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Arboricity and tree-packing in locally finite graphs

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Abstract

Nash-Williams' arboricity theorem states that a finite graph is the edge-disjoint union of at most k forests if no set of ℓ vertices induces more than $k(\ell - 1)$ edges. We prove a natural topological extension of this for locally finite infinite graphs, in which the partitioning forests are acyclic in the stronger sense that their Freudenthal compactification—the space obtained by adding their ends—contains no homeomorphic image of S^1. The strengthening we prove, which requires an upper bound on the end degrees of the graph, confirms a conjecture of Diestel [The cycle space of an infinite graph, Combin. Probab. Comput. 14 (2005) 59–79]. We further prove for locally finite graphs a topological version of the tree-packing theorem of Nash-Williams and Tutte.
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1. Introduction

A criterion for the smallest number of acyclic subgraphs of a finite graph whose union contains the entire graph is given by Nash-Williams' arboricity theorem:

Theorem 1 (*Nash-Williams [10]*). *Let $k \in \mathbb{N}$, and let G be a finite multigraph in which no set of ℓ vertices induces more than $k(\ell - 1)$ edges. Then G is the edge-disjoint union of at most k forests.*

Theorem 1 easily extends to locally finite graphs by compactness, if a forest is defined as a graph that contains no *finite* cycles. However, recent studies of the cycle space of infinite graphs—see Diestel [2]—suggest that in an appropriate infinite analogue of Nash-Williams' theorem the forests should not be allowed to contain 'infinite cycles' either. These are infinite subgraphs of G

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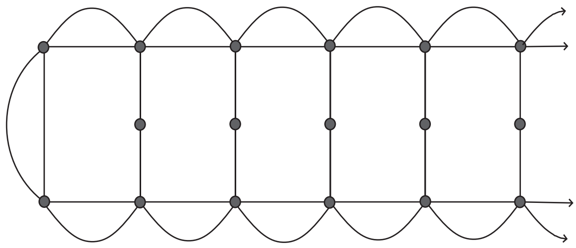

Fig. 1. Every two forests partitioning the multigraph contain infinite cycles.

whose closure in $|G|$, the compactification of G by its ends (see Section 2), is homeomorphic to the unit circle.

An infinite version of Theorem 1 based on such *topological forests* would be much stronger. So much so, in fact, that without additional constraints it is false. Indeed, consider the infinite ladder in which each rung except the first has been subdivided and all other edges duplicated (see Fig. 1). This multigraph satisfies the condition of Theorem 1 for $k = 2$, because it is an edge-disjoint union of two (ordinary) forests, but clearly, every two such forests must each contain a double ray. That double ray forms an infinite cycle as its closure contains the graph's single end, to which the double ray's subrays converge.

We can easily generalise this counterexample to arbitrary $k \in \mathbb{N}$ and simple graphs. Simply replace each of the subgraphs of the form $(\{v, w\}, \{vw, vw\})$ with a simple finite graph H that is the union of k edge-disjoint spanning trees (for example $H = K^{2k}$), identifying v, w with distinct vertices of H. Then, as before, G is an edge-disjoint union of k ordinary forests, and hence satisfies Nash-Williams' condition that no set of ℓ vertices spans more than $k(\ell - 1)$ edges. But any partition of G into k forests induces such a partition in each copy of H, i.e. into spanning trees of H. Each of these contains a v–w path, so each of our k forests contains a double ray and thus an infinite cycle. These counterexamples are due to Bruhn and Diestel (personal communication).

In order to generalise Theorem 1 to topological forests, we thus need to impose some further conditions. One natural way to do this is to require local sparseness not only for all finite subgraphs (as in Nash-Williams' condition) but also around ends, e.g. by placing an upper bound on their 'degrees'. The *degree* of an end ω is defined in [1] to be the supremum of the cardinalities of sets of edge-disjoint rays in ω (for details, see Section 2). So, the counterexamples above each have an end of degree $\geqslant 2k$. It is not difficult to construct others whose end has degree exactly $2k$. Just choose H such that it contains a vertex of degree k and identify this vertex with v.

The following conjecture of Diestel [2], whose proof is the main result of this paper, is therefore best possible in this sense:

Theorem 2. *Let $k \in \mathbb{N}$, and let G be a locally finite graph in which no set of ℓ vertices induces more than $k(\ell - 1)$ edges. Further, let every end of G have degree $< 2k$. Then $|G|$ is the edge-disjoint union of at most k topological forests in $|G|$.*

Although, as we have seen, the bound of $2k$ in Theorem 2 cannot be reduced, the theorem has no direct converse: a partition into k topological forests does not force all end degrees to be small. The $\mathbb{N} \times \mathbb{N}$ grid, for example, is an edge-disjoint union of two topological forests (its horizontal vs. its vertical edges), but its unique end has infinite degree.

Our second topic in this paper is how to extend the well-known tree-packing theorem of Nash-Williams and Tutte. This will be treated in Section 5, which has been entirely contributed by Bruhn and Diestel. We need a standard definition: an edge is said to *cross* a given vertex partition of a graph G if it has its endvertices in distinct partition sets.

Theorem 3 (*Nash-Williams [9], Tutte [13]*). *For a finite multigraph G the following statements are equivalent*:

(i) *G contains k edge-disjoint spanning trees*;
(ii) *every partition of $V(G)$, into $r \in \mathbb{N}$ sets say, is crossed by at least $k(r-1)$ edges of G.*

Nash-Williams [11, Conjecture A] conjectured that Theorem 3 should extend to countable graphs. This was disproved by Oxley [12], who constructed a locally finite graph that satisfies (ii) for $k = 2$ but has no two edge-disjoint spanning trees. Tutte pursued a different approach: he proved that in a locally finite graph every vertex partition into r parts is crossed by at least $k(r-1)$ edges if and only if the graph has k edge-disjoint 'semi-connected' spanning subgraphs. In our topological context, the condition acquires an unexpected natural interpretation: it turns out that the 'semi-connected' spanning subgraphs of a graph G are precisely those whose closure in the compactification of G is topologically connected. Furthermore, the closure of every such subgraph contains a *topological spanning tree*. This then leads to a near-verbatim generalisation of the tree-packing theorem:

Theorem 4. *For a locally finite multigraph G the following statements are equivalent*:

(i) *$|G|$ contains k edge-disjoint topological spanning trees*;
(ii) *every partition of $V(G)$, into $r \in \mathbb{N}$ sets say, is crossed by at least $k(r-1)$ edges of G.*

2. The topological space $|G|$

The basic terminology we use can be found in Diestel [3]. A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*, and the subrays of a ray are its *tails*. Two rays in a graph G are *equivalent* if no finite set of vertices separates them; the corresponding equivalence classes of rays are the *ends* of G. We denote the set of ends of G by $\Omega(G)$.

Let us define a topology on G together with its ends. This topology was first introduced by Freudenthal [7] and Jung [8]. We begin by endowing G itself with the usual topology of a 1-complex. (Thus, every edge is homeomorphic to the real interval $[0, 1]$, and the basic open neighbourhoods of a vertex v are the unions of half-open intervals $[v, z)$, one for every edge e at v with z an inner point of e.) In order to extend this topology to $\Omega(G)$, we take as a basis of open neighbourhoods of a given end $\omega \in \Omega(G)$ the sets of the form $C(S,\omega) \cup \Omega(S,\omega) \cup E'(S,\omega)$ where $S \subseteq V$ is a finite set of vertices, $C(S,\omega)$ is the component of $G - S$ in which every ray from ω has a tail, $\Omega(S,\omega)$ is the set of all ends $\omega' \in \Omega(G)$ whose rays have a tail in $C(S,\omega)$, and $E'(S,\omega)$ is any union of half-edges $(z, y]$, one for every S–$C(S,\omega)$ edge $e = xy$ of G, with z an inner point of e. Let $|G|$ denote the topological space thus defined.

It is not overly difficult to see that if G is locally finite and connected, then $|G|$ is compact (it is then also known as the *Freudenthal compactification* of G). We shall freely view G and its subgraphs either as abstract graphs or as subspaces of $|G|$. For details on $|G|$, see Diestel and Kühn [6].

A cut of G *separates* a set $S \subseteq V(G)$ from an end $\omega \in \Omega(G)$ if it meets every ray of ω that starts in S. For a subgraph $H \subseteq G$, the *boundary* ∂H of H is the cut $E(H, G - H)$; in particular, $\partial G = \emptyset = \partial\emptyset$. A *region* of G is an induced subgraph which is connected and whose boundary contains only finitely many edges. Note that given a subgraph $H \subseteq G$ and an end $\omega \in \Omega(G)$ with $\omega \notin \overline{H}$ its boundary ∂H separates ω from $V(H)$.

An *arc* is a set $A \subseteq |G|$ homeomorphic to the unit interval; a *circle* is a set $C \subseteq |G|$ homeomorphic to the unit circle. The subset $C \cap G$ may be viewed as a subgraph of G, and will then be called a *cycle* of G. Clearly, this definition includes traditional finite cycles but also allows for infinite cycles, which are disjoint unions of certain sets of double rays. Diestel and Kühn have shown in [5] that for a circle C the closure of $C \cap G$ coincides with C, we may thus speak of the unique defining circle of a cycle. We need the following theorem.

Theorem 5 (*Diestel and Kühn [4]*). *Let G be a locally finite graph, and let $Z \subseteq E(G)$. Then Z is a cycle in G if and only if $|F \cap Z|$ is even for every finite cut F of G.*

Having adapted the notion of a cycle to our topological viewpoint, we must do the same for forests and, in particular, spanning trees. The closure $\overline{H}$ in $|G|$ of a subgraph H of G is a *topological forest* if it contains no circles. A *topological spanning tree* is a path-connected topological forest in $|G|$ that contains all vertices of G (it then also contains all ends and all edges of which it contains inner points). See Diestel and Kühn [6] for more information on topological spanning trees.

A fundamental property of a tree is that it contains a path between any two of its vertices. That is the reason why topological spanning trees are required to be path-connected rather than only topologically connected. The next theorem shows that this makes no difference in our case.

Theorem 6 (*Diestel and Kühn [6]*). *If G is locally finite, then every closed connected subset of $|G|$ is path-connected.*

It is an open problem whether Theorem 6 extends to arbitrary subsets of $|G|$.

Finally, let us see how the vertex degree notion can be extended to the ends of a locally finite graph G. We define the *degree* $d(\omega) \in \mathbb{N} \cup \{\infty\}$ of an end $\omega \in \Omega(G)$ as the supremum of the cardinalities of sets of edge-disjoint rays in ω. This concept (and a generalisation to end degrees in subgraphs) was introduced in [1], where also the following Mengerian criterion for measuring end degrees can be found.

Lemma 7 (*Bruhn and Stein [1]*). *Let G be a locally finite graph, and let $\omega \in \Omega(G)$. Then $d(\omega) = k \in \mathbb{N}$ if and only if k is the smallest integer such that every finite set $S \subseteq V(G)$ can be separated from ω with a finite cut F of cardinality k.*

3. Finitely many small cuts cut off all ends

Given a locally finite graph G, a finite set $S \subseteq V(G)$ and an end $\omega \in \Omega(G)$ of finite degree $< k$, Lemma 7 yields a cut of cardinality $< k$ that separates S from ω, and therefore induces a region $K_\omega \subseteq G - S$ whose closure contains ω. Now, if instead of ω we consider finitely many ends, each of degree $< k$, we can easily choose the K_ω disjoint (because there is a finite set $S' \supseteq S$ which separates our ends pairwisely).

Clearly, there is also a (possibly infinite) set of regions $K \subseteq G - S$ with $|\partial K| < k$ such that *every* end of degree $< k$ lies in the closure of one of them. But are we still able to choose these regions disjoint? The next lemma gives a positive answer to this question.

Lemma 8. *Let $k \in \mathbb{N}$, let G be a locally finite graph and let $S \subseteq V(G)$ be finite. Then there is a set $\mathcal{K}$ of disjoint regions $K \subseteq G - S$ of G with $|\partial K| < k$, such that for every $\omega \in \Omega(G)$ with $d(\omega) < k$ there is a $K \in \mathcal{K}$ with $\omega \in \overline{K}$.*

Proof. If S is empty, then $\mathcal{K} := \{G\}$ is as desired, so assume $S \neq \emptyset$. We use induction on k to prove the existence of a set $\mathcal{K}^k$ of disjoint regions of G such that for all $K \in \mathcal{K}^k$:

(i) $K \subseteq G - S$ and $|\partial K| < k$;
(ii) there is no finite set $\mathcal{H}$ such that $V(K) = \bigcup_{H \in \mathcal{H}} V(H)$ and $|\partial H| < |\partial K|$ for all $H \in \mathcal{H}$.
In addition, we require for all regions $K' \subseteq G - S$:

(iii) if $|\partial K'| < k$ then $E(K') - \bigcup_{K \in \mathcal{K}^k} E(K)$ is finite.

Then $\mathcal{K}^k$ is the desired set $\mathcal{K}$ of the lemma. Indeed, consider an end $\omega \in \Omega(G)$ with $d(\omega) < k$, and let $R \in \omega$. By Lemma 7, there is a region $K' \subseteq G - S$ of G with $\omega \in \overline{K'}$ such that $|\partial K'| < k$. By (iii), $E(K') - \bigcup_{K \in \mathcal{K}^k} E(K)$ is finite. Hence, R has only finitely many of its edges outside $\bigcup_{K \in \mathcal{K}^k} E(K)$. Thus, since the $K \in \mathcal{K}$ are pairwise disjoint, there is a $K \in \mathcal{K}$ such that R has a tail in K, implying $\omega \in \overline{K}$, as desired.

Put $\mathcal{K}^1 := \emptyset$, a choice which trivially satisfies (i) and (ii), and also (iii) because $S \neq \emptyset$ and we may suppose G to be connected (thus, the only $K' \subseteq G - S$ with $|\partial K'| < 1$ is $K' = \emptyset$). So assume we already found a set $\mathcal{K}^{k-1}$ satisfying (i)–(iii); we show the existence of the set $\mathcal{K}^k$.

Let $H_1, H_2, \ldots$ be an enumeration of all regions $H \subseteq G - S$ of G with $|\partial H| < k$ and $E(H) - \bigcup_{K \in \mathcal{K}^{k-1}} E(K)$ infinite (there are only countably many such regions as $E(G)$ is countable). From the H_i and $\mathcal{K}^{k-1}$ we construct a sequence of subgraphs $L_i \subseteq G - S$ as follows. Put $L_0 := \bigcup_{K \in \mathcal{K}^{k-1}} K$, and let for $i \in \mathbb{N}$

$$L_i := L_{i-1} \cup H_i \quad \textit{if } \partial H_i \cap E(L_{i-1}) = \emptyset$$

and $L_i := L_{i-1}$ otherwise. It is easily shown by induction that for all $i \in \mathbb{N}$ each component of L_i is a region that sends less than k edges to the rest of G. Now, put $L := \bigcup_{i=1}^{\infty} L_i$ and let $\mathcal{K}^k$ be the set of the components of L. Note that $\bigcup_{K \in \mathcal{K}^k} K = L \subseteq G - S$ and that the $K \in \mathcal{K}^k$ are induced subgraphs of G. Furthermore, $|\partial K| < k$ for each $K \in \mathcal{K}^k$, as otherwise there would already have been a component K' of L_i with $|\partial K'| \geqslant k$ for some $i \in \mathbb{N}$ (just choose i such that L_i contains at least k vertices which are incident with edges in ∂K plus finite paths that connect these vertices pairwisely). Thus, $\mathcal{K}^k$ is a set of disjoint regions for which (i) holds.

Let us show (ii). Suppose there are a $K \in \mathcal{K}^k$ and a finite set $\mathcal{H}$ such that $V(K) = \bigcup_{H \in \mathcal{H}} V(H)$ and $|\partial H| < |\partial K|$ for all $H \in \mathcal{H}$. By (i), $|\partial K| < k$, thus, $|\partial H| < k-1$ for all $H \in \mathcal{H}$. Then (iii) for $k-1$ yields that $E(H) - \bigcup_{K' \in \mathcal{K}^{k-1}} E(K')$ is finite for all $H \in \mathcal{H}$. As $|\mathcal{H}| < \infty$ and ∂H is bounded for all $H \in \mathcal{H}$, also $E(K) - \bigcup_{H \in \mathcal{H}} E(H)$ is finite, implying that $E(K) - \bigcup_{K' \in \mathcal{K}^{k-1}} E(K')$ is finite. Hence K contains none of the H_i used in the construction of $\mathcal{K}^k$, and thus $K = K'$ for some $K' \in \mathcal{K}^{k-1}$, contradicting (ii) for $k - 1$.

Finally, we prove (iii). Suppose there is a region $K' \subseteq G - S$ of G with $|\partial K'| < k$ such that $E(K') - \bigcup_{K \in \mathcal{K}^k} E(K) = E(K') - E(L)$ is infinite. Assume K' is chosen with $|\partial K' \cap E(L)|$ minimal. Because $K' = H_j$ for some $j \in \mathbb{N}$, there is a $K \in \mathcal{K}^k$ that contains edges of $\partial K'$, as otherwise $\partial K' \cap E(L_{j-1}) = \emptyset$, resulting in $K' \subseteq L_j \subseteq L$, which contradicts our assumption that $E(K') - E(L)$ is infinite.

Hence $\partial K' \cap E(K) \neq \emptyset$, but $\partial K \cap E(L) = \emptyset$, implying that $|\partial(K \cup K') \cap E(L)|$, $|\partial(K' - K) \cap E(L)| < |\partial K' \cap E(L)|$. As G is connected, $K' - K$ has only finitely many components, one of which is a region $K'' \subseteq G - S$ such that $E(K'') - E(L)$ is infinite and $|\partial K'' \cap E(L)| < |\partial K' \cap E(L)|$. Also $K \cup K'$ is such a region, thus, the choice of K' ensures that $|\partial(K \cup K')|$, $|\partial(K' - K)| \geqslant k$. Then

$$\begin{aligned}
|\partial(K \cap K')| + k &\leqslant |\partial(K \cap K')| + |\partial(K \cup K')| \\
&= |E(K \cap K', K - K')| + |E(K \cap K', K' - K)| \\
&\quad + |E(K - K', G - (K \cup K'))| \\
&\quad + |E(K' - K, G - (K \cup K'))| \\
&\quad + 2|E(K \cap K', G - (K \cup K'))| \\
&= |\partial K| + |\partial K'| - 2|E(K - K', K' - K)| \\
&\leqslant |\partial K| + |\partial K'| \\
&< |\partial K| + k,
\end{aligned}$$

and similarly

$$\begin{aligned}
|\partial(K - K')| + k &\leqslant |\partial(K - K')| + |\partial(K' - K)| \\
&\leqslant |\partial K| + |\partial K'| \\
&< |\partial K| + k.
\end{aligned}$$

Thus, $|\partial(K \cap K')|$, $|\partial(K - K')| < |\partial K|$. But then each of the finitely many components of $K \cap K'$ and of $K - K'$ sends $< |\partial K|$ edges to the rest of G, while $V(K) = V(K \cap K') \cup V(K - K')$, a contradiction to (ii). □

If, as is the case in Theorem 2, $d(\omega)$ is bounded for all $\omega \in \Omega(G)$, the set $\mathcal{K}$ from Lemma 8 has to be finite:

Lemma 9. *Let $k \in \mathbb{N}$, let G be a locally finite, connected graph, and let $S \subseteq V(G)$ be finite. Suppose that every $\omega \in \Omega(G)$ has degree $< k$. Then there is a finite number of disjoint regions $K_1, K_2, \ldots, K_n \subseteq G - S$ with $|\partial K_i| < k$ for all $i = 1, \ldots, n$ such that for every $\omega \in \Omega(G)$ there is an $i \leqslant n$ with $\omega \in \overline{K}_i$.*

Proof. Lemma 8 supplies us with a set $\mathcal{K}$ of disjoint regions K of G that have the desired properties. Then $\mathcal{K}$ induces an open cover of the compact end space $\Omega(G)$, and hence has a finite subcover $\mathcal{K}'$. Since we may assume that the closure of each $K \in \mathcal{K}$ contains at least one end of G, it follows that $\mathcal{K} = \mathcal{K}'$ is finite, as desired. □

4. Arboricity

In our proof of Theorem 2 we successively define certain finite sets $S_1 \subseteq S_2 \subseteq \cdots$ of vertices, together with partitions of $E(G[S_i])$. In order to extend the partition of $E(G[S_i])$ to a partition of $E(G[S_{i+1}])$, we want to use Theorem 1 on the graph $\tilde{G}$ obtained from $G[S_{i+1}]$ by contracting S_i to a vertex, which we can do if the arboricity condition holds for $\tilde{G}$. The following lemma ensures that there is a way to choose the S_i so that it does:

Lemma 10. *Let $k \in \mathbb{N}$, and let G be a locally finite graph in which no set of ℓ vertices induces more than $k(\ell-1)$ edges. Then for every finite $S \subseteq V(G)$ there is a finite $S' \subseteq V(G)$ with $S' \supseteq S$ such that $\|G[X]\| + |E(X, S')| \leqslant k|X|$ for each $X \subseteq V(G - S')$.*

Proof. Put $S_0 := S$, and for $i \geqslant 1$ successively define S_i as $S_{i-1} \cup X_i$ if there is an $X_i \subseteq V(G - S_{i-1})$ such that $\|G[X_i]\| + |E(X_i, S_{i-1})| > k|X_i|$. Observe that then X_i is finite. Either the process stops at some $I \in \mathbb{N}$ in which case we put $S' := S_I$ and are done, or we obtain an infinite sequence $S_0 \subseteq S_1 \subseteq \cdots$ together with the corresponding X_i. In the latter case, consider for $n := k|S|$ the set S_n. By choice of the X_i,

$$\|G[S_n]\| \geqslant \sum_{i=1}^{n}(k|X_i|+1) = k\left(\sum_{i=1}^{n}|X_i| + |S|\right) = k|S_n| > k(|S_n|-1),$$

contradicting our assumption that the arboricity condition holds for G. □

We define for a vertex $v \in V(G)$ and for $i \in \mathbb{N}$ the set $N_i(v)$ to be the set of all vertices with distance i to v (thus, in particular, $N_1(v) = N(v)$). For the set $\{N_i(x) : x \in X\}$, where $X \subseteq V(G)$ and $i \in \mathbb{N}$, we write $N_i(X)$.

Proof of Theorem 2. We successively define for all $i \in \mathbb{N}$ finite sets $S_i \subseteq V(G)$, together with k edge-disjoint forests $F_1^i, \ldots, F_k^i$, such that

(i) $S_{i-1} \cup N(S_{i-1}) \subseteq S_i$, for $i \geqslant 2$;
(ii) $F_j^{i-1} \subseteq F_j^i$, for $j = 1, \ldots, k$ and $i \geqslant 2$;
(iii) $\bigcup_{j=1}^{k} E(F_j^i) = E(G[S_i])$; and
(iv) If $C \subseteq G$ is a cycle so that $C \cap G[S_i] \subseteq F^i_{i \bmod k}$, then $V(C) \cap S_{i-1} = \emptyset$, for $i \geqslant 2$.

We claim that the (by (ii) well-defined) unions $\bigcup_{i=1}^{\infty} F_1^i, \ldots, \bigcup_{i=1}^{\infty} F_k^i$ are the desired topological forests. Indeed, (i) and (iii) ensure that their edge sets partition $E(G)$, since we may assume G to be connected. Suppose that there is a $j \in \{1, \ldots, k\}$ so that $\bigcup_{i=1}^{\infty} F_j^i$ contains an infinite cycle C of G. Let v be a vertex in $V(C)$. By (i), we can choose $i \geqslant 2$ so that $v \in S_{i-1}$ and $j = i \bmod k$. This contradicts (iv).

A further condition is needed to make the successive choice of the forests F_j^i possible. We require that for $i \in \mathbb{N}$

(v) $\|G[X]\| + |E(X, S_i)| \leqslant k|X|$ for every $X \subseteq V(G - S_i)$.

Let S_1 be any one-elemented subset of $V(G)$ and put $F_1^1, \ldots, F_k^1 := \emptyset$; this choice obviously satisfies (iii) and (v), which is all we required for $i = 1$. So suppose $i \geqslant 2$, and that $S_\ell, F_1^\ell, \ldots, F_k^\ell$ are already defined for $\ell < i$ and satisfy (i)–(v). Since S_{i-1} is finite, Lemma 9 yields a finite number of regions $K_1, \ldots, K_n \subseteq G - S_{i-1}$ of G such that $|\partial K_m| < 2k$ for all $m = 1, \ldots, n$ and such that every end of G lies in the closure of one of the K_m. Then $T := V(G - \bigcup_{m=1}^{n} K_m)$ has only finitely many components, none of which may contain a ray. Thus T is finite, hence, as $|\bigcup_{m=1}^{n} \partial K_m| < \infty$, also $S := T \cup \bigcup_{m=1}^{n} N(G - K_m)$ is finite. So Lemma 10 yields a finite $S' \supseteq S$. Put $S_i := S'$ and observe that conditions (i) and (v) are satisfied.

In order to define the forests $F_1^i, \ldots, F_k^i$, we consider the multigraph $\tilde{G}$ obtained from $G[S_i]$ by contracting S_{i-1} to the vertex s_{i-1}, keeping multiple edges but deleting loops (if necessary, we first make S_{i-1} connected by adding some extra edges). Note that $K_m \cap G[S_i] = K_m \cap \tilde{G} \subseteq \tilde{G}$ and furthermore, as $\partial K_m \subseteq E(G[S_i])$, also $\partial K_m \subseteq E(\tilde{G})$. Condition (v) for $i-1$ (together with the arboricity condition for G) implies that in the finite multigraph $\tilde{G}$, no set of ℓ vertices induces more than $k(\ell - 1)$ edges. Hence, by Theorem 1 there is a partition of $E(\tilde{G})$ into the edge sets of k forests $\tilde{F}_1, \ldots, \tilde{F}_k \subseteq \tilde{G}$. Let $I := i \bmod k$ and assume the $\tilde{F}_j$ are chosen so that $|E(\tilde{F}_I) \cap \bigcup_{m=1}^{n} \partial K_m|$ is minimal.

We claim that for $m = 1, \ldots, n$:

$$\text{all edges in } E(\tilde{F}_I) \cap \partial K_m \text{ are incident with the same component of } \tilde{F}_I \cap K_m. \tag{1}$$

Then the partition of $E(\tilde{G})$ into $E(\tilde{F}_1), \ldots, E(\tilde{F}_k)$ corresponds to a partition of $E(G[S_i]) - E(G[S_{i-1}])$ into the edge sets of k forests $F_1, \ldots, F_k \subseteq G[S_i]$. Put $F_j^i := F_j^{i-1} \cup F_j$ for $j = 1, \ldots, k$, and observe that F_j^i is a forest since F_j^{i-1} as well as F_j is acyclic, and any cycle meeting both contains a subgraph that corresponds to a cycle of $\tilde{F}_j$. This choice satisfies (ii) and (iii). In order to see (iv), let $C \subseteq G$ be a cycle with $C \cap G[S_i] \subseteq F_I^i$, and suppose $V(C) \cap S_{i-1} \neq \emptyset$. By Theorem 5, C meets each ∂K_m in an even number of edges. For every two edges in $E(C) \cap \partial K_m$ there is a path in $F_I^i \cap K_m$ that connects their endvertices in K_m, because of (1). So, if $E(C) \cap \partial K_m \neq \emptyset$, we can substitute $C \cap K_m$ with the union of these paths. Doing so successively for all m, we obtain a finite subgraph of the forest F_I^i, that has only vertices of degree $\geqslant 2$, and thus contains a cycle, which is impossible. This establishes (iv).

So, let us prove (1). Consider an $m \in \{1, \ldots, n\}$. As otherwise (1) is clearly satisfied for m, suppose that $|E(\tilde{F}_I) \cap \partial K_m| \geqslant 2$. Because $|\partial K_m| < 2k$, there is then a $j \in \{1, \ldots, k\}$ such that $|E(\tilde{F}_j) \cap \partial K_m| \leqslant 1$. We may assume that there indeed is an edge $e \in E(\tilde{F}_j) \cap \partial K_m$, as otherwise taking any edge from $E(\tilde{F}_I) \cap \partial K_m$ and adding it to $\tilde{F}_j$ clearly yields a better choice of the forests $\tilde{F}_1, \ldots, \tilde{F}_k$. Let $e = vw$ with $v \in V(K_m)$ and $w \in V(\tilde{G} - K_m)$.

Now, consider the graph $\tilde{F}$ obtained from $\tilde{F}_I$ by contracting the components of $\tilde{F}_I \cap K_m$ and of $\tilde{F}_I - K_m$, deleting loops. Then $E(\tilde{F}) = E(\tilde{F}_I) \cap \partial K_m$; furthermore, $\tilde{F}$ is a forest, as $\tilde{F}_I$ is one. Let $\tilde{v} \in V(\tilde{F})$ be the vertex whose branch-set in $\tilde{F}_I$ contains v. Choose $X \subseteq V(\tilde{F})$ with $\tilde{v} \in X$ such that the branch-set of each $x \in X$ lies in K_m and that every non-trivial component of $\tilde{F}$ has exactly one vertex in X. Now, put $E_1 := E(X, N_1(X)) \cup E(N_2(X), N_3(X)) \cup \cdots$ and $E_2 := E(N_1(X), N_2(X)) \cup E(N_3(X), N_4(X)) \cup \cdots$; these two sets clearly partition $E(\tilde{F})$. Observe that in $\tilde{G}$

$$\text{each component of } \tilde{F}_I - K_m \text{ is adjacent to at most one edge of } E_1, \tag{2}$$

and

$$\text{each component of } \tilde{F}_I \cap K_m \text{ is adjacent to at most one edge of } E_2 \cup e. \tag{3}$$

Put $H_\ell := \tilde{F}_\ell$ for $\ell \in \{1, \ldots, k\} \setminus \{I, j\}$ and let H_I, H_j be subgraphs of $\tilde{G}$ with

$$E(H_I) := E(\tilde{F}_I - K_m) \cup E_1 \cup E(\tilde{F}_j \cap K_m),$$

$$E(H_j) := E(\tilde{F}_j - K_m) \cup E_2 \cup e \cup E(\tilde{F}_I \cap K_m).$$

We claim that H_I and H_j are forests. Indeed, any cycle in H_I contains edges of E_1, and thus a path in $\tilde{F}_I - K_m$ that connects two edges of E_1, which is impossible, by (2). On the other hand, any cycle in H_j must contain edges of $E_2 \cup e$, and thus a path in $\tilde{F}_I \cap K_m$ that connects two edges of $E_2 \cup e$, a contradiction to (3).

Hence, as $E(H_1), \ldots, E(H_k)$ clearly partition $E(\tilde{G})$, and $E(\tilde{F}_I \cap \bigcup_{m=1}^{n} \partial K_m) = E(H_I \cap \bigcup_{m=1}^{n} \partial K_m) \cup E_2$, the choice of $\tilde{F}_1, \ldots, \tilde{F}_k$ implies that $E_2 = \emptyset$. Suppose there is an edge

$e' \in E_1 = E_1 \cup E_2 = E(\tilde{F}_I) \cap \partial K_m$, with endvertex x in K_m, such that there is no v–x path in $\tilde{F}_I \cap K_m$. Then put $H'_I := (V(H_I), E(H_I) - \{e'\})$ and $H'_j := (V(H_j), E(H_j) + \{e'\})$. Observe that H'_j is a forest, as any cycle in H'_j contains both e and e', and thus a v–x path in $\tilde{F}_I \cap K_m$, which is impossible. But since H'_I has less edges in $\bigcup_{m=1}^{n} \partial K_m$ than $\tilde{F}_I$, this contradicts the choice of $\tilde{F}_1, \ldots, \tilde{F}_k$.

So every edge in $E(\tilde{F}_I) \cap \partial K_m$ is incident with the component of $\tilde{F}_I - K_m$ that contains v, establishing (1). □

5. Tree-packing

Clearly, (i) of Theorem 3 is equivalent to G being the edge-disjoint union of k spanning connected subgraphs. Tutte [13] weakens this version of condition (i) in order to extend the theorem to locally finite graphs. He replaces 'connected' with 'semi-connected', defined as follows: A subgraph H of a multigraph G is *semi-connected* in G if every bipartition of $V(G)$ is crossed either by an edge of H or by infinitely many edges of G.

Theorem 11 (*Tutte [13]*). *For a locally finite multigraph G the following statements are equivalent*:

(i) *G is the edge-disjoint union of k spanning semi-connected subgraphs*;
(ii) *every partition of* $V(G)$, *into* $r \in \mathbb{N}$ *sets say, is crossed by at least* $k(r-1)$ *edges of G*.

Considering Oxley's [12] counterexample, Tutte's infinite version of his theorem is in a sense best possible, but it certainly lacks the intuitive appeal of Theorem 3. However, in our topological setting the extension suddenly becomes quite natural: semi-connectedness of a spanning subgraph $H \subseteq G$ is the same as topological connectedness of its closure in $|G|$.

Lemma 12. *A spanning subgraph H of a locally finite multigraph G is semi-connected in G if and only if its closure* $\overline{H}$ *in* $|G|$ *is topologically connected.*

For the proof, we need the following lemma.

Lemma 13 (*Diestel and Kühn [4]*). *Let U be an infinite set of vertices in a locally finite, connected graph. Then there exists a ray R together with an infinite set of disjoint U-*$V(R)$ *paths.*

Proof of Lemma 12. If H is not semi-connected, then $V(G)$ can be partitioned into two non-empty sets U, W with no H-edge and only finitely many G-edges going from U to W. Then the closures of $H[U]$ and of $H[W]$ are disjoint open subsets of $\overline{H}$ (in the subspace topology of $\overline{H} \subseteq |G|$), hence $\overline{H}$ is not topologically connected.

To prove the forward implication, suppose H is semi-connected in G but $\overline{H}$ is not topologically connected. Then $\overline{H}$ is a disjoint union of two open non-empty subsets. Let U and W be the intersections of those two subsets with $V(G)$; then H contains no U–W edge, we have $U \cup W = V(H) = V(G)$, and $\overline{U} \cap \overline{W} = \emptyset$ (closures taken in $\overline{H}$, but that is the same as in $|G|$ since $\overline{H}$ contains all ends). Since H is semi-connected, there is an infinite set F of U–W edges in G; let U_F be the set of vertices in U incident with an edge of F. By Lemma 13, G contains a ray R together with an infinite set of disjoint U_F–$V(R)$ paths. As F is infinite but G is locally finite, there is also an infinite set of disjoint W–$V(R)$ paths. Hence every neighbourhood of the end ω that contains R meets both U and W, i.e. $\omega \in \overline{U} \cap \overline{W} \neq \emptyset$, a contradiction. □

So the semi-connected subgraphs of Theorem 11 have closures that are topologically connected subsets of $|G|$. We now show that each of these contains a topological spanning tree:

Lemma 14. *Let G be a locally finite multigraph, and let H be a spanning subgraph of G such that $\overline{H} \subseteq |G|$ is topologically connected. Then $\overline{H}$ contains a topological spanning tree of $|G|$.*

Proof. Since H spans G and $\overline{H}$ is topologically connected, G is connected and therefore countable. Let $e_1, e_2, \ldots$ be an enumeration of $E(H)$. Put $H_0 := H$, and for $i \geqslant 1$ let $H_i := H_{i-1} - e_i$ if $\overline{H_{i-1} - e_i}$ is topologically connected, and $H_i := H_{i-1}$ otherwise. Obviously, all $\overline{H}_i$ are topologically connected.

So $T := \bigcap_{i=1}^{\infty} H_i$ spans G, and its closure $\overline{T}$ does not contain any circles. Indeed, suppose there is a circle $C \subseteq \overline{T}$, which then contains an edge, e_i say. Hence $H_i = H_{i-1}$, i.e. $\overline{H_{i-1} - e_i}$ is not topologically connected, and thus allows a partition into two disjoint open sets U, V. As $\overline{H_{i-1}}$ is topologically connected, U and V each contain an endpoint of e_i. These are the endpoints of an arc $A \subseteq C$ that avoids all inner points of e_i. As U and V are open and disjoint, $A \setminus (U \cup V) \neq \emptyset$. On the other hand, $A \subseteq \overline{T - e_i} \subseteq \overline{H_{i-1} - e_i} \subseteq U \cup V$, yielding the desired contradiction.

Hence, if T is semi-connected, then by Lemma 12 (and Theorem 6), $\overline{T}$ is a topological spanning tree of G. If T is not semi-connected, then G has a finite cut F that contains no edges of T. Thus, there is an $i \in \mathbb{N}$ such that $F \cap E(H_i) = \emptyset$. This yields a partition of $\overline{H}_i$ into two disjoint open sets, contradicting our assumption that $\overline{H}_i$ is topologically connected. □

We finally prove our generalisation of the tree-packing theorem to locally finite graphs:

Proof of Theorem 4. From Lemma 14 follows that (i) holds if and only if G is the edge-disjoint union of k spanning subgraphs with topologically connected closure in $|G|$. The latter in turn is equivalent to (ii), by Theorem 11 and Lemma 12. □

The tree-packing theorem can also be generalised to some graphs that are not locally finite: those in which no two vertices are linked by infinitely many independent paths. Since no two vertices in such a graph dominate the same end (send infinite fans to its rays), we can identify every dominated end with the unique vertex that dominates it, without changing the topology on G itself. The resulting topology on $|G|$, ITOP, was introduced and studied in Diestel and Kühn [4]. The proof of Theorem 4 generalises to such graphs with ITOP (but not with TOP) in a straightforward way, which appears to be the most general natural form that the infinite tree packing theorem can take.

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