ via an edge. Such an edge exists since $V_{a_r^n}$ and $V_{b_r^n}$ are consecutive with respect to $\leq_T$ by property (2) and a_r^n as well as b_r^n are jumping vertices by property (1). Analogously, we add an edge if there exists a minimal partition class. Therefore, we can always obtain the desired Hamilton circle of G^2. □

4. Graphs without K^4 or $K_{2,3}$ as minor

We begin this section with a small observation which allows to strengthen Theorem 1.8 a bit by forbidding subgraphs isomorphic to a K^4 instead of minors.

Lemma 4.1. *For graphs without $K_{2,3}$ as a minor it is equivalent to contain a K^4 as a minor or as a subgraph.*

Proof. One implication is clear. So suppose for a contradiction, we have a graph without a $K_{2,3}$ as a minor that does not contain K^4 as a subgraph but as a subdivision, which is equivalent to containing a K^4 as a minor since K^4 is cubic. Consider a subdivided K^4 where at least one edge e of the K^4 corresponds to a path P_e in the subdivision whose length is at least two. Let v be an interior vertex of P_e and a, b be the endvertices of P_e. Let the other two branch vertices of the subdivision of K^4 be called c and d. Now we take $\{a, b, c, d, v\}$ as branch vertex set of a subdivision of $K_{2,3}$. The vertices a and b can be joined to c and d by internally disjoint paths using the ones of the subdivision of K^4 except the path P_e. Furthermore, the vertex v can be joined to a and b using the paths vP_ea and vP_eb. So we can find a subdivision of $K_{2,3}$ in the whole graph, which contradicts our assumption. □

Before we start with the proof of Theorem 1.8 we need to prepare two structural lemmas. The first one will be very convenient to control end degrees because it bounds the size of certain separators.

Lemma 4.2. *Let G be a 2-connected graph without $K_{2,3}$ as a minor and K_0 be a connected subgraph of G. Then $|N(K_1)| = 2$ holds for every component K_1 of $G - (K_0 \cup N(K_0))$.*

Proof. Let K_0, G and K_1 be defined as in the statement of the lemma. Since G is 2-connected, we know that $|N(K_1)| \geq 2$ holds. Now suppose for a contradiction that $N(K_1) \subseteq N(K_0)$ contains three vertices, say u, v and w. Pick neighbours u_i, v_i

and w_i of u, v and w, respectively, in K_i for $i \in \{0,1\}$. Furthermore, take a finite tree T_i in K_i whose leaves are precisely u_i, v_i and w_i for $i \in \{0,1\}$. This is possible because K_0 and K_1 are connected. Now we have a contradiction since the graph H with $V(H) = \{u,v,w\} \cup V(T_0) \cup V(T_1)$ and $E(H) = \bigcup_{i=0}^{1}(\{uu_i, vv_i, ww_i\} \cup E(T_i))$ forms a subdivision of $K_{2,3}$. □

For a connected graph G with a subgraph K let G_K denote the graph which is formed by taking G and contracting all components of $G - K$ where we delete multiple edges or loops. Obviously G_K is connected if G was connected. We can push this observation a bit further towards 2-connectedness with the following lemma.

Lemma 4.3. *Let K be a connected subgraph with at least three vertices of a 2-connected graph G. Then G_K is 2-connected.*

Proof. Suppose for a contradiction that G_K is not 2-connected for some G and K as in the statement of the lemma. Since K has at least three vertices, we obtain that G_K has at least three vertices too. So there exists a cut vertex v in G_K. If v is also a vertex of G and therefore does not correspond to a contracted component of $G - K$, then v would also be a cut vertex of G, which contradicts the assumption that G is 2-connected.

Otherwise v corresponds to a contracted component of $G - K$. Since vertices of G_K that correspond to contracted components of $G - K$ are not adjacent by definition of G_K and v, as a cut vertex in G_K, must have at least one neighbour in each component of $G_K - v$, we get in particular that v separates two vertices, say x and y, of G_K that do not correspond to contracted components of $G - K$. This yields a contradiction because K is connected and therefore contains an x–y path, which still exists in G_K and contradicts the statement that v separates x and y in G_K. □

With the lemmas above we are now prepared to prove Theorem 1.8.

Proof of Theorem 1.8. First we show that (ii) implies (i). Since G is Hamiltonian, we know by Corollary 2.8 that G is 2-connected. Suppose for a contradiction that G contains K^4 or $K_{2,3}$ as a minor. Then G has a finite subgraph H which already has K^4 or $K_{2,3}$ as a minor. Now take any finite connected subgraph K_0 of G which contains H and set $K = G[V(K_0) \cup N(K_0)]$. Next let us take an embedding of $|G|$ as in statement (ii) of this theorem. It is easy to see using Lemma 4.2 that our fixed embedding of $|G|$ induces an embedding of G_K into a closed disk such that all vertices of G_K lie on the boundary of the disk. This implies that G_K is outerplanar. So G_K can neither contain K^4 nor $K_{2,3}$ as a minor by Theorem 1.7, which contradicts that H is a subgraph of G_K.

Now let us assume (i) to prove the remaining implication. We set K_0 as an arbitrary connected subgraph of G with at least three vertices. Next we define $K_{i+1} = G[V(K_i) \cup N(K_i)]$ for every $i \geq 0$. By Lemma 4.3 we know that G_{K_i} is 2-connected for each $i \geq 0$. Furthermore, G_{K_i} contains neither K^4 nor $K_{2,3}$ as a minor for every $i \geq 0$ since it would also be a minor of G contradicting our assumption. So each G_{K_i} is outerplanar by Theorem 1.7. Using statement (ii) of Proposition 1.6 we obtain that each G_{K_i} has a unique Hamilton cycle C_i and that there is an embedding σ_i of G_{K_i} into a fixed closed disk D such that C_i is mapped onto the boundary ∂D of D. Set $E_i = E(C_i) \cap E(K_i)$ for every $i \geq 1$.

Next we define an embedding of G into D and extend it to the desired embedding of $|G|$. First take $\sigma_1 \restriction_{K_1}$. Note that $\sigma_1(E_1)$ lies on the boundary ∂D of D. Because of Lemma 4.2 we can extend $\sigma_1 \restriction_{K_1}$ using $\sigma_2 \restriction_{K_2 \setminus K_1}$, maybe after rescaling it, to obtain an embedding of K_2 such that the image of E_2 lies on ∂D. Proceeding in the same way, we get an embedding σ of all of G into D by $\sigma_1 \restriction_{K_1}$ together with rescaled embeddings $\sigma_{i+1} \restriction_{K_{i+1} \setminus K_i}$ for every $i \geq 1$ such that all vertices of G are mapped to ∂D. Furthermore, we may assume that σ has the following property:

Let $(M_i)_{i \geq 1}$ be an infinite sequence of components M_i of $G - K_i$ where $M_{i+1} \subseteq M_i$. Also, let $\{u_i, w_i\}$ be the neighbourhood of M_i in G. Then it holds that $(\sigma(u_i))_{i \geq 1}$ and $(\sigma(w_i))_{i \geq 1}$ converge to a common point on ∂D. $(*)$

It remains to extend this embedding σ to an embedding $\overline{\sigma}$ of all of $|G|$ into D. First we shall extend the domain of σ to all of $|G|$. For this we need to prove the following claim.

Claim 1. *For every end ω of G there exists an infinite sequence $(M_i)_{i \geq 1}$ of components M_i of $G - K_i$ with $M_{i+1} \subseteq M_i$ such that $\bigcap_{i \geq 1} \overline{M_i} = \{\omega\}$.*

Since K_i is finite, there exists a unique component of $G - K_i$ in which all ω-rays have a tail. Set this component as M_i. It follows from the definition that ω lies in $\overline{M_i}$. Furthermore, we get that $\bigcap_{i \geq 1} \overline{M_i}$ does neither contain any vertex nor an inner point of any edge. So suppose for a contradiction that $\bigcap_{i \geq 1} \overline{M_i}$ contains another end $\omega' \neq \omega$. We know there exists a finite set S of vertices such that all tails of ω-rays lie in a different component of $G - S$ than all tails of ω'-rays. By definition of the graphs K_i we can find an index j such that $S \subseteq V(K_j)$. So ω lies in $\overline{M_j}$ and ω' in $\overline{M'_j}$ where M'_j is the component of $G - K_j$ in which all tails of ω'-rays lie. Since G is locally finite, the cut $E(M_j, K_j)$ is finite. Using Lemma 2.3 we obtain that $\overline{M_j} \cap \overline{M'_j} = \emptyset$. Therefore, $\omega' \notin \overline{M_j} \supseteq \bigcap_{i \geq 1} \overline{M_i}$. This contradiction completes the proof of the claim.

Now let us define the map $\overline{\sigma}$. For every vertex or inner point of an edge x, we set $\overline{\sigma}(x) = \sigma(x)$. For an end ω let $(M_i)_{i \geq 1}$ be the sequence of components M_i of $G - K_i$ given by Claim 1 and $\{u_i, w_i\}$ be the neighbourhood of M_i in G. Using property $(*)$ we know that $(\sigma(u_i))_{i \geq 1}$ and $(\sigma(w_i))_{i \geq 1}$ converge to a common point p_ω on ∂D. We use this to set $\overline{\sigma}(\omega) = p_\omega$. This completes the definition of $\overline{\sigma}$.

Next we prove the continuity of $\overline{\sigma}$. For every vertex or inner point of an edge x, it is easy to see that an open set around $\overline{\sigma}(x)$ in D contains $\overline{\sigma}(U)$ for some open set U around x in $|G|$ because G is locally finite and so it follows from the definition of $\overline{\sigma}$ using the embeddings σ_i. Let us check continuity for ends. Consider an open set O around $\overline{\sigma}(\omega)$ in D, where ω is an end of G. Let $(M_i)_{i \geq 1}$ be a sequence as in Claim 1 for ω and $\{u_i, w_i\}$ be the neighbourhood of M_i in G. By property $(*)$ and the definition of $\overline{\sigma}$, we get that $(\sigma(u_i))_{i \geq 1}$ and $(\sigma(w_i))_{i \geq 1}$ converge to $\overline{\sigma}(\omega)$ on ∂D. So there exists a j such that O contains $\sigma(u_i)$ and $\sigma(w_i)$ for every $i \geq j$. By the definition of $\overline{\sigma}$ and σ using the embeddings σ_i, it follows that $\overline{\sigma}(\overline{M_j}) \subsetneq O$. Since $\overline{M_j}$ together with the inner points of the edges of $E(M_j, K_j)$ is a basic open set in $|G|$ containing ω whose image under $\overline{\sigma}$ is contained in O, continuity holds for ends too.

The next step is to check that $\overline{\sigma}$ is injective. If x and y are each either a vertex or an inner point of an edge, then they already lie in some K_j. By the definition of $\overline{\sigma}$ we get that $\overline{\sigma}(x) = \overline{\sigma}(y)$ if and only if there exists a j such that x and y are mapped to the same point by the embedding of K_j defined by $\sigma_1 \restriction_{K_1}$ and $\sigma_{i+1} \restriction_{K_{i+1} \setminus K_i}$ for every i with $1 \leq i \leq j - 1$. So x and y need to be equal.

For an and ω of G, let $(M_i)_{i \geq 1}$ be a sequence of components of $G - K_i$ such that $\bigcap_{i \geq 1} \overline{M_i} = \{\omega\}$, which exists by Claim 1, and $\{u_i, w_i\}$ be the neighbourhood of M_i in G. Since G is locally finite, there exists an integer j such that y lies in K_j if it is a vertex or an inner point of an edge, or y lies in $\overline{M'_j}$ for some component $M'_j \neq M_j$ of $G - K_j$ if y is an end of G. By the definition of $\overline{\sigma}$ and property $(*)$ we get that the arc on ∂D between $\sigma(u_j)$ and $\sigma(w_j)$ into which the vertices of M_j are mapped contains also $\overline{\sigma}(\omega)$ but not y. Hence, $\overline{\sigma}(\omega) \neq \overline{\sigma}(y)$ if $\omega \neq y$. This shows the injectivity of the map $\overline{\sigma}$.

To see that $\overline{\sigma}^{-1}$ is continuous, note that $|G|$ is compact by Proposition 2.1 and D is Hausdorff. So Lemma 2.2 immediately implies that $\overline{\sigma}^{-1}$ is continuous. This completes the proof that $\overline{\sigma}$ is an embedding.

It remains to show the existence of a unique Hamilton circle of G that is mapped onto ∂D by $\overline{\sigma}$. For this we first prove that $\partial D \subseteq \mathrm{Im}(\overline{\sigma})$. This then implies that $\overline{\sigma}^{-1} \restriction \partial D$ is a homeomorphism defining a Hamilton circle of G since it contains all vertices of G. We begin by proving the following claim.

Claim 2. *For every infinite sequence $(M_i)_{i\geq 1}$ of components M_i of $G - K_i$ with $M_{i+1} \subseteq M_i$ there exists an end ω of G such that $\bigcap_{i\geq 1} \overline{M_i} = \{\omega\}$.*

Let $(M_i)_{i\geq 1}$ be any sequence as in the statement of the claim. Since for every vertex v there exists a $j \in \mathbb{N}$ such that $v \in K_j$, we get that $\bigcap_{i\geq 1} \overline{M_i}$ is either empty or contains ends of G. Using that each M_i is connected and that $M_{i+1} \subseteq M_i$, we can find a ray R such that in every M_i lies a tail of R. Therefore, $\bigcap_{i\geq 1} \overline{M_i}$ contains the end in which R lies. The argument that $\bigcap_{i\geq 1} \overline{M_i}$ contains at most one end is the same as in the proof of Claim 1. This completes the proof of Claim 2.

Suppose a point $p \in \partial D$ does not already lie in $\mathrm{Im}(\sigma)$. Then it does neither lie in $\mathrm{Im}(\sigma_1 \restriction_{K_1})$ nor in any $\mathrm{Im}(\sigma_{i+1} \restriction_{K_{i+1} \setminus K_i})$. So there exists an infinite sequence $(M_i)_{i\geq 1}$ of components M_i of $G - K_i$ with $M_{i+1} \subseteq M_i$ such that p lies in the arc A_i of ∂D between $\sigma(u_i)$ and $\sigma(w_i)$ into which the vertices of M_i are mapped, where $\{u_i, w_i\}$ denotes the neighbourhood of M_i in G. Using Claim 2 we obtain that there exists an end ω of G such that $\bigcap_{i\geq 1} \overline{M_i} = \{\omega\}$. By property $(*)$ of the map σ the sequences $(\sigma(u_i))_{i\geq 1}$ and $(\sigma(w_i))_{i\geq 1}$ converge to a common point on ∂D, which must be p since the arcs A_i are nested. Now the definition of $\overline{\sigma}$ tells us that $\overline{\sigma}(\omega) = p$. Hence $\partial D \subseteq \mathrm{Im}(\overline{\sigma})$ and G is Hamiltonian.

We finish the proof by showing the uniqueness of the Hamilton circle of G. Suppose for a contradiction that G has two subgraphs C_1 and C_2 yielding different Hamilton circles $\overline{C_1}$ and $\overline{C_2}$. Then there must be an edge $e \in E(C_1) \setminus E(C_2)$. Let $j \in \mathbb{N}$ be chosen such that $e \in E(K_j)$. By Lemma 4.2 we obtain that $G_{K_j}[E(C_1) \cap E(G_{K_j})]$ and $G_{K_j}[E(C_2) \cap E(G_{K_j})]$ are two Hamilton cycles of G_{K_j} differing in the edge e. Note that G_{K_j} is a finite 2-connected outerplanar graph. The argument for this is the same as for G_K in the proof that (ii) implies (i). This yields a contradiction since G_{K_j} has a unique Hamilton cycle by statement (ii) of Proposition 1.6. □

Next we deduce Corollary 1.9.

Proof of Corollary 1.9. Let G be a locally finite 2-connected graph not isomorphic to a K^3 and not containing K^4 or $K_{2,3}$ as a minor. Further, let C be the subgraph of G such that $\overline{C}$ is the Hamilton circle of G. First we show that each edge $e \in E(C)$ is a 2-contractible edge. Note for this that the closure of the subgraph of G/e formed by the edge set $E(C) \setminus \{e\}$ is a Hamilton circle in $|G/e|$. Hence, G/e is 2-connected by Corollary 2.8.

It remains to verify that no edge of $E(G) \setminus E(C)$ is 2-contractible. For this we consider any edge $e = uv \in E(G) \setminus E(C)$. Let K be a finite connected induced subgraph of G containing at least four vertices as well as $N(u) \cup N(v)$, which is a finite set since G is locally finite. Then we know by Lemma 4.3 and by using the locally finiteness of G again that G_K is a finite 2-connected graph not containing K^4 or $K_{2,3}$ as a minor. So by Theorem 1.7 and Proposition 1.6 we get that G_K has a unique Hamilton cycle consisting precisely of its 2-contractible edges. However, as we have seen in the proof of Theorem 1.8, $G_K[E(C) \cap E(G_K)]$ is then the unique Hamilton cycle of G_K and it does not contain e. Since G_K is outerplanar, we get that the vertex of G_K/e corresponding to the edge e is a cut vertex in G_K/e. By our choice of K containing $N(u) \cup N(v)$, we get that the vertex in G/e corresponding to the edge e is a cut vertex of G/e too. So e is not 2-contractible. □

The question arises whether one could prove the more complicated part of Theorem 1.8, the implication $(i) \Longrightarrow (ii)$, by mimicking a proof for finite graphs. To see the positive answer for this question, let us summarize the proof for finite graphs except the part about the uniqueness.

By Theorem 1.7 every finite graph without K^4 or $K_{2,3}$ as a minor can be embedded into the plane such that all vertices lie on a common face boundary. Since every face of an embedded 2-connected graph is bounded by a cycle, we obtain the desired Hamilton cycle.

So for our purpose we would first need to prove a version of Theorem 1.7 for $|G|$ where G is a locally finite graph. This can be done similar to the way we have defined the embedding for the Hamilton circle in Theorem 1.8 by decomposing the graph into finite parts using Lemma 4.2. Since none of these parts contains a K^4 or a $K_{2,3}$ as a minor, we can fix appropriate embeddings of them and stick them together. In order to obtain an embedding of $|G|$, we need furthermore to ensure that the embeddings of finite parts that converge to an end in $|G|$ also converge to a point in the plane where we can map the corresponding end to.

The second ingredient of the proof is the following lemma pointed out by Bruhn and Stein, but which is a corollary of a stronger and more general result of Richter and Thomassen [23, Prop. 3].

Lemma 4.4. [2, Cor. 21] *Let G be a locally finite 2-connected graph with an embedding $\varphi : |G| \longrightarrow S^2$. Then the face boundaries of $\varphi(|G|)$ are circles of $|G|$.*

These observations show that the proof idea for finite graphs is still applicable for locally finite graphs.

5. A cubic infinite graph with a unique Hamilton circle

This section is dedicated to Theorem 1.11 by constructing an infinite graph with a unique Hamilton circle where all vertices and ends of the graph have degree or vertex- as well as edge-degree 3, respectively. The main ingredient in our construction is a finite graph T for which we know where all *Hamilton paths*, i.e., spanning paths, proceed after deleting certain vertices. This graph has been used by Tutte [30] to construct a counterexample to Tait's conjecture [27], which said that every 3-connected cubic planar graph is Hamiltonian. The following lemma captures the facts about T we shall need. The proof is straightforward, but involves several cases that need to be distinguished.

Lemma 5.1. *There is no Hamilton path in $T - u$, but there are precisely two in $T - r$ (see Figure 4).*

Proof. As mentioned already by Tutte [30], the graph $T - u$ does not have a Hamilton path. It remains to show that $T - r$ has precisely two Hamilton paths. For this we need to check several cases, but afterwards we can precisely state the Hamilton paths. For convenience, we label each edge with a number as depicted in Figure 5 and refer to the edges just by their labels for the rest of the proof.

Obviously, the edges incident with ℓ and u would need to be in every Hamilton path of $T - r$ since these vertices have degree 1. Furthermore, the edges 2 and 3

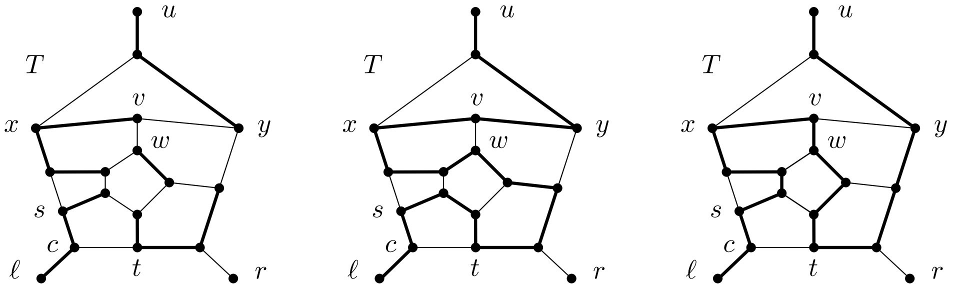

FIGURE 4. The fat edges in the most left picture are in every Hamilton path of $T - r$. The fat edges in the other two pictures mark the two Hamilton paths of $T - r$.

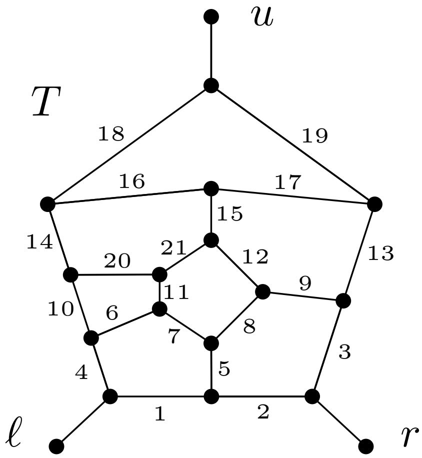

FIGURE 5. Our fixed labelling of the relevant edges of T.

need to be in every Hamilton path of $T - r$ since the vertex incident with 2 and 3 has degree 2 in $T - r$.

Claim 3. *The edge* 4 *needs to be in every Hamilton path of* $T - r$.

Suppose for a contradiction that there is a Hamilton path P in $T - r$ that does not use 4. Then it needs to contain 1. Since it also contains 2, we know $5 \notin E(P)$. This implies further that $7, 8 \in E(P)$. We can use $4 \notin E(P)$ also to deduce that $6, 10 \in E(P)$ holds. Now we get $11 \notin E(P)$ since $6, 7 \in E(P)$. This implies $20, 21 \in E(P)$. But now $14 \notin E(P)$ holds because $10, 20 \in E(P)$. From this we get then $16, 18 \in E(P)$. So 19 cannot be contained in P, which implies $13, 17 \in E(P)$. Now we arrived at a contradiction since the edges incident with l and u together with the edges of the set $\{1, 2, 3, 13, 17, 16, 18\}$ form a ℓ-u path in $T - r$ that is contained in P and needs therefore to be equal to P. Then, however, P would not be a Hamilton path $T - r$. This completes the proof of Claim 3

We immediately get from Claim 3 that 5 needs to be in every Hamilton path of $T - r$ and since 8 and 9 can not both be contained in any Hamilton path of $T - r$, because they would close a cycle together with $5, 2$ and 3, we also know that 12 needs to be in every Hamilton path of $T - r$.

Claim 4. *The edges* 14 *and* 16 *lie in every Hamilton path of* $T - r$.

Suppose for a contradiction that the claim is not true. Then there is a Hamilton path P of $T - r$ containing 18. So P cannot contain 19, which implies $13, 17 \in E(P)$. Since $3, 13 \in E(P)$, we obtain $9 \notin E(P)$, from which we follow that $8 \in P$ holds. Furthermore, 15 cannot be contained in P, because then the edges $15, 17, 13, 3, 2, 5, 8, 12$ would form a cycle in P. Therefore, 16 is an edge of P. From $5, 8 \in E(P)$ we can deduce that $7 \notin E(P)$ holds. So 6 and 11 are edges of P, which that implies $10 \notin E(P)$. Then $14, 20 \in E(P)$ needs to be true. Now, however, we have a contradiction, because P would have a vertex incident with three vertices, namely $14, 16$ and 18. This completes the proof of Claim 4

It follows from Claim 4 that 19 is contained in every Hamilton path of $T - r$. We continue with another claim.

Claim 5. *The edges* 6 *and* 20 *lie in every Hamilton path of* $T - r$.

Suppose for a contradiction that the claim is not true. Then there is a Hamilton path P of $T - r$ containing 10. This immediately implies that $6 \notin E(P)$, yielding $7, 11 \in E(P)$, and $20 \notin E(P)$, yielding $21 \in E(P)$. We note that 8 cannot be an edge of P since P would then contain a cycle spanned by the edge set $\{8, 7, 11, 21, 12\}$. Therefore, $9 \in E(P)$ must hold. Here we arrive at a contradiction, since P now contains a cycle spanned by the edge set $\{9, 3, 2, 5, 7, 11, 21, 12\}$. This completes the proof of Claim 5

Using all the observations we have made so far, we can now show that $T - r$ has precisely two Hamilton paths and state them by looking at the edge 11. Assume that 11 is contained in a Hamilton path P_1 of $T - r$. Then $7, 21 \notin E(P_1)$ follows, because $6, 20 \in E(P_1)$ holds by Claim 5. Since we could deduce from Claim 3 that $5, 12 \in E(P_1)$ holds, we get furthermore $8, 15 \in E(P_1)$. This now yields $9, 17 \notin E(P_1)$ and, therefore, $13 \in E(P_1)$. As we can see, the assumption that 11 is contained in a Hamilton path P_1 of $T - r$ is true. Also, P_1 is uniquely determined with respect to this property and consists of the fat edges in the most right picture of Figure 4.

Next assume that there is a Hamilton path P_2 of $T - r$ that does not contain the edge 11. Then 7 and 21 have to be edges of P_2. Using again that $5, 12 \in E(P_2)$ holds, we deduce $8, 15 \notin E(P_2)$. Then, however, we get $9, 17 \in E(P)$ and have already uniquely determined P_2, which corresponds to the fat edges in the middle picture of Figure 4. □

Using Lemma 5.1 we will now prove Theorem 1.11 by constructing a prescribed graph.

Proof of Theorem 1.11. We construct a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ inductively and obtain the desired one G as a limit of the sequence. We start with $G_0 = T_0^1 = T$.

Now suppose we have already constructed G_n for $n \geq 0$. Furthermore, let $\{T_n^i \,;\, 1 \leq i \leq 2^n\}$ be a specified set of disjoint subgraphs of G_n each of which each is isomorphic to T. We define G_{n+1} as follows. Take G_n and two copies T_c and T_v of T for each $T_n^i \subseteq G_n$. Then identify for every i the vertices of T_c that correspond to u, ℓ and r, respectively, with the vertices of the related $T_n^i \subseteq G_n$ corresponding to ℓ, s and t, respectively. Also identify for every i the vertices of T_v corresponding to u, ℓ and r, respectively, with the ones of the related $T_n^i \subseteq G_n$ corresponding to w, x and y, respectively. Finally, delete in each $T_n^i \subseteq G_n$ the vertices corresponding to c and v, see Figure 6. This completes the definition of G_{n+1}. It remains to fix the set of 2^{n+1} many disjoint copies of T that occur as disjoint subgraphs in G_{n+1}. For this we take the set of all copies T_c and T_v of T that we have inserted in the subgraphs T_n^i of G_n.

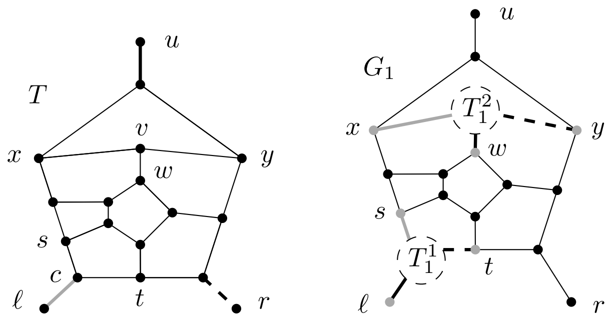

FIGURE 6. A sketch of the construction of G_1. The fat black, grey and dashed edges incident with the grey vertices in the right picture correspond to the ones in the left picture.

Using the graphs G_n we define a graph $\hat{G}$ as a limit of them. We set

$$\hat{G} = G[\hat{E}] \text{ where } \hat{E} = \left\{ e \in \bigcup_{n \in \mathbb{N}} E(G_n) \ ; \ \exists N \in \mathbb{N} : e \in \bigcap_{n \geq N} E(G_n) \right\}.$$

Note that an edge $e \in E(G_n)$ is an element of $\hat{E}$ if and only if it was not deleted during the construction of G_{n+1} as an edge incident with one of the vertices that correspond to c or v in T_n^i for some i. Finally we define G as the graph obtained from $\hat{G}$ by identifying the three vertices that correspond to u, ℓ and r of T_0^1.

Next let us verify that every vertex and every end of G has degree or vertex- as well as edge-degree, respectively, 3. Since every vertex of T except u, ℓ and r has degree 3, the construction ensures that every vertex of G has degree 3 too. In order to analyse the end degrees, we have to make some observations first. The edges of G that are adjacent to vertices corresponding to u, ℓ and r of any T_n^i define a cut $E(A_n^i, B_n^i)$ of G. Note that for any finite cut of a graph all rays in one end of the graph have tails that lie completely on one side of the cut. Therefore, the construction of G ensures that for every end ω of G there exists a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(n) \in \{1, \ldots, 2^n\}$ such that all rays in ω have tails in $B_n^{f(n)}$ for each $n \in \mathbb{N}$ and $B_n^{f(n)} \supseteq B_{n+1}^{f(n+1)}$ with $\bigcap_{n \in \mathbb{N}} B_n^{f(n)} = \emptyset$. Using that $|E(A_n^i, B_n^i)| = 3$ for every n and i, this implies that every end of G has edge-degree at most 3. Since there are three disjoint paths from $\{u, \ell, r\}$ to $\{s, \ell, t\}$ as well as to $\{x, w, y\}$ in T, we can also easily construct three disjoint rays along the cuts $E(A_n^i, B_n^i)$ that belong to an arbitrary chosen end of G. So every end of G has vertex-degree 3. In total this yields that every end of G has vertex- as well as edge-degree 3.

It remains to prove that G has precisely one Hamilton circle. We begin by stating the edge set of the subgraph C defining the Hamilton circle $\overline{C}$ of G. Let $E(C)$ consist of those edges of $E(G) \cap T_n^i$ for every n and i that correspond to the fat edges of T in the most right picture of Figure 4. Now consider any finite cut D of G. The construction of G yields that there exists an $N \in \mathbb{N}$ such that D is already a cut of the graph obtained from G_n by identifying the vertices corresponding to u, ℓ and r of $T_0^1 \subseteq G_n$ for all $n \geq N$. Using this observation we can easily see that every vertex of G has degree 2 in $\overline{C}$ and that every finite cut is met at least twice, but always in an even number of edges of C. By Lemma 2.5 we get that $\overline{C}$ is topologically and also arc-connected. Therefore, every end of G has edge-degree at least 1 and at most 3 in $\overline{C}$. Together with Theorem 2.6 this implies that every end of G has edge-degree 2 in $\overline{C}$. Hence, Lemma 2.7 tells us that $\overline{C}$ is a circle, which is Hamiltonian since it contains all vertices of G.

We finish the proof by showing that $\overline{C}$ is the unique Hamilton circle of G. Since any Hamilton circle $\overline{H}$ of G hits each cut $E(A_n^i, B_n^i)$ precisely twice, $\overline{H}$ induces a path through T that contains all vertices of T except one out of the set $\{u, \ell, r\}$. By Lemma 5.1 we know that such paths must contain the edge adjacent to u. Let us consider any T_n^i in G_n and let T_{n+1}^j be the copy of T whose vertices of degree 1 we have identified with the vertices corresponding to the neighbours of c in T_n^i during the construction of G_{n+1}. The way we have identified the vertices implies that path induced by $\overline{H}$ through T_n^i must also use the edge adjacent to ℓ since the induced path in T_{n+1}^j must use the edge adjacent to u. With a similar argument we obtain that the induced path inside T_n^i must use the edge corresponding to vw. We know from Lemma 5.1 that there is a unique Hamilton path in $T - r$ that uses the edges ℓc and vw, namely the one corresponding to the fat edges in the most right picture of Figure 4. So the edges which must be contained in every Hamilton circle are precisely those of C. □

Acknowledgement

I would like to thank Tim Rühmann for reading an early draft of this paper and giving helpful comments.

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