# Contractible edges in 2-connected locally finite graphs

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Submitted: May 30, 2014; Accepted: May 25, 2015; Published: Jun 15, 2015
Mathematics Subject Classifications: 05C40, 05C45, 05C63

**Abstract**

In this paper, we prove that every contraction-critical 2-connected infinite graph has no vertex of finite degree and contains uncountably many ends. Then, by investigating the distribution of contractible edges in a 2-connected locally finite infinite graph $G$, we show that the closure of the subgraph induced by all the contractible edges in the Freudenthal compactification of $G$ is 2-arc-connected. Finally, we characterize all 2-connected locally finite outerplanar graphs nonisomorphic to $K_3$ as precisely those graphs such that every vertex is incident to exactly two contractible edges as well as those graphs such that every finite bond contains exactly two contractible edges.

**Keywords:** contractible edge, Hamilton cycle, outerplanar, infinite graph

## 1 Introduction

Since the pioneering work of Tutte [11] who proved that every 3-connected finite graph nonisomorphic to $K_4$ contains a contractible edge, a lot of research has been done on contractible edges in finite graphs. One may consult the survey paper by Kriesell [8] for details.

For any 2-connected graph nonisomorphic to $K_3$, we have the well-known fact that every edge can either be deleted or contracted so that the resulting graph remains 2-connected. This immediately leads to the following result.

**Theorem 1.** *Let $G$ be a 2-connected finite graph nonisomorphic to $K_3$. Then the subgraph induced by all the contractible edges in $G$ is 2-connected.*

Wu [12] investigated the distribution of contractible elements in matroids and extended Theorem 1 to simple 2-connected matroids. He also characterized all simple 2-connected matroids $M$ having exactly $r(M)+1$ contractible elements (where $r(M)$ is the rank of $M$) as those matroids isomorphic to a graphic matroid of an outerplanar Hamiltonian graph.

**Theorem 2** (Wu [12])**.** *Let $G$ be a 2-connected finite graph nonisomorphic to $K_3$. Then every vertex of $G$ is incident to exactly two contractible edges if and only if $G$ is outerplanar.*

On the other hand, only a few results were known for contractible edges in infinite graphs. For example, Mader [10] showed that every contraction-critical locally finite infinite graph has infinitely many triangles. Kriesell [9] provided a method of constructing contraction-critical $k$-connected infinite graphs ($k \geqslant 2$). In Section 3, we will prove that every contraction-critical 2-connected infinite graph contains vertices of infinite degree only and has uncountably many ends.

A natural way to extend Theorems 1 and 2 is to consider locally finite infinite graphs. Notice that Theorem 1 is no longer true as demonstrated by the infinite double ladder (the cartesian product of a double ray and $K_2$). The subgraph $G_C$ induced by all the contractible edges is the disjoint union of two double rays and is not even connected. Interestingly, the situation changes dramatically by looking at the graph from a topological viewpoint as introduced by Diestel and Kühn [4, 5, 6]. By adding the two ends of the double ladder to $G_C$, the resulting closure $\overline{G_C}$ is a circle and is 2-arc-connected. In Section 4, we will prove that for every 2-connected locally finite infinite graph $G$, $\overline{G_C}$ is 2-arc-connected.

Returning to Theorem 2, the backward direction is straightforward. For the forward direction, by Theorem 1, $G_C$ is spanning and 2-connected. Since every vertex is incident to exactly two contractible edges, $G_C$ is a Hamilton cycle. Then it is easy to see that $G$ is outerplanar. When extending to locally finite infinite graphs, we now need the non-trivial statement that if $G$ is a 2-connected locally finite infinite graph such that every vertex is incident to exactly two contractible edges, then $\overline{G_C}$ is a Hamilton circle. This will be proved in Section 5. We will use it to prove an infinite analog of Theorem 2 for any 2-connected locally finite graph $G$ nonisomorphic to $K_3$. Also we will show that $G$ is outerplanar if and only if every finite bond of $G$ contains exactly two contractible edges.

## 2 Definitions

All basic graph-theoretical terminology can be found in Diestel [3]. Unless otherwise stated, all graphs considered in this paper can be finite or infinite. An edge of a $k$-connected graph is said to be *$k$-contractible* if its contraction results in a $k$-connected graph. Otherwise, it is called *$k$-non-contractible.* A $k$-connected graph in which every edge is $k$-non-contractible is called *contraction-critical* $k$-connected. We simply write 2-contractible as *contractible.* Let $G = (V, E)$ be a 2-connected graph. Denote the set of all contractible edges in $G$ by $E_C$ and the subgraph induced by all the contractible edges by $G_C := (V, E_C)$. Let $X$ and $Y$ be two disjoint subsets of $V$. An *$X$-$Y$ path* $P$ is a path such that only the starting vertex of $P$ lies in $X$ and only the ending vertex of $P$ lies in $Y$. Denote the set of all edges between $X$ and $Y$ by $E_G(X, Y)$. If $X$ and $Y$ form a partition of $V$, then $E_G(X, Y)$ is called a *cut*. A minimal non-empty cut is a *bond*. Denote the set of all edges incident to a vertex $x$ by $E_G(x)$ and the set of all neighbors of $x$ by $N_G(x)$. Define $N_G(X) := (\bigcup_{x\in X} N_G(x)) \setminus X$. A set $S$ of $k$ vertices is called an $k$-*separator* if $G-S$ is not connected.

Let $G$ be a locally finite graph. A *ray* is a 1-way infinite path, a *double ray* is a 2-way infinite path, and the subrays of a ray or double ray are its *tails*. An *end* is an equivalence class of rays where two rays are *equivalent* if no finite set of vertices separates them. Denote the set of the ends by $\Omega(G)$. We define a topological space, denoted by $|G|$, on $G$ together with its ends, which is known as the *Freudenthal compactification* of $G$ as follows. View $G$ as a 1-complex. Thus, every edge is homeomorphic to the unit interval. The basic open neighborhoods of a vertex $x$ consists of a choice of half-open half edges $[xz)$, one for each incident edge $xy$, where $z$ is any interior point of $xy$. For an end $\omega \in \Omega(G)$, we take as a basic open neighborhood the set of the form: $\hat{C}(S,\omega) := C(S,\omega) \cup \Omega(S,\omega) \cup \mathring{E}(S,\omega)$, where $S \subseteq V$ is a finite set of vertices, $C(S,\omega)$ is the component of $G-S$ in which every ray from $\omega$ has a tail, $\Omega(S,\omega)$ is the set of all ends whose rays have a tail in $C(S,\omega)$, and $\mathring{E}(S,\omega)$ is the set of all interior points of edges between $S$ and $C(S,\omega)$. Let $H$ be a subgraph of $G$. Then the closure of $H$ in $|G|$ is called a *standard subspace* and is denoted by $\overline{H}$. We say $H$ *contains* a point $x$ of $|G|$ if $x \in \overline{H}$.

Let $X$ and $Y$ be two topological spaces. A continuous map from the unit interval $[0, 1]$ to $X$ is a *path* in $X$. A homeomorphic image of $[0, 1]$ in $X$ is called an *arc* in $X$. This induces an ordering $<$ for the points in the arc. The images of 0 and 1 are the *endpoints* of the arc. An arc in $X$ with endpoints $x$ and $y$ is called an $x$-$y$ *arc*. A homeomorphic image of the unit circle in $X$ is called a *circle* in $X$. A (*path-*)*component* of $X$ is a maximal (path-)connected set in $X$. $X$ is 2-*connected* (2-*arc-connected*) if for all $x \in X$, $X \setminus x$ is connected (arc-connected). We say $X$ can be *embedded* in $Y$ if there exists an injective continuous function $\phi : X \to Y$ such that $X$ is homeomorphic to $\phi(X)$ in the subspace topology of $Y$. Then $\phi$ is called an *embedding* of $X$ in $Y$. Take $Y$ to be $\mathbb{R}^2$. A component of $\mathbb{R}^2 \setminus \phi(X)$ is called a *face* of $\phi(X)$ in $\mathbb{R}^2$. A graph $G$ is *planar* if $G$ can be embedded in $\mathbb{R}^2$. A graph $G$ is *outerplanar* if there exists an embedding $\phi$ of $G$ in $\mathbb{R}^2$ such that there is a face $f$ of $\phi(G)$ in $\mathbb{R}^2$ whose boundary $\partial f$ contains all the vertices of $G$. Chartrand and Harary [2] characterized outerplanar finite graphs as precisely those graphs that do not contain a $K_{2,3}$- or $K_4$- subdivision.

Suppose $A$ is an arc in $|G|$ and $x$ is a vertex in $A$. Then the vertex immediately before $x$ in $A$ if exists is denoted by $x^-$ and the vertex immediately after $x$ in $A$ if exists is denoted by $x^+$. An arc in $|G|$ is an $\omega$-*arc* if the end $\omega$ is one of its endpoints and unless otherwise stated, it corresponds to the image of 1. Following Bruhn and Stein [1], we define the *end degree* of an end $\omega$ in $G$ as the supremum over the cardinalities of sets of edge-disjoint rays in $\omega$, and denote this number by $deg_G(\omega)$. In fact, they proved that this is equal to the supremum over the cardinalities of sets of edge-disjoint $\omega$-arcs in $|G|$. For a subgraph $H$ of $G$, define the degree of $\omega$ in $H$ as the supremum over the cardinalities of sets of edge-disjoint $\omega$-arcs in $\overline{H}$ which is denoted by $deg_H(\omega)$.

## 3 Contraction-critical 2-connected infinite graphs

It is well-known that the only contraction-critical 2-connected finite graph is $K_3$. However, there are infinitely many contraction-critical 2-connected infinite graphs as shown by the following construction due to Kriesell [9]. Define $G_0 := \emptyset$ and let $G_1$ be any 2-connected finite graph. Suppose we have constructed $G_n$ such that $G_{n-1} \subsetneq G_n$. For each edge $xy$ in $E(G_n) \setminus E(G_{n-1})$, add a new $x$-$y$ path of length at least 2. The resulting graph is $G_{n+1}$. Repeat the process inductively. Then the graph $G := \bigcup_{i \geqslant 1} G_i$ is a contraction-critical 2-connected infinite graph. Note that $G$ has no vertex of finite degree and has uncountably many ends. We will show that this holds in general for any contraction-critical 2-connected infinite graph. First, we state a fundamental fact about contractible edges in 2-connected graphs.

**Lemma 3.** *Let $G$ be a 2-connected graph nonisomorphic to $K_3$ and $e$ be an edge of $G$. Then $G - e$ or $G/e$ is 2-connected.*

Now, we can develop some tools that will be used for the rest of the paper.

**Lemma 4.** *Let $G$ be a 2-connected graph nonisomorphic to $K_3$, and $e$ and $f$ be two non-contractible edges of $G$. Then $f$ is a non-contractible edge of $G - e$.*

*Proof.* By Lemma 3, $G - e$ is 2-connected. Since $V(f)$ is a 2-separator of $G$, $V(f)$ is also a 2-separator of $G - e$ and $f$ is a non-contractible edge of $G - e$. □

**Lemma 5.** *Let $G$ be a 2-connected graph nonisomorphic to $K_3$ and $F$ be a finite subset of $E(G)$.*

(a) *If $G - F$ is disconnected, then $F$ contains at least two contractible edges.*

(b) *If $G - F$ is connected but not 2-connected, then $F$ contains at least one contractible edge.*

*Proof.* For (a), suppose $F$ contains at most one contractible edge. Then by Lemma 3 and 4, we can delete all the non-contractible edges in $F$ and the resulting graph is still 2-connected, a contradiction.

For (b), suppose all edges in $F$ are non-contractible. Then by Lemma 3 and 4, we can delete all edges in $F$ and $G - F$ is still 2-connected, a contradiction. □

**Lemma 6.** *Let $G$ be a 2-connected graph nonisomorphic to $K_3$. Let $\{x, y\}$ be a 2-separator of $G$ and $C$ be a component of $G - x - y$. If $|E_G(x, C)|$ is finite, then $E_G(x, C)$ contains a contractible edge.*

*Proof.* Note that $y$ is a cutvertex of $G - E_G(x, C)$. By Lemma 5(b), $E_G(x, C)$ contains a contractible edge. □

**Lemma 7.** *Let $G$ be a 2-connected graph nonisomorphic to $K_3$ and $x$ be a vertex of $G$. Suppose all edges incident to $x$ are non-contractible. Then*

(a) *$x$ has infinite degree.*

(b) *For any edge $xy$ incident to $x$, every component of $G-x-y$ contains infinitely many neighbors of $x$.*

*Proof.* For (a), Suppose $x$ has finite degree. By applying Lemma 5(a) to $E_G(x)$, $x$ is incident to at least two contractible edges, a contradiction.

For (b), let $C$ be a component of $G-x-y$. By Lemma 6, $E_G(x, C)$ contains infinitely many edges. □

**Theorem 8.** *Let $G$ be a contraction-critical 2-connected infinite graph. Then every vertex of $G$ has infinite degree and $G$ has uncountably many ends.*

*Proof.* By Lemma 7(a), every vertex of $G$ has infinite degree.

Next, we will construct a rooted binary infinite tree $T$ in $G$ together with edges incident to each vertex of $T$ with the following properties:

(1) The root of $T$ is denoted by $x$.

(2) The vertices of $T$ are denoted by $x_{n_1n_2\ldots n_k}$ where $k \in \mathbb{N}$ and $n_i \in \{0,1\}$ for $1 \leqslant i \leqslant k$. For $k = 0$, define $x_{n_1n_2\ldots n_k} := x$.

(3) Each vertex $x_{n_1n_2\ldots n_k}$ of $T$ is adjacent to two vertices $x_{n_1n_2\ldots n_k0}$ and $x_{n_1n_2\ldots n_k1}$ in $T$.

(4) For each vertex $x_{n_1n_2\ldots n_k}$ of $T$, there exists an edge $x_{n_1n_2\ldots n_k}y_{n_1n_2\ldots n_k}$ in $G$ such that $y_{n_1n_2\ldots n_k}$ does not lie in $T$.

(5) The subtree of $T$ rooted at $x_{n_1n_2\ldots n_k}$ is defined as

$T_{n_1n_2\ldots n_k} := T[\bigcup_{i=0}^{\infty}\bigcup_{(m_1,m_2,\ldots,m_i)\in\{0,1\}^i} x_{n_1n_2\ldots n_km_1m_2\ldots m_i}]$.

For fixed $n_1, n_2, \ldots, n_k$, $\bigcup_{j=0}^{k}\{x_{n_1n_2\ldots n_j}, y_{n_1n_2\ldots n_j}\}$ separates $T_{n_1n_2\ldots n_k0}$ and $T_{n_1n_2\ldots n_k1}$ in $G$.

Each ray in $T$ starting at $x$ is of the form: $xx_{n_1}x_{n_1n_2}x_{n_1n_2n_3}\ldots$. Let $R := xx_{n_1}x_{n_1n_2}$ $x_{n_1n_2n_3}\ldots$ and $Q := xx_{m_1}x_{m_1m_2}x_{m_1m_2m_3}\ldots$ be two distinct rays in $T$. Then there exists a smallest $k$ such that $n_i = m_i$ for all $i \leqslant k$ and $n_{k+1} \neq m_{k+1}$. By property (5) above, $\bigcup_{j=0}^{k}\{x_{n_1n_2\ldots n_j}, y_{n_1n_2\ldots n_j}\}$ separates $R$ and $Q$ in $G$. Therefore, each ray in $T$ starting at $x$ belongs to a unique end of $G$, and $G$ has uncountably many ends.

Now, it remains to construct $T$ inductively. Let $x$ be any vertex in $G$. Define $T_0 := (\{x\}, \emptyset)$. Choose any edge incident to $x$ in $G$, say $xy$. Let $C_0$ and $C_1$ be any two components of $G-x-y$. Let $x_0$ be a neighbor of $x$ in $C_0$ and $x_1$ be a neighbor of $x$ in $C_1$. Define $T_1 := (\{x, x_0, x_1\}, \{xx_0, xx_1\})$. Note that $N_G(C_0) \subseteq \{x, y\}$ and $N_G(C_1) \subseteq \{x, y\}$. Also, $G - C_0$ and $G - C_1$ are both connected.

Suppose we have constructed the rooted binary tree $T_k$ where

$V(T_k) := \bigcup_{i=0}^{k}\bigcup_{(n_1,n_2,\ldots,n_i)\in\{0,1\}^i} x_{n_1n_2\ldots n_i}$ and

$E(T_k) := \bigcup_{i=0}^{k-1}\bigcup_{(n_1,n_2,\ldots,n_i)\in\{0,1\}^i}\{x_{n_1n_2\ldots n_i}x_{n_1n_2\ldots n_i0}, x_{n_1n_2\ldots n_i}x_{n_1n_2\ldots n_i1}\}$ such that

(i) each vertex $x_{n_1n_2\dots n_i}$ $(0 \leqslant i \leqslant k)$ lies in a connected subgraph $C_{n_1n_2\dots n_i}$ of $G$ (for $i=0$, $x_{n_1n_2\dots n_i} := x$, $y_{n_1n_2\dots n_i} := y$ and $C_{n_1n_2\dots n_i} := G$),

(ii) for each vertex $x_{n_1n_2\dots n_i}$ $(0 \leqslant i < k)$, we have found an edge $x_{n_1n_2\dots n_i}y_{n_1n_2\dots n_i}$ that lies in $C_{n_1n_2\dots n_i}$ such that $C_{n_1n_2\dots n_i0}$ and $C_{n_1n_2\dots n_i1}$ are two components of $C_{n_1n_2\dots n_i} - x_{n_1n_2\dots n_i} - y_{n_1n_2\dots n_i}$ that are adjacent to $x_{n_1n_2\dots n_i}$,

(iii) for fixed $n_1, n_2, \dots, n_i$ $(1 \leqslant i \leqslant k)$, $N_G(C_{n_1n_2\dots n_i}) \subseteq \bigcup_{j=0}^{i-1}\{x_{n_1n_2\dots n_j}, y_{n_1n_2\dots n_j}\}$,

(iv) for fixed $n_1, n_2, \dots, n_i$ $(1 \leqslant i \leqslant k)$, $G - C_{n_1n_2\dots n_i}$ is connected.

Now, for each vertex $x_{n_1n_2\dots n_k}$ in $T_k$, since it has infinite degree and $N_G(x_{n_1n_2\dots n_k}) \setminus C_{n_1n_2\dots n_k} \subseteq N_G(C_{n_1n_2\dots n_k})$ is finite by (iii), all but finitely many neighbors of $x_{n_1n_2\dots n_k}$ lie in $C_{n_1n_2\dots n_k}$. Let $z$ be a neighbor of $x_{n_1n_2\dots n_k}$ in $C_{n_1n_2\dots n_k}$ and $B := C_{n_1n_2\dots n_k} - x_{n_1n_2\dots n_k} - z$. Suppose $B$ is connected. Since $B' := G - C_{n_1n_2\dots n_k}$ is connected by (iv), $B$ and $B'$ are the only two components of $G - x_{n_1n_2\dots n_k} - z$. By Lemma 7(b), $B'$ contains infinitely many neighbors of $x_{n_1n_2\dots n_k}$ contradicting $N_G(x_{n_1n_2\dots n_k}) \setminus C_{n_1n_2\dots n_k} \subseteq N_G(C_{n_1n_2\dots n_k})$ which is finite by (iii). Therefore, $B$ is not connected.

Note that at least one component of $B$ is adjacent to $x_{n_1n_2\dots n_k}$. If not, then, by the 2-connectedness of $G$, each component of $B$ has a neighbor in $N_G(C_{n_1n_2\dots n_k}) \subseteq G - C_{n_1n_2\dots n_k}$. By (iv), this implies $G - x_{n_1n_2\dots n_k} - z$ is connected, a contradiction. Suppose there are two components of $B$, say $D$ and $D'$, that are both adjacent to $x_{n_1n_2\dots n_k}$. Then choose $y_{n_1n_2\dots n_k} := z$, $C_{n_1n_2\dots n_k0} := D$ and $C_{n_1n_2\dots n_k1} := D'$. Suppose only one component of $B$ is adjacent to $x_{n_1n_2\dots n_k}$, say $C$. Each component of $B$ other than $C$ is adjacent to $z$ by the connectedness of $C_{n_1n_2\dots n_k}$ and has a neighbor in $N_G(C_{n_1n_2\dots n_k}) \subseteq G - C_{n_1n_2\dots n_k}$ by the 2-connectedness of $G$. Denote the union of components of $B$ other than $C$ by $C'$. Since $G - C_{n_1n_2\dots n_k}$ is connected by (iv), $C'' := G[(G - C_{n_1n_2\dots n_k}) \cup C']$ is connected. Hence, $C$ and $C''$ are the only two components of $G - x_{n_1n_2\dots n_k} - z$ and $N_G(C) = \{x_{n_1n_2\dots n_k}, z\}$. Let $z'$ be a neighbor of $x_{n_1n_2\dots n_k}$ in $C$. Then one component $D$ of $C_{n_1n_2\dots n_k} - x_{n_1n_2\dots n_k} - z'$ contains $z$ and $C'$. Since $G - x_{n_1n_2\dots n_k} - z'$ is not connected, $D$ cannot be the only component of $C_{n_1n_2\dots n_k} - x_{n_1n_2\dots n_k} - z'$. Let $D'$ be any component of $C_{n_1n_2\dots n_k} - x_{n_1n_2\dots n_k} - z'$ other than $D$. Then $D'$ lies in $C$ and $N_G(D') \subseteq \{x_{n_1n_2\dots n_k}, z'\} \cup (N_G(C) - z) = \{x_{n_1n_2\dots n_k}, z'\}$. Now, choose $y_{n_1n_2\dots n_k} := z'$, $C_{n_1n_2\dots n_k0} := D$ and $C_{n_1n_2\dots n_k1} := D'$.

In both cases, $x_{n_1n_2\dots n_k}y_{n_1n_2\dots n_k}$ lies in $C_{n_1n_2\dots n_k}$, and $C_{n_1n_2\dots n_k0}$ and $C_{n_1n_2\dots n_k1}$ are two components of $C_{n_1n_2\dots n_k} - x_{n_1n_2\dots n_k} - y_{n_1n_2\dots n_k}$ that are adjacent to $x_{n_1n_2\dots n_k}$. Let $x_{n_1n_2\dots n_k0}$ be a neighbor of $x_{n_1n_2\dots n_k}$ in $C_{n_1n_2\dots n_k0}$ and $x_{n_1n_2\dots n_k1}$ be a neighbor of $x_{n_1n_2\dots n_k}$ in $C_{n_1n_2\dots n_k1}$. Since $C_{n_1n_2\dots n_k0} \subseteq C_{n_1n_2\dots n_k}$ and $C_{n_1n_2\dots n_k1} \subseteq C_{n_1n_2\dots n_k}$, $N_G(C_{n_1n_2\dots n_k0}) \subseteq N_G(C_{n_1n_2\dots n_k}) \cup \{x_{n_1n_2\dots n_k}, y_{n_1n_2\dots n_k}\} \subseteq \bigcup_{j=0}^{k}\{x_{n_1n_2\dots n_j}, y_{n_1n_2\dots n_j}\}$ and $N_G(C_{n_1n_2\dots n_k1}) \subseteq N_G(C_{n_1n_2\dots n_k}) \cup \{x_{n_1n_2\dots n_k}, y_{n_1n_2\dots n_k}\} \subseteq \bigcup_{j=0}^{k}\{x_{n_1n_2\dots n_j}, y_{n_1n_2\dots n_j}\}$ by (iii). By the connectedness of $C_{n_1n_2\dots n_k}$, every component of $C_{n_1n_2\dots n_k} - x_{n_1n_2\dots n_k} - y_{n_1n_2\dots n_k}$ has a neighbor in $\{x_{n_1n_2\dots n_k}, y_{n_1n_2\dots n_k}\}$. For $n_{k+1} \in \{0,1\}$, denote the union of the components of $C_{n_1n_2\dots n_k} - x_{n_1n_2\dots n_k} - y_{n_1n_2\dots n_k}$ other than $C_{n_1n_2\dots n_kn_{k+1}}$ by $U_{n_{k+1}}$. Then $G[x_{n_1n_2\dots n_k}y_{n_1n_2\dots n_k} \cup U_{n_{k+1}}]$ is connected. Since $x_{n_1n_2\dots n_{k-1}} \in G - C_{n_1n_2\dots n_k}$, $x_{n_1n_2\dots n_{k-1}}x_{n_1n_2\dots n_k} \in E(G)$ and $G - C_{n_1n_2\dots n_k}$ is connected by (iv), $G - C_{n_1n_2\dots n_kn_{k+1}} := G[(G - C_{n_1n_2\dots n_k}) \cup x_{n_1n_2\dots n_k}y_{n_1n_2\dots n_k} \cup U_{n_{k+1}}]$ is connected.

Define $T_{k+1}$ where $V(T_{k+1}) := \bigcup_{i=0}^{k+1} \bigcup_{(n_1,n_2,\ldots,n_i)\in\{0,1\}^i} x_{n_1n_2\ldots n_i}$ and $E(T_{k+1}) := \bigcup_{i=0}^{k} \bigcup_{(n_1,n_2,\ldots,n_i)\in\{0,1\}^i} \{x_{n_1n_2\ldots n_i}x_{n_1n_2\ldots n_i0}, x_{n_1n_2\ldots n_i}x_{n_1n_2\ldots n_i1}\}$.

Finally, define $T := \bigcup_{k=0}^{\infty} T_k$. It is easy to see that $T$ satisfies properties (1) through (4). Let $z_0$ and $z_1$ be any vertices in $T_{n_1n_2\ldots n_k0}$ and $T_{n_1n_2\ldots n_k1}$ respectively. Then $z_0$ is of the form $x_{n_1n_2\ldots n_k0p_1p_2\ldots p_i}$ while $z_1$ is of the form $x_{n_1n_2\ldots n_k1q_1q_2\ldots q_j}$. We have $z_0 = x_{n_1n_2\ldots n_k0p_1p_2\ldots p_i} \in C_{n_1n_2\ldots n_k0p_1p_2\ldots p_i} \subseteq C_{n_1n_2\ldots n_k0p_1p_2\ldots p_{i-1}} \subseteq \ldots \subseteq C_{n_1n_2\ldots n_k0}$ and $z_1 = x_{n_1n_2\ldots n_k1q_1q_2\ldots p_j} \in C_{n_1n_2\ldots n_k1q_1q_2\ldots q_j} \subseteq C_{n_1n_2\ldots n_k1q_1q_2\ldots q_{j-1}} \subseteq \ldots \subseteq C_{n_1n_2\ldots n_k1}$. Therefore, $T_{n_1n_2\ldots n_k0} \subseteq C_{n_1n_2\ldots n_k0}$ and $T_{n_1n_2\ldots n_k1} \subseteq C_{n_1n_2\ldots n_k1}$. Since $\bigcup_{j=0}^{k}\{x_{n_1n_2\ldots n_j}, y_{n_1n_2\ldots n_j}\}$ contains both $N_G(C_{n_1n_2\ldots n_k0})$ and $N_G(C_{n_1n_2\ldots n_k1})$, it separates $C_{n_1n_2\ldots n_k0}$ and $C_{n_1n_2\ldots n_k1}$ in $G$ and thus separates $T_{n_1n_2\ldots n_k0}$ and $T_{n_1n_2\ldots n_k1}$ in $G$. Hence, Property (5) holds for $T$ and the proof is complete. □

## 4 Subgraph induced by all the contractible edges

In this section, we will extend Theorem 1 to any 2-connected locally finite infinite graph $G$ and prove that $\overline{G_C}$ is 2-arc-connected. Note that Lemma 5(a) implies that every vertex is incident to at least two contractible edges. Hence, $G_C$ is spanning. Using the following two lemmas, it is easy to see that $\overline{G_C}$ is arc-connected.

**Lemma 9** (Diestel [3]). *Let $G$ be a locally finite graph. Then a standard subspace of $|G|$ is connected if and only if it contains an edge from every finite cut of $G$ of which it meets both sides.*

**Lemma 10** (Diestel and Kühn [6]). *If $G$ is a locally finite graph, then every closed connected subspace of $|G|$ is arc-connected.*

**Theorem 11.** *Let $G$ be a 2-connected locally finite infinite graph and $G_C$ be the subgraph induced by all the contractible edges in $G$. Then $\overline{G_C}$ is arc-connected.*

*Proof.* Let $F$ be any finite cut of $G$. By Lemma 5(a), $F$ contains at least two edges in $G_C$. Hence, $\overline{G_C}$ is connected by Lemma 9. By Lemma 10, $\overline{G_C}$ is arc-connected. □

Next, we prove that $\overline{G_C}$ is 2-connected.

**Lemma 12.** *Let $G$ be a 2-connected locally finite infinite graph and $x$ be a point of $|G|$. Suppose there is a partition $(X, X')$ of $V(G\setminus x)$ such that $E_G(X, X')$ is non-empty and all edges in $E_G(X, X')$ are non-contractible. Then $G$ contains a subdivision of a 1-way infinite ladder $L$ consisting of two disjoint rays: $R := x_0P_1x_1P_2x_2\ldots$ and $R' := x'_0P'_1x'_1P'_2x'_2\ldots$ with the rungs of the ladder being $x_0x'_0, x_1x'_1, x_2x'_2, \ldots$, all of which are $X$-$X'$ edges such that $x \notin \overline{L}$.*

*Proof.* Since $G$ is 2-connected, $|E_G(X, X')| \geqslant 2$ unless $x$ is a vertex and $|E_G(X, X')| = 1$. Consider any $X$-$X'$ edge $x_0x'_0$ that does not contain $x$. Let $C$ be the component of $G - x_0 - x'_0$ containing $x$ and $C_1$ be a component of $G - x_0 - x'_0$ not containing $x$.

Suppose we have constructed the finite ladder $L_k$ consisting of two disjoint paths $R_k := x_0P_1x_1P_2x_2\ldots x_{k-1}P_kx_k$ and $R'_k := x'_0P'_1x'_1P'_2x'_2\ldots x'_{k-1}P'_kx'_k$ with the rungs of the ladder being $x_0x'_0, x_1x'_1, \dots x_kx'_k$, all of which are $X$-$X'$ edges such that $L_k \subseteq G[C_1 \cup x_0 \cup x'_0]$ and $G[C \cup L_k - x_k - x'_k]$ is connected. Let $C_{k+1}$ be a component of $G - x_k - x'_k$ not containing $x$. Then $C_{k+1} \subseteq C_1$ and $C_{k+1} \cap L_k = \emptyset$. By applying Lemma 6 to $E_G(x_k, C_{k+1})$ and $E_G(x'_k, C_{k+1})$, there exist contractible edges $x_ky_{k+1}$ and $x'_ky'_{k+1}$ where $y_{k+1} \in C_{k+1}$ and $y'_{k+1} \in C_{k+1}$. Since $x \notin C_{k+1}$ and all edges in $E_G(X, X')$ are non-contractible, $y_{k+1} \in X$ and $y'_{k+1} \in X'$. Choose a path $Q_{k+1}$ in $C_{k+1}$ between $y_{k+1}$ and $y'_{k+1}$. Then there exists an $X$-$X'$ edge $x_{k+1}x'_{k+1}$ on $Q_{k+1}$ such that $V(y_{k+1}Q_{k+1}x_{k+1}) \subseteq X$ and $x'_{k+1} \in X'$. Define $P_{k+1} := x_ky_{k+1} \cup y_{k+1}Q_{k+1}x_{k+1}$, $P'_{k+1} := x'_ky'_{k+1} \cup y'_{k+1}Q_{k+1}x'_{k+1}$, $R_{k+1} := R_k \cup P_{k+1}$, $R'_{k+1} := R'_k \cup P'_{k+1}$ and $L_{k+1} := L_k \cup P_{k+1} \cup P'_{k+1} \cup x_{k+1}x'_{k+1}$. Note that $L_{k+1} \subseteq G[C_1 \cup x_0 \cup x'_0]$ and $G[C \cup L_{k+1} - x_{k+1} - x'_{k+1}]$ is connected.

Define $R := \bigcup_{k \geqslant 0} R_k$, $R' := \bigcup_{k \geqslant 0} R'_k$ and $L := \bigcup_{k \geqslant 0} L_k$. Then $L \subseteq G[C_1 \cup x_0 \cup x'_0]$ and $x \notin \overline{L}$. □

**Theorem 13.** *Let $G$ be a 2-connected locally finite infinite graph and $G_C$ be the subgraph induced by all the contractible edges in $G$. Then $\overline{G_C}$ is 2-connected.*

*Proof.* Suppose $\overline{G_C}$ is not 2-connected. Then there exists a point $x$ in $\overline{G_C}$ such that $\overline{G_C} \setminus x$ is not connected. Let $U$ and $U'$ be two disjoint non-empty open sets in $|G|$ such that $\overline{G_C} \setminus x \subseteq U \cup U'$, $(\overline{G_C} \setminus x) \cap U \neq \emptyset$ and $(\overline{G_C} \setminus x) \cap U' \neq \emptyset$. Define $X := (\overline{G_C} \setminus x) \cap U \cap V(G)$ and $X' := (\overline{G_C} \setminus x) \cap U' \cap V(G)$. Since $G_C$ is spanning, $X \cup X' = V(G \setminus x)$. Suppose $U$ contains an interior point $a$ of an edge $bc$ of $G_C$. Then $\overline{G_C} \setminus x$ contains half edges $[ba]$ or $[ca]$ of $bc$. By the connectedness of half edge, $U$ contains $b$ or $c$. Suppose $U$ contains an end $\omega$ of $|G|$. Then $U$ contains a basic open neighborhood of $\omega$, say $\hat{C}(S, \omega)$, and thus contains infinitely many vertices. The same arguments hold for $U'$. Therefore, both $X$ and $X'$ are non-empty. Since $G \setminus x$ is connected, $E_G(X, X')$ is non-empty.

Suppose $x$ is a vertex or an end of $G$. Then all edges in $E_G(X, X')$ are non-contractible and $(X, X')$ is a partition of $V(G \setminus x)$. Suppose $x$ is an interior point of an edge $e$. Then all edges in $E_G(X, X')$ are non-contractible unless $e \in E_G(X, X') \cap E_C$. Note that, $E_G(X, X') - e$ is non-empty as $G$ is 2-connected and every edge in $E_G(X, X') - e$ is non-contractible. Let $e = yy'$ where $y \in X$ and $y' \in X'$. Suppose $X = \{y\}$. By Lemma 5(a), since $y$ is incident to at least two contractible edges, there is a contractible $X$-$X'$ edge other than $e$, which is impossible. Therefore, $|X| \geqslant 2$. Now, all edges in $E_G(X - y, X')$ are non-contractible and $(X - y, X')$ is a partition of $V(G \setminus y)$. In both cases, by Lemma 12, $G$ contains a subdivision of a 1-way infinite ladder $L$ such that $x \notin \overline{L}$.

Let $\omega$ be the end of $|G|$ containing $R$ and $R'$. Note that $\omega \neq x$. Since $G_C$ is spanning, $\overline{G_C} \setminus x$ contains all the ends of $|G|$ except possibly $x$. Without loss of generality, assume $\omega \in U$. Since $U$ is open, there exists a basic open neighborhood $\hat{C}(S, \omega) \subseteq U$. Since $x'_0, x'_1, x'_2, \dots \in X' \subseteq U'$ converge to $\omega$, all but finitely many of them lie in $\hat{C}(S, \omega)$, contradicting $U \cap U' = \emptyset$. □

Finally, we prove the main result of this section, namely, $\overline{G_C}$ is 2-arc-connected. This follows from a theorem by Georgakopoulos [7] concerning connected but not path-connected subspaces of locally finite graphs. Note that since $|G|$ is Hausdorff, path-connectedness is equivalent to arc-connectedness.

**Theorem 14** (Georgakopoulos [7]). *Given any locally finite connected graph $G$, a connected subspace $X$ of $|G|$ is path-connected unless it satisfies the following assertions:*

(1) *$X$ has uncountably many path-components each of which consists of one end only;*

(2) *$X$ has infinitely many path-components that contain a vertex; and*

(3) *every path-component of $X$ contains an end.*

**Theorem 15.** *Let $G$ be a 2-connected locally finite infinite graph and $G_C$ be the subgraph induced by all the contractible edges in $G$. Then $\overline{G_C}$ is 2-arc-connected.*

*Proof.* Suppose $\overline{G_C}$ is not 2-arc-connected. Then there exists a point $x$ in $\overline{G_C}$ such that $\overline{G_C}\backslash x$ is not arc-connected. Note that $\overline{G_C}\backslash x$ is connected by Theorem 13. By Theorem 14, $\overline{G_C}\backslash x$ has uncountably many path-components each of which consists of one end only. Let $\omega$ and $\omega'$ be two such path-components of $\overline{G_C} \setminus x$. Since $\overline{G_C}$ is arc-connected by Theorem 11, there exists an arc $A$ joining $\omega$ and $\omega'$ in $\overline{G_C}$. Now, $x$ must lie in $A$ for otherwise $\omega$ and $\omega'$ would lie in the same path-component of $\overline{G_C} \setminus x$. But the path-component of $\overline{G_C} \setminus x$ containing $\omega$ also contains $[\omega A x)$, a contradiction. □

## 5 Outerplanarity of 2-connected locally finite graphs

As mentioned in the introduction, in order to extend Theorem 2 to locally finite infinite graphs, we would like to prove that for any 2-connected locally finite infinite graph $G$, if every vertex is incident to exactly two contractible edges, then $\overline{G_C}$ is a Hamilton circle. This requires several lemmas listed below.

**Lemma 16.** *Let $G$ be a locally finite graph. Then every arc in $|G|$ whose two endpoints are ends contains a vertex.*

*Proof.* Suppose $A$ is an arc in $|G|$ whose two endpoints are ends $\omega_1$ and $\omega_2$. Then there exists a finite set $S$ of vertices such that $\hat{C}(S, \omega_1)$ and $\hat{C}(S, \omega_2)$ are distinct. By the connectedness of $A$, $A$ contains a vertex of $S$. □

**Lemma 17.** *Let $G$ be a locally finite graph and $\omega$ be an end in $|G|$. Then every $\omega$-arc $A$ in $|G|$ contains a vertex, say $z$, and $zA$ contains a ray starting with $z$.*

*Proof.* Denote the starting point of $A$ by $a$. First, we show that $A$ contains a vertex. If $a$ is a vertex, then we are done. If $a$ is an end, then it is true by Lemma 16. If $a$ is an interior point of an edge $xy$, then by the connectedness of $A$, $A$ contains $x$ or $y$.

Let $z$ be a vertex in $A$. By the connectedness of $zA$, $zA$ contains an interior point of an edge incident to $z$, say $zz_1$. Then the connectedness of $zA$ implies $zz_1$ lies in $zA$. Repeat the above argument for $z_1A$ and so on. We obtain a ray that starts with $z$ and lies in $zA$. □

**Lemma 18.** *Let $G$ be a locally finite graph and $\omega$ be an end in $|G|$. Let $A_1$ and $A_2$ be two $\omega$-arcs in $|G|$ that are disjoint except at $\omega$. Then, for all finite subset $S$ of $V(G)$, $\hat{C}(S,\omega)$ contains a subarc $A_1'\omega A_2'$ of $A_1\omega A_2$ and there is an $A_1'$-$A_2'$ path in $C(S,\omega)$.*

*Proof.* Let $x_1$ be the last point of $A_1$ that lies in $S$ and $x_2$ be the last point of $A_2$ that lies in $S$. By Lemma 17, $x_1A$ contains a ray $R_1$ starting with $x_1$ and $x_2A$ contains a ray $R_2$ starting with $x_2$. Let $y_1$ be the neighbor of $x_1$ in $R_1$ and $y_2$ be the neighbor of $x_2$ in $R_2$. Then $y_1A_1\omega A_2y_2$ lies in $\hat{C}(S,\omega)$. Also there is a $y_1$-$y_2$ path in $C(S,\omega)$ which automatically contains a $y_1A_1$-$y_2A_2$ path. □

We also need a result on the characterization of a topological circle in $|G|$ in terms of its vertex and end degrees.

**Lemma 19** (Bruhn and Stein [1])**.** *Let $C$ be a subgraph of a locally finite graph $G$. Then $\overline{C}$ is a circle if and only if $\overline{C}$ is connected and every vertex and end of $|G|$ in $\overline{C}$ has degree two in $\overline{C}$.*

Now, we can proceed with the proof.

**Theorem 20.** *Let $G$ be a 2-connected locally finite infinite graph and $G_C$ be the subgraph induced by all the contractible edges in $G$. If every vertex of $G$ is incident to exactly two contractible edges, then $\overline{G_C}$ is a Hamilton circle.*

*Proof.* Since $G_C$ is spanning, $\overline{G_C}$ contains all vertices and ends of $|G|$. By Theorem 11, $\overline{G_C}$ is arc-connected. Obviously, every vertex of $G$ has degree two in $\overline{G_C}$. Therefore, it remains to prove that every end of $|G|$ has degree two in $\overline{G_C}$.

**Claim 21.** *Let $A$ be an arc in $\overline{G_C}$ and $x$ be a vertex in $A$. Suppose that both $x^-$ and $x^+$ exist in $A$. Let $y$ be any neighbor of $x$ other than $x^-$ and $x^+$. Then every $x^-$-$x^+$ arc in $|G|$ intersects $\{x,y\}$.*

*Proof.* Since $xx^-$ and $xx^+$ are the only contractible edges incident to $x$, $xy$ is non-contractible. Lemma 6 implies that $G-x-y$ has exactly two components, and $x^-$ and $x^+$ lie in different components. By the connectedness of an arc, every $x^-$-$x^+$ arc in $|G|$ intersects $\{x,y\}$. □

**Claim 22.** *Let $\omega$ be an end in $|G|$. Suppose $A_1$ and $A_2$ are two edge-disjoint $\omega$-arcs in $\overline{G_C}$. Then $A_1$ and $A_2$ can intersect only at the ends of $|G|$ with the only possible exception being that the starting points of $A_1$ and $A_2$ are the same vertex.*

*Proof.* Obviously, $A_1$ and $A_2$ cannot intersect at an interior point of an edge. Suppose $A_1$ and $A_2$ intersect at a vertex $x$. If $x$ is not the starting point for both $A_1$ and $A_2$, then the degree of $x$ in $\overline{G_C}$ is at least three, a contradiction. □

**Claim 23.** *Let $\omega$ be an end in $|G|$. Suppose $A_1$ and $A_2$ are two edge-disjoint $\omega$-arcs in $\overline{G_C}$ such that the starting points of $A_1$ and $A_2$ are distinct vertices. Then there exists an end $\omega'$ in $|G|$ such that there are three $\omega'$-arcs in $\overline{G_C}$ that are disjoint except at $\omega'$ unless $A_1 \cap A_2 = \{\omega\}$.*

*Proof.* Suppose $A_1 \cap A_2 \neq \{\omega\}$. Let $\omega'$ be the first point of $A_2$ that intersects $A_1$. By Claim 22, $\omega'$ is an end in $|G|$ different from $\omega$. Then $A_1\omega'$, $A_2\omega'$ and $\omega A_1 \omega'$ are the required three $\omega'$-arcs. ☐

**Claim 24.** *Let $\omega$ be an end in $|G|$. Suppose there are three edge-disjoint $\omega$-arcs in $\overline{G_C}$. Then there exists an end $\omega'$ in $|G|$ such that there are three $\omega'$-arcs in $\overline{G_C}$ that are disjoint except at $\omega'$.*

*Proof.* Let $A_1, A_2, A_3$ be three edge-disjoint $\omega$-arcs in $\overline{G_C}$. By Lemma 17, for each $i \in \{1,2,3\}$, $A_i$ contains a ray $R_i$. Denote the first edge of $R_i$ by $x_i y_i$. By Claim 22, $y_1, y_2, y_3$ are all distinct. Therefore, without loss of generality, we can assume that the starting points of $A_1, A_2, A_3$ are all distinct vertices. Consider $A_1$ and $A_2$. If $A_1 \cap A_2 \neq \{\omega\}$, then the claim follows from Claim 23. Suppose $A_1 \cap A_2 = \{\omega\}$. If $A_2 \cap A_3 \neq \{\omega\}$, then again the claim follows from Claim 23. Therefore, suppose $A_2 \cap A_3 = \{\omega\}$. But then, $A_1, A_2, A_3$ are the desired three $\omega$-arcs. ☐

**Claim 25.** *For each end $\omega$ in $|G|$, $deg_{G_C}(\omega) \leqslant 2$.*

*Proof.* Suppose there are three edge-disjoint $\omega$-arcs in $\overline{G_C}$. By Claim 24, there exists an end $\omega'$ in $|G|$ such that there are three $\omega'$-arcs in $\overline{G_C}$ that are disjoint except at $\omega'$. Denote these three $\omega'$-arcs by $A_1, A_2, A_3$. By Lemma 16, without loss of generality, we can assume $A_1, A_2, A_3$ start with vertices $a_1, a_2, a_3$ respectively.

By applying Lemma 18 to $A_1$ and $A_2$ with $S = \{a_1, a_2\}$, we obtain an $a_1^+ A_1$-$a_2^+ A_2$ path $P$. Let $x_1 = P \cap A_1$, $x_2 = P \cap A_2$ and $x$ be the neighbor of $x_1$ in $Q$. If $P$ intersects $A_3$, then interchange $A_2$ and $A_3$. Therefore, without loss of generality, there is an $a_1^+ A_1$-$a_2^+ A_2$ path $P$ that does not intersect $A_3$.

Now, apply Lemma 18 to $x_2 A_2$ and $A_3$ with $S = V(P)$. We obtain an $x_2^+ A_2$-$A_3$ path $Q$ not intersecting $P$. Let $y_2 = Q \cap x_2^+ A_2$ and $y_3 = Q \cap A_3$. By Claim 21, $Q$ cannot intersect $A_2 x_2^-$, and $Q$ cannot intersect both $A_1 x_1^-$ and $x_1^+ A_1$. Suppose $Q \cap A_1 x_1^- \neq \emptyset$. Let $y$ be the first vertex of $Q$ that lies in $A_1 x_1^-$. Then $x_1^- A_1 y Q y_2 A_2 \omega' A_1 x_1^+$ is an $x_1^-$-$x_1^+$ arc not intersecting $\{x_1, x\}$, contradicting Claim 21. Suppose $Q \cap x_1^+ A_1 \neq \emptyset$. Then there is an $x_1^+ A_1$-$A_3$ subpath in $Q$ not intersecting $A_2$, and we interchange $A_1$ and $A_2$. Therefore, without loss of generality, we can assume that there is an $x_2^+ A_2$-$A_3$ path $Q$ that does not intersect $P \cup A_1 \cup A_2 x_2^-$. Let $u_2$ be the neighbor of $y_2$ in $Q$ and $u_3$ be the neighbor of $y_3$ in $Q$.

Finally, apply Lemma 18 to $x_1 A_1$ and $y_2 A_2$ with $S = V(P \cup Q)$. We obtain an $x_1^+ A_1$-$y_2^+ A_2$ path $R$ not intersecting $P \cup Q$. Let $z_1 = R \cap x_1^+ A_1$ and $z_2 = R \cap y_2^+ A_2$. By Claim 21, $R$ cannot intersect $A_1 x_1^-$ and $R$ cannot intersect $A_2 y_2^-$. Also, $R$ cannot intersect both $A_3 y_3^-$ and $y_3^+ A_3$. Suppose $R \cap A_3 y_3^- \neq \emptyset$. Let $z$ be the last vertex of $R$ that lies in $A_3 y_3^-$. Then $y_3^- A_3 z R z_2 A_2 \omega' A_3 y_3^+$ is an $y_3^-$-$y_3^+$ arc not intersecting $\{y_3, u_3\}$, contradicting Claim 21. Suppose $R \cap y_3^+ A_3 \neq \emptyset$. Let $z'$ be the first vertex of $R$ that lies in $y_3^+ A_3$. Then $y_2^- A_2 x_2 P x_1 A_1 z_1 R z' A_3 \omega' A_2 y_2^+$ is an $y_2^-$-$y_2^+$ arc not intersecting $\{y_2, u_2\}$, contradicting Claim 21. Therefore, $R \cap (A_1 x_1^- \cup A_2 y_2^- \cup A_3 \cup P \cup Q) = \emptyset$. But, $y_2^- A_2 x_2 P x_1 A_1 z_1 R z_2 A_2 y_2^+$ is an $y_2^-$-$y_2^+$ arc not intersecting $\{y_2, u_2\}$, contradicting Claim 21. ☐

**Claim 26.** *For each end $\omega$ in $|G|$, $deg_{G_C}(\omega) = 2$.*

*Proof.* Let $x$ be a vertex in $G_C$. Since $\overline{G_C}$ is arc-connected, there is an $\omega$-arc $A$ in $\overline{G_C}$ joining $x$ to $\omega$. Let $y$ be the neighbor of $x$ in $A$ and $a$ be an interior point of $xy$. Since $\overline{G_C}$ is 2-connected, $\overline{G_C} \setminus a$ is connected. Suppose $\overline{G_C - xy}$ is not connected. Then there exist two disjoint nonempty open sets $U$ and $V$ in $|G|$ such that $\overline{G_C - xy} \subseteq U \cup V$, $U \cap \overline{G_C - xy} \neq \emptyset$ and $V \cap \overline{G_C - xy} \neq \emptyset$. If $x, y \in U$, then $U \cup [x, a) \cup [y, a)$ and $V$ are two disjoint open sets in $|G|$ both intersecting $\overline{G_C} \setminus a$, and their union contains $\overline{G_C} \setminus a$, which is impossible. If $x \in U$ and $y \in V$, then $U \cup [x, a)$ and $V \cup [y, a)$ are two disjoint open sets in $|G|$ both intersecting $\overline{G_C} \setminus a$, and their union contains $\overline{G_C} \setminus a$, which is also impossible. Therefore, $\overline{G_C - xy}$ is connected and is arc-connected by Lemma 10. Let $A'$ be an $x$-$\omega$ arc in $\overline{G_C - xy}$. If $yA\omega \cap A'$ contains a vertex $u$, then $u$ has degree at least three in $G_C$, a contradiction. Let $\omega'$ be the first point in $yA\omega \cap A'$ which is an end. If $\omega' \neq \omega$, then $deg_{G_C}(\omega') \geqslant 3$ contradicting Claim 25. Therefore, $\omega' = \omega$ and we have $deg_{G_C}(\omega) \geqslant 2$. By Claim 25, $deg_{G_C}(\omega) = 2$. □

We are now ready to prove the infinite analog of Theorem 2.

**Theorem 27.** *Let $G$ be a 2-connected locally finite graph nonisomorphic to $K_3$. Then the following are equivalent:*

(1) *Every vertex of $G$ is incident to exactly two contractible edges.*

(2) *Every finite bond of $G$ contains exactly two contractible edges.*

(3) *$G$ is outerplanar.*

*Proof.*

(2) $\Rightarrow$ (1) Trivial.

(1) $\Rightarrow$ (3) By Theorem 2, this is true for finite $G$. Therefore, assume $G$ is infinite. Suppose every vertex of $G$ is incident to exactly two contractible edges. By Theorem 20, $\overline{G_C}$ is a Hamilton circle. All edges in $E(G) \setminus E_C$ are chords of $\overline{G_C}$ and are non-contractible. Consider any chord $xy$ of $\overline{G_C}$. Since every vertex of $G$ is incident to exactly two contractible edges, by Lemma 6, $G - x - y$ consists of exactly two components $C_1$ and $C_2$. Without loss of generality, assume that $x^+\overline{G_C}y^- \subseteq \overline{C_1}$ and $y^+\overline{G_C}x^- \subseteq \overline{C_2}$. Then there is no chord of $\overline{G_C}$ between $x^+\overline{G_C}y^-$ and $y^+\overline{G_C}x^-$. Hence, no chords of $\overline{G_C}$ are overlapping. Embed $\overline{G_C}$ in a circle of $\mathbb{R}^2$ and denote the embedding by $\phi$. Now, draw every chord $xy$ of $\overline{G_C}$ as a straight line segment between $\phi(x)$ and $\phi(y)$ in $\mathbb{R}^2$. This shows that $G$ is outerplanar.

(3) $\Rightarrow$ (2) Let $B$ be a finite bond of $G$ between two components $X$ and $Y$ of $G - B$. By Lemma 5(a), $B$ contains at least two contractible edges. Suppose $B$ contains three contractible edges $x_1y_1, x_2y_2, x_3y_3$ such that $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Since $X$ and $Y$ are connected, there exists a path $P$ in $X$ joining $x_1$ to $x_2$ and a path $Q$ in $Y$ joining $y_1$ to $y_2$. Let $C := P \cup x_1y_1 \cup Q \cup x_2y_2$. Obviously, $x_3y_3 \notin E(C)$. Take any $x_3$-$P$ path $P'$ in $X$ joining $x_3$ to $P$ at $x$ and any $y_3$-$Q$ path $Q'$ in $Y$ joining $y_3$ to $Q$ at $y$. Let $R' := P' \cup x_3y_3 \cup Q'$. If $R' = x_3y_3$, then $x_3y_3$ is a chord of $C$. Since $x_3y_3$ is contractible, the two components of $C - x_3 - y_3$ are joined by a path, say $R$. Then $C \cup R \cup R'$ is a

$K_4$-subdivision and $G$ is not outerplanar. Suppose $R' \neq x_3y_3$. If both $x$-$y$ paths in $C$ are not edges, then $C \cup R'$ is a $K_{2,3}$-subdivision and $G$ is not outerplanar. If one of the two $x$-$y$ paths in $C$ is an edge, then without loss of generality, assume $x_2 = x$ and $y_2 = y$. Since $x_2y_2$ is contractible, the two components of $(C \cup R') - x_2 - y_2$ are joined by a path, say $R$. Then $C \cup R \cup R'$ is a $K_4$-subdivision and $G$ is not outerplanar. $\square$


### Acknowledgements

The author would like to thank Henning Bruhn for suggesting Theorem 8, and the referees for comments that greatly improve the accuracy and presentation of the paper.


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