# Infinite Highly Connected Planar Graphs of Large Girth

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## Abstract

We construct infinite planar graphs of arbitrarily large connectivity and girth, and study their separation properties. Every finite cycle in the graphs separates them, but they corroborate Diestel's conjecture that every $k$-connected locally finite graph contains a possibly infinite cycle — see [3] — whose deletion leaves it $(k-3)$-connected.

## Introduction

For finite graphs, it is well known that large enough connectivity forces arbitrarily large complete minors [2]. For infinite graphs, this is not true: there are planar graphs of arbitrarily high finite connectivity. (Planar graphs cannot be infinitely connected, as infinite connectivity clearly forces an infinite complete topological minor.) In this paper we construct planar graphs whose girth, as well as connectivity, exceeds any given finite bound.

We present constructions of two quite different types of such planar graphs. The graphs of the first type, presented in Section 1, are one-ended, whereas the graphs of the second type, presented in Section 2, have continuum many thin ends.

By a theorem of Thomassen [2], every finite $(k+3)$-connected graph of girth at least 4 contains a cycle whose deletion leaves the graph $k$-connected. Aharoni & Thomassen [1] showed that this is not true for infinite graphs, even for locally finite ones. Their counter-example is constructed by a non-trivial recursion. Our graphs of the second type are also counter-examples, but this requires no further proof: their planarity implies at once that every cycle separates them.

Diestel [3] conjectured that Thomassen's theorem might generalise to locally finite graphs if we allow infinite cycles (as defined in [3]). We do not prove this but show that our counter-examples confirm Diestel's conjecture: each of these graphs contains an infinite cycle whose deletion reduces the connectivity by at most two.

## 1. Arbitrarily Large Connectivity and Girth I

The problem of constructing infinite planar graphs with arbitrarily large connectivity and girth was suggested by Diestel (personal communication). The following 1-ended example is due to Diestel and Stein.

For a given $k \in \mathbb{N}$, let $\Gamma^k$ be the graph constructed as follows. Begin with a k-cycle $C_1$, add a new vertex $x$, and join it to all vertices of $C_1$. Then perform $\omega$ steps of the following type: In step $i, i > 1$, substitute every edge of $C_{i-1}$ with a path of length $k-2$. Call the cycle that consists of the vertices of $C_{i-1}$ and the newly added paths $C'_{i-1}$. Then add to the graph a new cycle $C_i$ of length $k|C'_{i-1}|$ and join every vertex of $C'_{i-1}$ to $k$ consecutive vertices of $C_i$, so that consecutive vertices of $C'_{i-1}$ are joined to consecutive $k$-tuples of vertices of $C_i$ (fig 1.1).

For every vertex $v \neq x$ of $\Gamma^k$, let $l(v)$ be the unique integer such that $v \in C'_{l(v)}$, and let $l(x) = 0$. For any finite set $U \subset V(\Gamma^k)$ let $l(U) = \max_{v \in U} l(v)$.

**Theorem 1: $\Gamma^k$ is a $k$- connected 1-ended planar graph of girth $k$.**

**Proof:** $\Gamma^k$ is obviously planar. To see that it is 1-ended, note that every finite set of vertices is contained in a finite set of the form $W = \{x\} \cup C'_1 \cup \ldots \cup C'_n$ so that $\Gamma^k - W$ is connected. Thus, no finite set of vertices can seperate two rays.

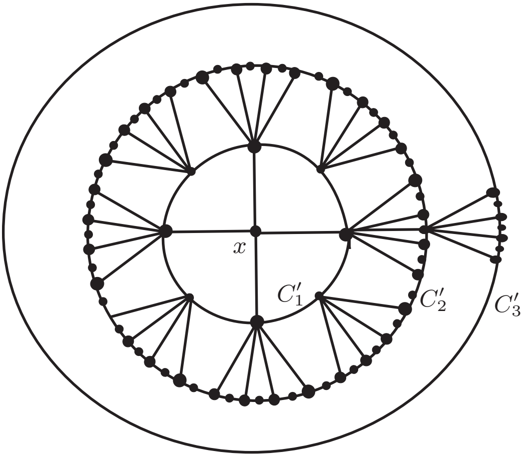


Figure 1.1: An embedding of $\Gamma^k$ for $k = 4$. The thin vertices represent vertices in $C'_i$ but not in $C_i$.

To prove that $\Gamma^k$ is $k$-connected, consider a set $B \subset V(\Gamma^k)$ with $|B| \leq k-1$ and pick any two vertices $a, b$ of $\Gamma^k - B$. We will show that there is an $a$-$b$-path in $\Gamma^k - B$. Because in $\Gamma^k$ any vertex $v$ has $k$ neighbours in $C'_{l(v)+1}$, at least one of them is not in $B$ and thus there are in $\Gamma^k - B$ paths from $a, b$ to $C'_L$, where $L = max\{\, l(B)+1, l(a), l(b)\,\}$. Because no vertex of $C'_L$ lies in $B$, this yields a path from $a$ to $b$ in $\Gamma^k - B$.

Finally, we show that $\Gamma^k$ has girth $k$. Consider a cycle $C$ of $\Gamma^k$ and let $n = l(V(C))$. Either all vertices of $C$ lie on $C'_n$, in which case it is obvious that its length is $> k$, or $C$ contains an edge $vw$, with $v \in C'_n$ and $w \in C'_{n-1}$. By construction, no vertex of $C'_n$ sends more than one edge to $C'_{n-1}$, so $C$ must contain another edge $v'w'$, with $v \neq v' \in C'_n$, $w' \in C'_{n-1}$, and a path from $v$ to $v'$ that consists of edges of $C'_n$ only. By construction, any path between distinct vertices of $C'_n$ that send edges to $C'_{n-1}$ has length at least $k-2$. This means that $C$ has length $\geq k$.
□

## 2. Arbitrarily Large Connectivity and Girth II

In this section we describe another type of infinite planar graphs, also suggested by Diestel. Their girth can be chosen arbitrarily large. We confirm Diestel's conjecture that their connectivity grows in proportion to their girth. Unlike the graphs of the previous section, these graphs have no thick end but continuum many thin ones.

Let $\Delta^k_M$, $k \in \mathbb{N}$, $M \leq \aleph_0$ be the graph constructed as follows (we define $\Delta^k_M$ as a plane graph, but we will be using the same symbol to denote the corresponding abstract graph, letting the context decide which of the two we are reffering to):

Starting with a plane cycle $\mathcal{C}$ of length $2k+1$, perform $\omega$ steps of the following type. At each step, add a new vertex in every inner face of the current plane graph (we call these vertices *centers*), and join it by independent paths of length $k$ to all the vertices on the boundary of that face that are less than $M$ steps old (we call such a path a *radius*). Clearly, this graph has girth $2k+1$. We show that:

**Theorem 2: $\Delta^k_M$ has continuum many, thin ends.**

and that:

**Theorem 3: For every $k$ there is an $M_0 \in \mathbb{N}$ such that $\Delta^k_M$ is $2k-1$-connected for $M > M_0$.**

We will call the cycles that bound a face at some step *primitive.* In what follows $C$ denotes an arbitrary primitive cycle.

Define the *father* of $C$ as the primitive cycle whose interior contains $C$ and which was constructed in the step immediately preceding the step in which $C$ was constructed. $C$ is a *child* of $C'$ if $C'$ is $C$'s father. Define the *ancestor* relation between primitive cycles as the reflexive transitive closure of the father relation.

**proof of theorem 2:** Note that because any ray can cross a given primitive cycle only finitely often, it must have a subray that lies entirely inside or entirely outside it, and a ray that lies inside a primitive cycle cannot be equivalent with one that lies outside it. Thus any infinite sequence of primitive cycles each of which is a child of the previous one, uniquely specifies an end, namely the class of rays equivalent with a ray that lies inside all cycles of the sequence. Because there are continuum many such sequences, $\Delta_M^k$ has continuum many ends. Since no infinite set of independent rays can enter any given primitive cycle, the ends are thin.

□

We now set off to prove theorem 3. In what follows we will consider $k$ fixed and prove lemmata 1-4 for a sufficiently large $M$. Our main task is to show the following lemma:

**Lemma 1: Any center can be connected to any other with (at least) $2k-1$ independent paths.**

Then, because any non-center is connected to a lot of centers with independent paths, if $M$ is large enough theorem 3 will follow easily.

In order to prove lemma 1, we will make use of lemmata 2 and 3 (to be proved later).

For any primitive cycle and any center $c$ that lies in its inside, lemma 2 allows us to draw them as shown in fig 2.1, where the heavy dots represent the young vertices. This lemma makes use of the following definitions:

The construction of any primitive cycle $C$ other than $\mathcal{C}$ is completed with the addition of $2k-1$ vertices in one step (that belong to two radii). We call them the *young vertices* of $C$. The rest of $C$ 's vertices we call its *old* vertices. Pick any two consecutive vertices of $\mathcal{C}$ and call them its old vertices, calling the rest its young vertices. Note that the young vertices of any primitive cycle form a subpath of it.

If $C$ is any (plane) cycle, let $\hat{C} = C \cup \dot{C}$ where $\dot{C}$ is the bounded component of $\mathbb{R}^2 \backslash C$.

Call the first center constructed inside $C$ the *center of* $C$, and call $C$ the *father* of its center. $C$ is an *ancestor of a center* if it is an ancestor of its father.

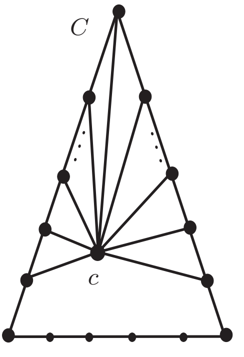


Figure 2.1: The paths described in lemma 2.

**Lemma 2 : For every center $c$ and any of its ancestors $C$, there are in $\hat{C}$ independent paths from $c$ to all of its young vertices, meeting $C$ only at their endpoints.**

Lemma 3 allows us to draw any primitive cycle as shown in fig 2.2, where the heavy dots represent the young vertices.

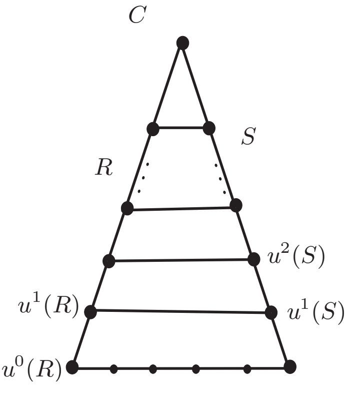


Figure 2.2: The paths described by lemma 3.

For any vertex $a$ of $C$, we denote the radius that connects the center of $C$ to $a$, if one exists, by $R_a = R_a(C)$.

If $R_a$ is any radius, then name its vertices with $u^i = u^i(R_a)$, $0 \leq i \leq k$, so that $R = u^0(= a)u^1u^2 \dots u^k$.

If $R, S$ are radii of $C$, let $C(R, S)$ denote the cycle bounded by these radii and by that path on $C$ that connects their endpoints that contains less young vertices of $C$ (fig 2.3).

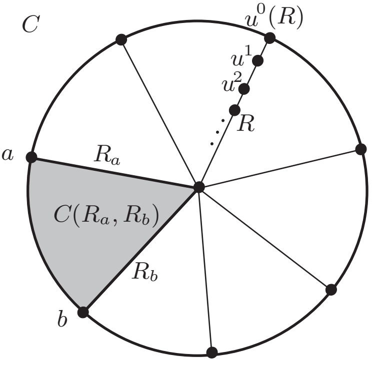


Figure 2.3

**Lemma 3 : For every primitive cycle** $C = C(R, S)$ **there are** $k-1$ **disjoint paths between** $\{\, u^i(R) | 1 \le i \le k-1 \,\}$ **and** $\{\, u^i(S) | 1 \le i \le k-1 \,\}$ **that lie in** $\hat{C}$ **meeting it only at their endpoints.**

Figures 2.4 a and 2.4 b show how we make use of fig 2.1 (lemma 2) and fig 2.2 (lemma 3) to prove lemma 1. The former figures correspond to the two possible relative positions of any two centers within a primitive cycle. In both of them, $x, y$ denote the arbitrary centers to be connected with $k-1$ independent paths.

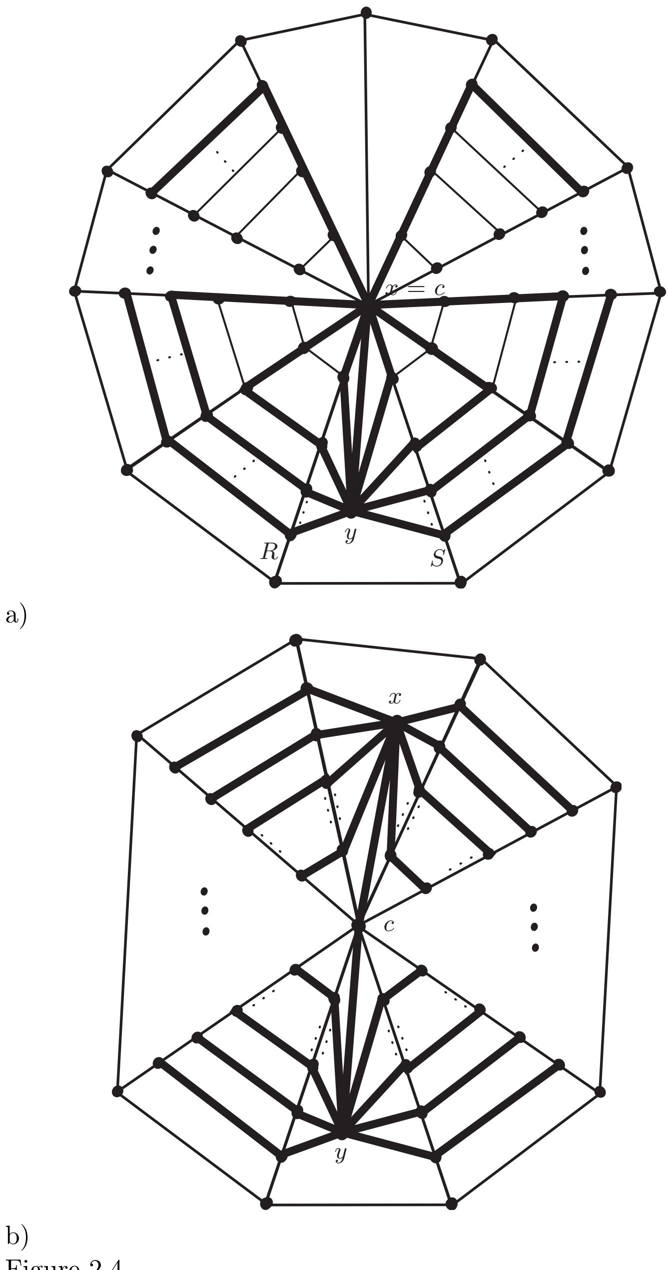


Figure 2.4

Our next lemma, will help prove both lemmata 2 and 3, and is thus the cornerstone of the whole proof.

We will say that a primitive cycle $C = C(R,S)$ satisfies $\mathcal{A}^l$, if there are $l$ disjoint paths in $\hat{C}$ from the first $l$ inner vertices of $R$ (i.e. $\{\, u^i(R) | 1 \le i \le l\,\}$) to the first $l$ vertices of $S$ (i.e. $\{\, u^i(S) | 0 \le i \le l-1\,\}$)

that meet $C$ only at their endpoints. (Note that by construction the roles of $R$ and $S$ can be interchanged.)

Similarly, we will say $C$ satisfies $\mathcal{B}^l$, if there are $l$ disjoint paths in $\hat{C}$ from the first $l$ vertices of $R$ to the first $l$ vertices of $S$ that meet $C$ only at their endpoints.

$C$ is called *young* if it does not meet any old vertices of its father.

**Lemma 4: Every young primitive cycle satisfies $\mathcal{A}^{k-1}$ and $\mathcal{B}^{k-1}$.**

**Proof:** We will perform induction.

Let $C' = C(R, S)$ be an arbitrary primitive cycle and let $a_i = u^i(R)$ and $b_i = u^i(S)$ for $0 \leq i \leq k$.

It is obvious that $C'$ satisfies $\mathcal{A}^1$ (respectively $\mathcal{B}^1$) if $M > 1$: The desired path can be constructed by joining $R_{a_1}$ to $R_{b_0}$ (respectively $R_{a_1}$ to $R_{b_1}$).

So suppose that every young primitive cycle satisfies $\mathcal{A}^m$ and $\mathcal{B}^m$ for some $m < k - 1$, and pick any young primitive cycle $C$. We will prove that $C$ satisfies $\mathcal{A}^{m+1}$ and $\mathcal{B}^{m+1}$.

In order to prove that $C$ satisfies $\mathcal{A}^{m+1}$ (respectively $\mathcal{B}^{m+1}$), join $R_{a_1}$ to $R_{b_0}$ (respectively $R_{a_1}$ to $R_{b_1}$) to get one of the desired paths. The rest of the paths will be constructed in three steps.

For the first step, for $2 \leq i \leq m + 1$ the primitive cycle $C_i = C(R_{a_i}, R_{a_{i+1}})$ is young, so by the induction hypothesis it satisfies $\mathcal{A}^m$, which means that there are in $\hat{C}_i$ disjoint paths from the first $i - 1$ vertices of $R_{a_i}$ to the first $i-1$ inner vertices of $R_{a_{i+1}}$. The union of these paths for all $i$ gives a set of disjoint paths between $\{\, a_i | 2 \leq i \leq m+1 \,\}$ and $\{\, u^i(R_{a_{m+2}}) | 1 \leq i \leq m \,\}$ (fig 2.5).

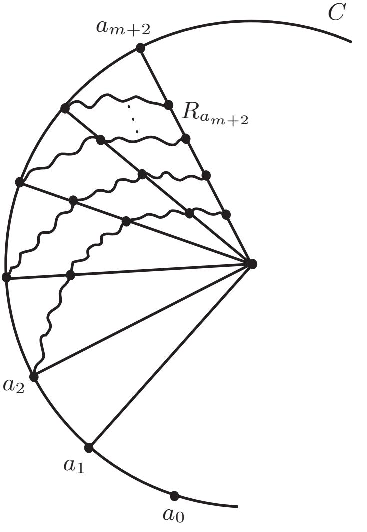

Figure 2.5: The wavy curves represent the paths of the first step.

In the second step, repeating the argumentation of the first on the children of $C$ between $R_{b_1}$ and $R_{b_{m+1}}$ (respectively between $R_{b_2}$ and $R_{b_{m+2}}$ in order to prove that $C$ satisfies $\mathcal{B}^{m+1}$), we obtain a set of disjoint paths between $\{ b_i | 1 \leq i \leq m \}$ and $\{ u^i(R_{b_{m+1}}) | 1 \leq i \leq m \}$ (respectively between $\{ b_i | 2 \leq i \leq m+1 \}$ and $\{ u^i(R_{b_{m+2}}) | 1 \leq i \leq m \}$).

For the third step, note that all children of $C$ in $\hat{C}(R_{a_{m+2}}, R_{b_{m+1}})$ (respectively $\hat{C}(R_{a_{m+2}}, R_{b_{m+2}})$) are young. By joining the paths provided by the satisfaction of $\mathcal{B}^m$ for every of those, we obtain disjoint paths between the first $m$ inner vertices of $R_{a_{m+2}}$ and the first $m$ inner vertices of $R_{b_{m+1}}$ (respectively $R_{b_{m+2}}$) (fig 2.6). Note that if $m = k-2$ then $R_{a_{m+2}} = R_{b_{m+2}}$ and no paths are constructed in this step.

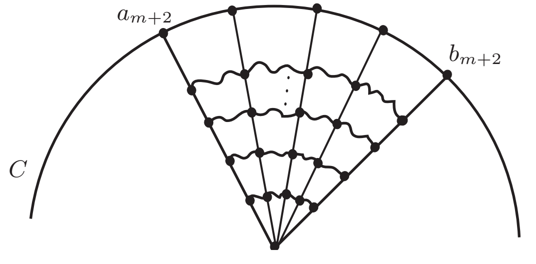


Figure 2.6: The paths of the third step.

By joining the paths constructed in all three steps we obtain the desired paths (fig 2.7 shows the paths that prove the satisfaction of $\mathcal{A}^{m+1}$). Because $C$ was chosen arbitrarily, every primitive cycle satisfies $\mathcal{A}^{m+1}$ and $\mathcal{B}^{m+1}$. This concludes the inductive step.

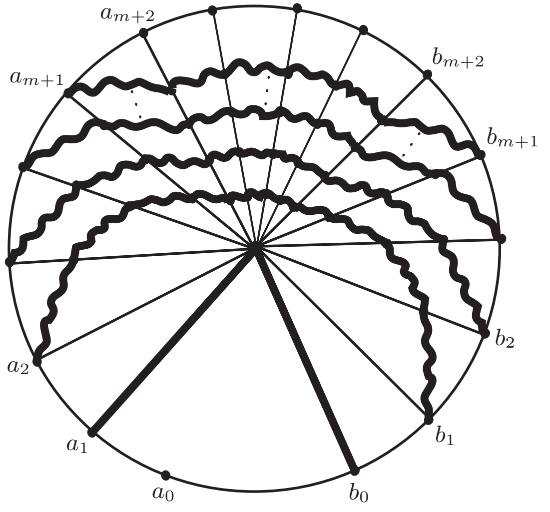

Figure 2.7: The paths obtained by joining the paths of steps 1-3 (thick).

□

Lemma 3 seems at first glance to be a special case of lemma 4 but it is not, because the latter demands that the cycle be young, so we have to prove the former separately:

**Proof of Lemma 3:** Let $a_i = u^i(R)$ and $b_i = u^i(S)$ for $1 \leq i \leq k-1$.

All children of $C$ in $\hat{C}_1 = \hat{C}(R_{a_1}, R_{a_k})$ are young, so we can imitate the first step of the proof of lemma 4 (this time using directly the fact that all primitive cycles satisfy $\mathcal{A}^{k-1}$ instead of any induction hypothesis) to construct a set of disjoint paths in $\hat{C}_1$ between $\{ a_i | 1 \leq i \leq k-1 \}$ and the first $k-1$ inner vertices of $R_{a_k}$. In the same way we can construct a set of disjoint paths in $\hat{C}_2 = \hat{C}(R_{b_1}, R_{b_k})$ between $\{ b_i | 1 \leq i \leq k-1 \}$ and the first $k-1$ inner vertices of $R_{b_k}$. Since $R_{a_k} = R_{b_k}$ joining the paths of these two sets in pairs yields the desired paths.

□

**Proof of lemma 2:** If $C$ is the father of $c$, then the radii of $C$ do the job. So all that we need to show is that for any primitive cycle $C' \neq \mathcal{C}$ there is a set of disjoint paths that connect every young vertex of $C'$ to a young vertex of its father $C''$ and lie in $\hat{C}''$ and out of $\hat{C}'$. If this is true, we can perform induction on the number of generations between $c$ and $C$ to prove the lemma.

So let $C'$ be any primitive cycle and $C''$ its father. Let $R_a, R_b$ be the radii of $C''$ that delimit $C'$, where $a, b$ are vertices of $C''$ (and $C'$).

Let $a_1 a_2 \ldots a_k (= b_k) b_{k-1} \ldots b_1$ be the subpath of $C''$ that consists of its young vertices with $a_1$ nearest (possibly equal) to $a$ and $b_1$ nearest to $b$.

Applying lemma 3 to all children of $C''$ except $C'$ and joining the resulting paths, we obtain a set of disjoint paths $\{ P_i | 1 \leq i \leq k-1 \}$, where $P_i$ connects $u^i(R_a)$ to $u^i(R_b)$. Then the paths $u^i(R_a) P_i u^i(R_{a_i}) R_{a_i} a_i, 1 \leq i \leq k-1$ and $u^i(R_b) P_i u^i(R_{b_i}) R_{b_i} b_i, 1 \leq i \leq k-1$ are disjoint. Adding the path $R_{a_k}$, which is also disjoint to the former, we have the desired set of paths (fig 2.8).

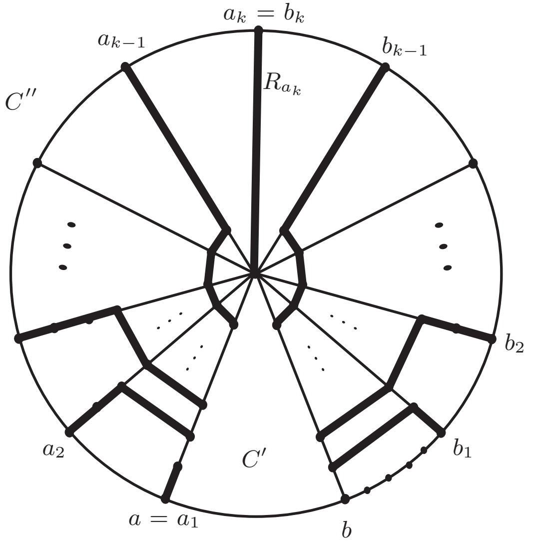


Figure 2.8: The paths between the young vertices of $C''$ and $C'$ (thick). In the figure $a$ happens to coincide with $a_1$. The thin vertices next to $b$ represent old vertices of $C''$.

□

We now have all what we need to prove lemma 1.

**Proof of Lemma 1:** Let $x, y$ be any two centers, and let $C$ be their last common ancestor and $c$ its center. Suppose without loss of generality that $y \neq c$. There are two cases:

1. $x = c$ (fig 2.4 a).

Let $C' = C(R, S)$ be the child of $C$ in whose inside $y$ lies. Lemma 2 gives us for each young vertex $v$ of $C'$ an independent $y - v$-path $P_v$ in $\hat{C}'$. Our plan is to find a $v - x$ path $P'_v$ outside $C'$, so that for $v' \neq v$ $P'_v$ and $P'_{v'}$ are independent. Joining these paths to the aforementioned in pairs, we obtain the $2k - 1$ paths we are looking for.

One of the young vertices of $C'$ coincides with $x$, so for this one we simply set $P'_v = x$.

Applying lemma 3 to all children of $C$ except $C'$ and joining the resulting paths, we obtain a set of disjoint paths $\{ P_i | 1 \leq i \leq k - 1 \}$, where $P_i$ connects $u^i(R)$ to $u^i(S)$.

For $v = u^i(R)$ (respectively $u^i(S)$) $P'_v$ is constructed as follows: Travel $k - 1 - i$ steps along $P_i$, one step being a subpath between two consecutive radii of $C$, and then use the radius of $C$ on which you landed to reach $x$. (fig 2.4 a) Note that $C$ has at least $2k-1$ radii, so a path of this form that begins at a $R$ vertex cannot meet one that begins at a $S$ vertex.

2. $x \neq c$ (fig 2.4 b).

Let $C_x$ (respectively $C_y$) be the child of $C$ in whose inside $x$ (respectively $y$) lies. $C_x$ cannot coincide with $C_y$, because if this was the case we would have chosen $C_x$ rather than $C$ as the first common ancestor of $x, y$.

Again making use of lemma 2 we get inside $C_x$ a set of independent paths from $x$ to all young vertices of $C_x$, and similarly for $y$ and $C_y$. One path from each of those sets ends at $c$, so joining these two we obtain our first $x-y$-path. Apply lemma 3 to all children of $C$ except $C_x$ and $C_y$. Joining the resulting paths and the former yields the rest of the desired paths.

□

Theorem 3 now follows easily:

**Proof of Theorem 3:** Pick $M_0$ so large that at least $2k-1$ radii begin at any non-center of $\Delta^k_{M_0}$ and lemma 1 holds.

Suppose there is a set $B$ of at most $2k-2$ vertices that disconnects $\Delta^k_M$. By Lemma 1, all centers of $\Delta^k_M - B$ lie in the same component $K$. Pick any vertex $v$ from some component $K' \neq K$. This must be a non-center, and by the choice of $M$ at least $2k-1$ radii begin at it. Since these radii are independent paths from $v$ to a center, at least one center is still connected to $v$ in $\Delta^k_M - B$, contradicting the fact that $K' \neq K$.

□

## 3. Separation Properties

Let $\dot{\Delta}^k_M$ be the graph obtained from $\Delta^k_M$ by copying the inner face of $\mathcal{C}$ on its outer. Then Theorem 3 transfers verbatim to $\dot{\Delta}^k_M$:

**Theorem 3': For every $k$ there is an $M_0 \in \mathbb{N}$ such that $\dot{\Delta}^k_M$ is $2k-1$-connected for $M > M_0$.**

**Proof:** Set $M_0$ equal to the respective value of theorem 3, and suppose there is a set $B$ of $\leq 2k-2$ vertices that disconnects $\dot{\Delta}^k_M$. By theorem 3, there is a component $K$ that contains all vertices in $\hat{\mathcal{C}}$ not it $B$. As $|V(\mathcal{C})| > |V(B)|$, $K$ meets $\mathcal{C}$ and again by theorem 3 contains all vertices outside $\mathcal{C}$ and not in $B$ as well and thus $K = \dot{\Delta}^k_M - B$, a contradiction.

□

$\dot{\Delta}^k_M$ has yet another interesting property. By a theorem of Thomassen [2], every $(k+3)$-connected finite graph of girth at least 4 contains a cycle after whose deletion the resulting graph is still $k$-connected. This is not true for infinite graphs, even locally finite ones. Aharoni & Thomassen constructed a locally finite counter-example [1]. Their graph is constructed by a fairly complicated recursion, where in each step they attach a copy of some fixed graph to all the cycles in the graph of the previous step. It is easy to see that $\dot{\Delta}^k_M$ is also such a counter-example for $k \geqslant 3$ and $M$ large enough but finite: because no cycle of this graph bounds a face, the deletion of every cycle separates the graph.

Diestel & Kühn [4,5] have suggested a topological generalization of finite cycles for infinite graphs called *circles.* These are defined as the homeomorphic images of $S^1$ in the graph's — seen as a 1-complex — Freudenthal compactification $|G|$ (see [3] for details; we do not need more here). By replacing cycles with these circles it has been possible to extend to infinite graphs some standard results about finite graphs, that would otherwise fail. In this context, Diestel [3] poses the following question in an attempt to extend Thomassen's theorem:

**Problem 1: If $G$ is $(k+3)$-connected, does $|G|$ contain a circle $C$ such that $G-C$ is $k$-connected?.**

We want to show that the answer to this question is positive for the graphs that we have just seen to be counter-examples to the naive extension of Thomassen's theorem.

An example of a circle is a double ray whose rays are disjoint and belong to the same end, together with this end. Let $G = \dot{\Delta}^k_M$, where $M$ is finite but large enough to guarantee that $G$ is $(2k-1)$-connected. We will find such a circle $C$ in $G$, and prove that $G-C$ is $(2k-4)$-connected.

We will define $C$ recursively. Pick an edge $w_1v_1$ of $\mathcal{C}$ and let $C_1$ be the child of $\mathcal{C}$ that contains $w_1, v_1$. For the recursive step, suppose that we have already defined a path $w_n \dots w_2w_1v_1v_2 \dots v_n$ that lies on some primitive cycle $C_n$ and for any $i \in \{1, \dots, n-1\}$, $v_i, w_i$ are at least $M$ steps older than $v_n$ and $w_n$. Because by construction no radius reaches $v_i, w_i$ for $1 < i < n$ after the appearance of $w_n, v_n$, any primitive cycle that contains such a path, has a child that also contains the path. So let $C_{n+1}$ be the most distant descendant of $C_n$ that has young vertices that send edges to $w_n$ and $v_n$, and name these vertices $w_{n+1}$ and $v_{n+1}$ respectively (note that this guarantees that $w_{n+1}, v_{n+1}$ are at least $M$ steps younger than the other $w_i, v_i$). Repeating ad infinitum we define a double ray $D = \ldots w_2 w_1 v_1 v_2 \ldots$. Because the young vertices of the $C_i$ form disjoint paths between its two rays, the latter belong to the same end. Let $C$ be this double ray together with that end.

**Theorem 4:** $G - C$ **is** $(2k-3)$**-connected.**

**Proof:** Pick two vertices of $G - C$ arbitrarily, and let $P$ be a set of $2k - 1$ independent paths between them in $G$ ($P$ exists by theorem 3'). We will show that $C$ cannot meet more that two elements of $P$, and thus that there are at least $(2k-3)$ independent paths between any two vertices of $G - C$.

Pick a $j \in \mathbb{N}$ such that $C_j$ (as described in the definition of $C$) was constructed later than all (finitely many) vertices in $\hat{\mathcal{C}}$ that lie on a path in $P$. Then, as no primitive cycle contains vertices constructed before it, no such path meets $\hat{C}_j - C_j$. Since the Graph is plane, these paths are curves that meet only at their endpoints, and so $\hat{C}_j$ can meet at most two of them.

Because $C_{i+1}$ is a child of $C_i$, $\hat{C}_{i+1}$ always lies in $\hat{C}_i$, so $D$ lies in $\hat{C}_i$ for all $i$, in particular in $\hat{C}_j$. This means that $D$ as well cannot meet more that two elements of $P$.

□

## References

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