Uniqueness of electrical currents in a network of finite total resistance

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Abstract

We show that if the sum of the resistances of an electrical network N is finite, then there is a unique electrical current in N provided we do not allow, in a sense, any flow to escape to infinity.

1 Introduction

Electrical networks are physical objects, but also useful tools in mathematics. For example, they are closely related to random walks [21], they find applications in the study of Riemannian manifolds [22], and they are related to various problems in combinatorics [4]. An electrical network N has an underlying graph G and a function $r : E(G) \to \mathbb{R}^+$ assigning resistances to the edges of G. If G is finite, then the electrical current in N —between two fixed vertices p, q and with fixed intensity I— is the unique flow satisfying Kirchhoff's cycle law. If G is infinite then there may be several such flows, and one of the standard problems in the study of infinite electrical networks is to specify under what conditions such a flow is unique, see e.g. [24, 26].

Our main result is that if the sum of all resistances in a network N is finite, then there is a unique electrical current in N, provided we do no allow any flow to escape to infinity; more precisely, we require that for every finite edge-cut F of G that does not separate the source p from the sink q the net flow through F is zero. A flow satisfying this condition is said to be *non-elusive*. To see the necessity of this requirement consider the network of Figure 1. This network admits several

Figure 1: A flow escaping to infinity

flows, all of which satisfy Kirchhoff's cycle law as there are no cycles: one of

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these flows runs only along the edge pq; another can be obtained by sending a flow from p all the way to the left end, and collecting the same amount of flow from the right end into q as depicted in Figure 1. We consider the latter flow to be rather pathological, since it is permitted even if we disconnect the graph by removing the edge pq. And indeed, it is not a non-elusive flow. The interested reader will experiment with more complicated networks, and convince himself that requiring a flow to be non-elusive is a natural way to prevent a flow from using infinity in an abusive manner. Non-elusive flows allowed a generalisation of the well-known Max-Flow Min-Cut theorem to infinite networks [1] (non-elusive flows were called finite-cut-respecting flows in the latter paper).

We can now state our main result:

Theorem 1.1. *Let $N = (G, r, p, q, I)$ be a locally finite network with $\sum_{e \in E} r(e) < \infty$. Then there is a unique non-elusive p–q flow with intensity I and finite energy in N that satisfies Kirchhoff's cycle law.*

The energy of a flow f is defined by $W(f) := \sum_{e \in E(G)} f^2(e) r(e)$. The requirement that the energy be finite is necessary (see Section 5), and very common in the literature.

The essential part of Theorem 1.1 is the uniqueness rather than the existence of the desired flow. Let us briefly consider the case when G is finite. To prove uniqueness in that case, suppose there are two flows with the required properties, and consider their difference z. Then z is a circulation in G, and so there must be some cycle along which z is always positive in the same direction. But such a cycle yields a contradiction to the fact that z must satisfy Kirchhoff's cycle law, as it is the difference of two flows that do. Now back to the case when G is infinite, the above argument breaks down as an infinite circulation need not traverse any finite cycle in the same direction. However, it is possible to prove that if a circulation is non-elusive then it must traverse some (finite or infinite), *topological circle*; that is, a homeomorphic image of the real unit circle S^1 in the *end-compactification* $|G|$ of G. Such circles, introduced by Diestel and Kühn [12], have been the object of intense study recently [9, Section 8.5], and some of the acquired machinery is used here; see Section 2.1 for more about circles. As an intermediate result we obtain that these circles must also satisfy Kirchhoff's cycle law if all finite cycles do.

Our main result also relies on some recent results from [15] about topologies on graphs induced by an assignment of lengths to the edges; see Section 2.3 for details.

The proof of Theorem 1.1 spans Sections 3 to 5. In Section 6 we discuss the complementary case of finite total conductance. In Section 7 we extend to non-locally-finite graphs. Finally, in Section 8 we discuss the relation of our results to stochastic processes and offer a conjecture related to the Dirichlet problem.

2 Definitions and basic facts

We will be using the terminology of Diestel [9] for graph theoretical terms and the terminology of [2] for topological ones.

A network is a tuple $N = (G, r, p, q, I)$, where G is an (undirected) (multi–) graph, r is a mapping assigning a *resistance* $r(e) \in \mathbb{R}^+$ to each edge e of G, $p, q \in V(G)$, and $I \in \mathbb{R}$ is a constant. A *flow* in N is a real-valued *p–q flow*

in G with *intensity* I (intuitively p and q are connected to a battery generating a constant current of intensity I). A flow f satisfies, by definition, Kirchhoff's node law at every vertex except p and q, and the net flow leaving p is the *intensity* of f. More formally, we have

Kirchhoff's node law: For every vertex $x \in V(G)$ there holds

$$\sum_{xy \in E(G)} f(x,y) = \begin{cases} 0 & \text{if } x \neq p, q, \\ I & \text{if } x = p, \\ -I & \text{if } x = q. \end{cases} \qquad (K1)$$

(Recall that f is a function from V^2 to $\mathbb{R}$ satisfying $f(x,y) = -f(y,x)$ for every $xy \in E(G)$.) A *circulation* is a flow of intensity 0.

A flow f in N is called *non-elusive* if for every finite cut (X, X') such that both p, q lie in X (or both lie in X') there holds $f(X, X') = 0$, where $f(X, X') = \sum_{xy \in E, x \in X, y \in X'} f(x,y)$. It follows that if $p \in X$ and $q \in X'$ then $f(X, X') = f(\{p\}, V(G) - p) = -f(\{q\}, V(G) - q) = I$. Note that if G is finite or 1-ended (see Section 2.1 for the definition of an end) then every flow is non-elusive.

If G is finite, then one defines the *electrical current* i to be the flow in N satisfying Kirchhoff's cycle law, and it is well-known —and not hard to prove— that this flow always exists and that it is unique.

A *directed cycle* is a cycle together with a choice of one of its two possible orientations. If C is a directed cycle then we let $\vec{E}(C)$ denote the set of ordered pairs (x,y) such that xy is an edge of C traversed from x to y in the chosen orientation of C.

Kirchhoff's cycle law: For every directed cycle C in G there holds

$$\sum_{\vec{e} \in \vec{E}(C)} v(\vec{e}) = 0 \qquad (K2)$$

where $v(\vec{e}) := i(\vec{e})r(\vec{e})$ is the *voltage drop* or *potential difference* induced by i along e (in physics, the latter equation is known as *Ohm's law*).

An important concept for both finite and infinite electrical networks is that of energy: the *energy of the flow f* is defined by $W(f) := 1/2 \sum_{xy \in E(G)} f^2(x,y)r(e)$. ($W(f)$ is usually called "energy" in mathematics, but in physics it is called "power".)

The following theorem is well known, and indicates the importance of the concept of energy for electrical networks. See Section 4 for a proof.

Theorem 2.1. *Let N be a finite network and let $W = W(I, p, q)$ be the infimum of $W(f)$ over all flows f in N. Then, there is a unique flow i in N satisfying $W(i) = W$. This flow satisfies Kirchhoff's cycle law.*

2.1 Ends, the Freudenthal compactification and wild circles in graphs

Let G be a graph, fixed throughout this section.

A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*. A *tail* of a ray R is an infinite (co-final) subpath of R. Two rays R, L in G are *equivalent* if no finite set of edges separates them. The corresponding equivalence classes of rays are the *ends* of G. We denote the set of ends of G by $\Omega = \Omega(G)$.

We now endow the space consisting of G, considered as a 1-complex, and its ends with the topology $|G|$. Firstly, every edge $e \in E(G)$ inherits the open sets corresponding to open sets of $[0, 1]$. Moreover, for every finite edge-set $S \subset E(G)$, we declare all sets of the form

$$C(S, \omega) \cup \hat{\Omega}(S, \omega) \cup E'(S, \omega) \tag{1}$$

to be open, where $C(S, \omega)$ is any component of $G - S$ and $\hat{\Omega}(S, \omega)$ denotes the set of all ends of G having a ray in $C(S, \omega)$ and $E'(S, \omega)$ is any union of half-edges $(z, y]$, one for every edge $e = xy$ in S with y lying in $C(S, \omega)$. Let $||G||$ denote the topological space of $G \cup \hat{\Omega}$ endowed with the topology generated by the above open sets. Moreover, let $|G|$ denote the space obtained from $||G||$ by identifying any two points that have the same open neighbourhoods. (Our notation is slightly non-standard: our "ends" are usually called edge-ends in the literature, and the symbol $|G|$ often denotes a different space if G is non-locally-finite. However, for a locally finite G these differences disappear and our notation agrees with the mainstream.) If a point x of $|G|$ resulted from the identification of a vertex with some other points (possibly also vertices), then we will, with a slight abuse, still call x a *vertex*. It is easy to see that two vertices v, w of G are identified in $|G|$ if and only if there are infinitely many edge-disjoint v–w paths.

It it well-known (see [11]) that $|G|$ coincides with the *Freudenthal compact-ification* [14] of the 1-complex G if G is locally finite.

The study of $|G|$, in particular of topological circles therein, has been a very active field recently. It has been demonstrated by the work of several authors ([5, 6, 7, 8, 12, 13, 17, 16, 18, 25]) that many well known results about paths and cycles in finite graphs can be generalised to locally finite ones if the classical concepts of path and cycle are interpreted topologically, i.e. replaced by the concepts of a (topological) arc and circle in $|G|$; see Section 2.2 for some definitions and [9, Section 8.5] for an exposition of this field. An example of such a topological circle is formed by a double ray both rays of which converge to the same end together with that end. There can however be much more exciting circles in $|G|$: in Figure 2, the $\aleph_0$ many thick double rays together with the continuum many ends of the graph combine to form a single topological circle W, the so-called *wild* circle. The double rays are arranged within W like the rational numbers within the reals: between any two there is a third one; see [12] for a more precise description of W.

For a circle C we let $E(C)$ denote the set of edges it traverses. We define directed circles and the notation $\vec{E}(C)$ similarly to the case when C is a finite cycle.

2.2 Topological paths, circles, etc.

A *circle* in a topological space X is a homeomorphic copy of the unit circle S^1 of $\mathbb{R}^2$ in X. An *arc* R in X is a homeomorphic image of the real interval $[0, 1]$ in X. Its *endpoints* are the images of 0 and 1 under any homeomorphism from $[0, 1]$ to R. If $x, y \in R$ then xRy denotes the subarc of R with endpoints x, y. A *topological path* in X is a continuous map from a closed real interval to X.

Let $\sigma : [a, b] \to X$ be a topological path in a metric space (X, d). For a finite sequence $S = s_1, s_2, \ldots, s_k$ of points in $[a, b]$, let $\ell(S) := \sum_{1 \le i < k} d(\sigma(s_i), \sigma(s_{i+1}))$,
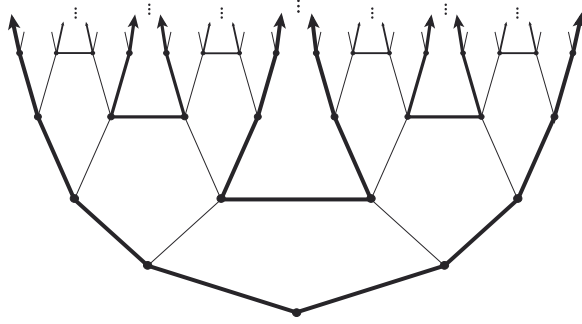
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Figure 2: The 'wild' circle of Diestel and Kühn, formed by infinitely many (thick) double rays and continuum many ends.

and define the *length* of σ to be $l(\sigma) := \sup_S \ell(S)$, where the supremum ranges over all finite sequences $S = s_1, s_2, \ldots, s_k$ with $a = s_1 < s_2 < \ldots < s_k = b$. If C is an arc or a circle in (X, d), then we define its length $l(C)$ to be the length of a surjective topological path $\sigma : [0, 1] \to C$ that is injective on $(0, 1)$; it is easy to see that $l(C)$ does not depend on the choice of σ.

2.3 ℓ-TOP

Fix a graph G and a function $\ell : E(G) \to \mathbb{R}_*^+$. This naturally gives rise to a distance function d_ℓ between the points of G, and we let $|\mathrm{G}|_\ell$, also called ℓ-TOP, denote the corresponding metric space.

To make this more precise, for each edge $e \in E(G)$ fix a homeomorphism σ_e from e to the real interval $[0, \ell(e)]$; by means of σ_e, any *half-edge* f, i.e. any connected subset of an edge, with endpoints a, b obtains a length $\ell(f)$, namely $\ell(f) := |\sigma_e(a) - \sigma_e(b)|$. Now use ℓ to define a distance function on G: for any $x, y \in V(G)$ let $d_\ell(x, y) = \inf_{P \text{ is an } x-y \text{ path}} \ell(P)$, where $\ell(P) := \sum_{e \in E(P)} \ell(e)$. For points $x, y \in G$ that might lie in the interior of an edge we define $d_\ell(x, y)$ similarly, but instead of graph-theoretical paths we consider arcs in the 1-complex G: let $d_\ell(x, y) = \inf_{P \text{ is an } x-y \text{ arc}} \left(\sum_{f \text{ is an edge or half-edge in } P} \ell(f) \right)$. By identifying any two vertices x, x' of G for which $d_\ell(x, x') = 0$ holds we obtain a metric space $(\overline{G}, d_\ell)$. Note that if G is locally finite then $\overline{G} = G$. Let $|\mathrm{G}|_\ell$ be the completion of $(\overline{G}, d_\ell)$.

The *boundary points* of G are the elements of the set $\partial^\ell G := |\mathrm{G}|_\ell \backslash \pi(\overline{G})$, where π is the canonical embedding of $\overline{G}$ in its completion $|\mathrm{G}|_\ell$.

The space $|\mathrm{G}|_\ell$ was introduced in [15], where several important special cases were found, and many basic facts were proved. Here we list some of these facts that we are going to use in this paper. For the first of them, it is easy to realize the connection to the current paper by interpreting the resistance of an edge as its length:

Theorem 2.2 ([15]). *If* $\sum_{e \in E(G)} \ell(e) < \infty$ *then* $|G|_\ell \approx |G|$.

(Where "$\approx$" means that the identity on G extends to a homeomorphism between the two spaces.)

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For our next lemma, fix an enumeration $e_0, e_1, \ldots$ of $E(G)$, and let $E_n := \{e_0, \ldots, e_n\}$. Moreover, let $\mathring{e}_n$ denote the set of inner points of the edge e_n, and let $\mathring{E}_n := \bigcup\{\mathring{e}_0, \ldots, \mathring{e}_n\}$.

Lemma 2.3 ([15])**.** *Let C be a circle or arc in $|G|_\ell$ such that $E(C)$ is dense in C. Then, for every $\epsilon \in \mathbb{R}^+$ there is an $n \in \mathbb{N}$ such that for every subarc of C in $|G|_\ell \backslash \mathring{E}_n$ connecting two vertices v, w there holds $d_\ell(v, w) < \epsilon$.*

Lemma 2.4 ([15])**.** *If $\sum_{e \in E(G)} \ell(e) < \infty$ then for every circle or arc C in $|G|_\ell$ there holds $l(C) = \sum_{e \in E(C)} \ell(e)$.*

3 Intermediate results

3.1 Splitting infinite circles into cycles

In this section we show that it is possible to write any infinite circle in a graph as a sum of a sparse family of finite cycles. We will later use this fact to show that the infinite circles of a network must also satisfy $(K2)$ if all finite ones do, which will play an important role in the proof of our main result in Section 5.

Call a family of edge sets *sparse* if no edge appears in more than three members of the family.

Lemma 3.1. *Let G be a countable graph and let C be a directed circle in $|G|$. Then there is a sparse family $\{D_n\}_{n \in \mathbb{N}}$ of finite directed cycles D_n in G such that $\vec{E}(C) = \sum_n \vec{E}(D_n)$,*

where the latter sum is a formal sum of directed edges with coefficients in $\mathbb{Z}$ in which two edges cancel out if they correspond to the same edge with opposite orientations. (Lemma 3.1 was proved implicitly also in [10].)

Proof. Let $e_0, e_1, \ldots$ be an enumeration of $E(G)$ and let $\vec{f}_0, \vec{f}_1, \ldots$ be an enumeration of $\vec{E}(C)$. Pick an assignment $\ell : E(G) \to \mathbb{R}_*^+$ such that $\sum_{e \in E(G)} \ell(e) < \infty$, and recall that by Theorem 2.2 there holds $|G|_\ell \approx |G|$.

We will construct $\{D_n\}_{n \in \mathbb{N}}$ recursively, in ω steps. To begin with, Let D_0 be any finite cycle containing f_0, directed in the direction of $\vec{f}_0$. Combined with C this cycle gives rise to what we call a *quasi-circle*: that is, a continuous image of S^1 comprising a finite path (in this case $D_0 \backslash f_0$), called the *green* part of the quasi-circle, and a subarc of C (in this case $C \backslash f_0$), called its *blue* part. Note that the green and blue part of a quasi-circle might have some common vertices and edges. We now construct the other D_i inductively, in ω steps. For $i = 1, 2, \ldots$, suppose we have already constructed $D_0, \ldots D_{i-1}$, and specified a set $\mathcal{Q}_{i-1}$ of quasi-circles (let $\mathcal{Q}_0$ be the singleton containing the above quasi-circle) so that

(i) for every $\vec{f} \in \vec{E}(C)$, either $\vec{f} \in \sum_{j < i} \vec{E}(D_j)$ (in the right direction) or f is contained in the blue part of some quasi-circle in $\mathcal{Q}_{i-1}$;

(ii) for any two distinct elements L, M of $\mathcal{Q}_{i-1}$, the blue parts of M and L are disjoint and the green parts of M and L are edge-disjoint.

(iii) the green part of every element of $\mathcal{Q}_{i-1}$ is contained in $\bigcup_{j < i} D_j$.

Then, in step i, let $k = k(i)$ be the least index such that $\vec{f}_k \notin \sum_{j<i} \vec{E}(D_j)$ (we choose this k even if $\overleftarrow{f_k} \in \sum_{j<i} \vec{E}(D_j)$). By (i) there is an $M = M(i) \in \mathcal{Q}_{i-1}$ the blue part of which contains f_k, and by (ii) this M is unique.

Let $h = h(i) := \min\{\ell(e) \mid e \in \bigcup_{j<i} E(D_j) \cup \{f_k\}\}$. By our choice of ℓ and Lemma 2.2, we can apply Lemma 2.3 to C, which yields an $n = n(i)$ such that for every subarc of C in $|G|_\ell \setminus \mathring{E}_n$ connecting two vertices v, w there holds $d_\ell(v, w) < h$. As the blue part M_b of M is, by construction, a subarc of C, the latter assertion also holds for M_b. Let x, y be the endpoints of M_b, and let M_b^x (respectively, M_b^y) be the component of $M_b \setminus \mathring{f}_k$ containing x (resp. y). We may assume without loss of generality that

$$h < d(M_b^x, M_b^y)/2, \tag{2}$$

where $d(M_b^x, M_b^y)$ is the minimum distance of a point in M_b^x from a point in M_b^y, for otherwise we could have chosen a smaller bound than h before applying Lemma 2.3.

Our aim now is to construct the directed trail D_i that contains $\vec{f}_k$ as well as the green part M_g of M, the latter traversed in the opposite direction. To achieve this, we need to construct two paths P_x, P_y, edge-disjoint from each other and from M_g, each path joining an endvertex of f_k to an endvertex of M_g.

To construct P_x, let $\vec{g}_1, \ldots, \vec{g}_r$ be an enumeration of the directed edges in $E_n \cap E(M_b^x)$ in the order and orientation they appear on M_b^x as we move from f_k to x. Let P_x^1 be a path from the endvertex of f_k in M_b^x to the tail of $\vec{g}_1$, and for every $1 < j \leq r$ let P_x^j be a path from the head of $\vec{g}_j$ to the tail of $\vec{g}_{j+1}$. Finally, let P_x^{r+1} be a path from the head of $\vec{g}_r$ to x (some of these paths may be trivial). We can now combine these paths with the edges $g_1, \ldots, g_r$ to obtain a directed f_k-x walk, and shortcutting this walk if necessary we can transform it into a path, which path we call P_x. Note that shortcutting a walk does not influence the order and direction in which the remaining edges are traversed. We construct P_y similarly. Note that by our choice of E_n, we could have chosen the paths P_x^j so that $\ell(P_x^j) < h$ for every j. This, and the choice of h, implies that no P_x^j can contain an edge in $\bigcup_{j<i} D_j$ as any such edge is longer than P_x^j, from which we obtain that

$$P_x \cup P_y \text{ contains no edge in } \left(\bigcup_{j<i} D_j\right) \setminus (E_n \cap E(C)). \tag{3}$$

Moreover, by (2) we may assume that P_x and P_y are disjoint; indeed, if some P_x^j has a vertex in common with some $P_y^{j'}$, then their union $P_x^j \cup P_y^{j'}$ contains an M_b^x-M_b^y path of length at most $\ell(P_x^j) + \ell(P_y^{j'}) < 2h < d(M_b^x, M_b^y)$, a contradiction.

Let $D_i := \vec{f}_k P_x(-M_g)(-P_y)$, where $(-P)$ denotes the path P traversed in the inverse direction. Note that D_i is not necessarily a cycle, as we would like it to be, but rather a closed walk, but we will later modify it into a sum of cycles. By the construction of D_i we have

$$D_i \text{ traverses } M_g, \text{ and does so in the opposite direction as } M \text{ does.} \tag{4}$$

Moreover, D_i traverses no edge in $E(G) \setminus E(C)$ more than once and traverses each edge in $E(C)$ at most twice (the latter can occur for an edge that happens to lie in $E_n \cap E(C) \cap M_g$).

To complete step i, it remains to define $\mathcal{Q}_i$. To obtain $\mathcal{Q}_i$ from $\mathcal{Q}_{i-1}$, we remove M and add the quasi-circles obtained as follows. Consider the subspace $M_b^x \backslash (E(P_x) \cap \{\mathring{g}_1, \dots, \mathring{g}_r\})$ of M_b^x; note that for every component K of this subspace there is a subpath P_K of P_x connecting the endvertices of K. Now $K \cup P_K$ defines a quasi-circle whose blue part is K and whose green part is P_K. Similarly for M_b^x and P_y. Add all these quasi-circles to $\mathcal{Q}_{i-1} \backslash M$ to obtain $\mathcal{Q}_i$. It follows from our choice of P_K, (iii) and (3) that

> no edge appears in the green part of more than one element of $\mathcal{Q} := \bigcup \mathcal{Q}_i$. (5)

Moreover, it is easy to check that

> D_i is the edge-disjoint union of edges in $\{g_1, \dots, g_r\} \subset E(C)$ and green parts of elements of $\mathcal{Q}_i$. (6)

Finally, it is easy to see that (i), (ii) and (iii) are all satisfied by $\mathcal{Q}_i$ if they were satisfied by $\mathcal{Q}_{i-1}$ (which is the case by our inductive hypothesis).

We have thus constructed the family $\{D_n\}_{n \in \mathbb{N}}$, and by (5) this family has the desired property that no edge is traversed more than three times (in fact, an edge in $E(G) \backslash E(C)$ is traversed either twice (in opposite directions) or not at all, and an edge in $E(C)$ is traversed either once or three times). Note that by our choice of the edge f_k, every edge of $E(C)$ will eventually appear in the sum $\sum_{j<i} \vec{E}(D_j)$, and in fact with the right orientation, for some step i. It also follows from our choice of the edge f_k that every quasi-circle in $\mathcal{Q}$ will be considered as $M(i)$ for some step i, and so by (4) its green part will eventually disappear from the sum of the D_n. It follows from these observations and (6) that $\sum_{n \in \mathbb{N}} \vec{E}(D_n) = \vec{E}(C)$.

Thus, the family $\{D_n\}_{n \in \mathbb{N}}$ has all the desired properties except that D_n is not necessarily a cycle but rather a closed walk that might traverse some edges twice and visit vertices more often. This, however, is easy to amend: if D_n visits some vertex x more than once, then we can split it into closed subwalks that each visit x only once, while traversing each edge in the same direction as D_n does. Performing this operation recursively, we can split D_n into a family $D'_1, \dots D'_k$ of directed cycles such that $\sum \vec{E}(D'_j) = \vec{E}(D_n)$, and so replacing each D_n by such a family we obtain the desired result. $\square$

Interestingly, in the last proof we used $\ell\text{-}TOP$ to prove an assertion that at first sight does not seem to be related to it. I would be interested to see a graph-theoretical proof of Lemma 3.1.

Lemma 3.1 motivates the following problem.

Problem 3.1. *Let G be a locally finite graph, and let C be a circle in $|G|$. Prove that G has a planar subgraph H containing $E(C)$ such that $E(C)$ is also the edge-set of a circle in $|H|$ (so that H can be drawn in the plane with $E(C)$ bounding a face).*

3.2 Shortcutting topological paths

It is a well-known and useful fact that the image of a topological path in a Hausdorff space contains an arc with the same endpoints [19]. For our uniqueness proof in Section 5 we will need a stronger version of this fact, saying, intuitively,

that this arc can be chosen so that its points are traversed in the same order as in the original topological path. In order to state this more formally, let $\sigma, \tau : [0, 1] \to X$ be topological paths in a Hausdorff space X. We say that τ *shortcuts* σ if there is a monotone increasing injection $m : [0, 1] \to [0, 1]$ such that $\tau(x) = \sigma \circ m(x)$ for every $x \in [0, 1]$. We will prove that

Lemma 3.2. *Let $\sigma : [0, 1] \to X$ be a topological path in a Hausdorff space X, and suppose its endpoints $u := \sigma(0), v := \sigma(1)$ are distinct. Then, there is an injective topological u–v path in X that shortcuts σ.*

For this we are going to need the following

Lemma 3.3 ([19, Lemmas 5.12, 5.13]). *Let X be a Hausdorff space, and let $\sigma : [0, 1] \to X$ be a topological path in X. Then, there exists a closed subset F of $[0, 1]$ such that*

(i) $\sigma(0), \sigma(1) \in \sigma(F)$;

(ii) *If C is any component of $[0, 1] \backslash F$ and x, y are the end points of C, then $\sigma(x) = \sigma(y)$;*

(iii) *No proper closed subset of F satisfies both (i) and (ii), and*

(iv) $\sigma(F)$ *is an arc.*

We now proceed with the proof of Lemma 3.2.

Proof. Apply Lemma 3.3 to σ to obtain a subset F of $[0, 1]$ with properties (i)-(iv). By (iv) $A := \sigma(F)$ is an arc, so let $b : A \to [0, 1]$ be a homeomorphism. Define the metric d_A on A by $d_A(x, y) := |b(x) - b(y)|$; clearly, d_A is compatible with the topology of A.

We now define a mapping $\sigma' : [0, 1] \to X$ that will be an intermediate step towards the construction of τ. For every point $x \in F$ let $\sigma'(x) = \sigma(x)$. For every component C of $[0, 1] \backslash F$, recall that by (ii) $\sigma(x) = \sigma(y) = p$ where x, y are the end points of C, and let $\sigma(z) := p$ for every $z \in C$. Obviously, σ' is a topological path. Moreover, it is easy to see that σ' shortcuts σ; indeed, just define $m : [0, 1] \to [0, 1]$ to map any point $x \in F$ to itself and any point $z \notin F$ to the endpoint of the component of $[0, 1] \backslash F$ in which it lies.

Note that σ' is *almost injective*, that is, the preimage of any point is an interval of $[0, 1]$. This implies that the length h of σ' with respect to d_A is finite, since, easily, any almost injective path in $[0, 1]$ has finite length. Thus we may define a mapping $\tau : [0, 1] \to \sigma(F)$ by mapping any point $x \in [0, 1]$ to a point $\sigma'(y)$ such that the restriction $\sigma' \restriction [0, y]$ of σ' to $[0, y]$ has length x/h (with respect to d_A). Note that by the definition of the length of a topological path τ is well defined, i.e. if $y, y' \in [0, 1]$ are such that $l(\sigma' \restriction [0, y]) = l(\sigma' \restriction [0, y'])$ then $\sigma'(y) = \sigma'(y')$. It is straightforward to check that τ shortcuts σ', and thus τ also shortcuts σ since σ' shortcuts σ. Note that $\tau(0) = u$ and $\tau(1) = v$.

It is also easy to prove that τ is injective; for if $x \leq y \in [0, 1]$ then σ' contains a topological path σ'' from $\tau(x)$ to $\tau(y)$ such that $l(\sigma'') = (y - x)/h > 0$, and as σ'' is almost transitive this means that its endpoints $\tau(x), \tau(y)$ are distinct.

We claim next that τ is continuous. For this, pick a point $x \in [0, 1]$ and let O be an open ball of radius $\epsilon \in \mathbb{R}^+$ with respect to d_A around $\tau(x)$. We have to show that $[0, 1]$ has an open set $U \ni x$ such that $\tau(U) \subseteq O$. But this

is easy: let $U := [x - \frac{\epsilon}{h}, x + \frac{\epsilon}{h}]$. For every $y \in U$ there is a topological $\tau(x)$–$\tau(y)$ path σ'' contained in σ' such that $l(\sigma'') = (y - x)/h$. As the length of any topological path is, by definition, at least the distance of its endvertices, we have $d_A(\tau(x), \tau(y)) \leq (y - x)/h$, and thus $\tau(y) \in O$. Since y was arbitrary, we have $\tau(U) \subseteq O$ as required. This proves that τ is continuous. $\qquad\square$

4 Existence

The existence of a flow of finite energy satisfying Kirchhoff's cycle law in a locally finite network is a well-known fact, and there are several standard techniques to prove it. Here we will see two such techniques, and point out that they can also be employed when looking for a non-elusive flow with the above properties. Both these techniques start by showing that in any network $N = (G, r, p, q, I)$, where G is locally finite, there is a p–q flow i of intensity I that has minimum energy among all such flows; it is then an easy step to show that i satisfies Kirchhoff's cycle law: if C were a directed cycle with $\sum_{\vec{e} \in \vec{E}(C)} v(\vec{e}) > 0$, then subtracting from i a constant circular flow around C of sufficiently small intensity yields a flow i' with $W(i') < W(i)$, a contradiction (note that $W(i) < \infty$ since there are flows of finite energy in N: just pick a p–q path P and send a constant flow of intensity I along P).

Thus the interesting part is to show the existence of a (non-elusive) flow of minimum energy. Our first technique does so using the following well-known fact.

Lemma 4.1 ([23, Theorem 4.10]). *If C is a non-empty, closed, convex subset of a Hilbert space, then there is a unique point $y \in C$ of minimum norm among all elements of C.*

In order to apply it, let H be the space of all functions $f : E(G) \to \mathbb{R}$ endowed with the inner product $\langle f, g \rangle := \sum_{e \in E(G)} f(e)r(e)g(e)$, and note that the corresponding norm is $||f|| = \sqrt{W(f)}$. It is straightforward to check that this is a Hilbert space, and that the subset C consisting of non-elusive flows is closed and convex. Thus Lemma 4.1 yields a non-elusive flow of minimum energy.

Our second technique uses a standard compactness argument. Recall that there is at least one flow in N with finite energy W. This W yields, for every edge e, an upper bound for the amount of flow $i(e)$ that a flow i of minimum energy, if one exists, can ever send along e: this upper bound is $\hat{i}(e) := \sqrt{W/r(e)}$. Now consider for every edge e a topological space X_e homeomorphic to the real interval $[-\hat{i}(e), \hat{i}(e)]$, and let $X := \Pi_e X_e$ be the product of these spaces. Every (non-elusive) flow f in N can be represented as a point p_f in X: from each component X_e choose the point of $[-\hat{i}(e), \hat{i}(e)]$ corresponding to $f(e)$ (we are assuming a choice of an orientation for each edge). Let $(f_i)_{i \in \mathbb{N}}$ be a sequence of non-elusive flows in N whose energies converge to the infimum M of the energies of all such flows. By Tychonoff's theorem, X is a compact space; thus, $(f_i)_{i \in \mathbb{N}}$ has an accumulation point w in X, and it is straightforward to check that the function $i : E(G) \to \mathbb{R}$ corresponding to w is a flow and in fact a non-elusive one. It is also not hard to see that $W(i) = M$.

We have thus given two proofs of the existence part of our main result:

Theorem 4.2. *Let $N = (G, r, p, q, I)$ be a locally finite network. Then there is a unique non-elusive flow i of minimum energy in N. This flow satisfies Kirchhoff's cycle law.*

(The uniqueness of i only follows from our first proof.)

Next, we show that the flow i provided by Theorem 4.2 can be obtained as a limit of electrical currents in a sequence of finite networks converging to N. The aim of this fact is a better understanding of the concept of non-elusive flows, but the reader may choose to skip to the next section as we will not make explicit use of this fact later.

Let $G_n, n \in \mathbb{N}$, be the (finite) subgraph of G spanned by the vertices at distance at most n from p, and let G_n^{**} be the graph obtained from G by contracting each component K of $G - G_n$ into a vertex v_K (keeping multiple edges that may result from these contractions). We will, with a slight abuse, use G_n and G_n^{**} to also denote the corresponding networks, not just the graphs. For a finite network H we denote by $i(H)$ the unique electrical current in H.

Proposition 4.3. *Let N be locally finite network, and let i be the non-elusive flow in N with minimum energy. Then $i = \lim_n i(G_n^{**})$; in particular, the latter limit exists.*

Proof. Define $i_n^{**} : E(G_n^{**}) \to \mathbb{R}$ by $i_n^{**}(e) = i(e)$ for every n; since i is non-elusive, it follows that i_n^{**} satisfies Kirchhoff's node law for every contracted vertex. Thus i_n^{**} is a p–q flow of value I in G_n^{**}, and thus $W(i(G_n^{**})) \le W(i_n^{**}) \le W(i)$ for every n.

Let g be an accumulation point of the sequence $(i(G_n^{**}))_{n \in \mathbb{N}}$ in the product space X defined as above; here we have to check whether the components X_e are large enough that X can accommodate the elements of the latter sequence, but this is indeed the case since we have shown that $W(i(G_n^{**})) \le W(i)$ for every n. The latter inequality also implies

$$W(g) \le W(i). \tag{7}$$

As in our compactness proof of Theorem 4.2 it is easy to prove that g is a flow in N satisfying Kirchhoff's cycle law. We moreover claim that g is non-elusive. Indeed, for every finite cut (X, X') that does not separate p from q there is an $n \in \mathbb{N}$ such that every edge in (X, X') is contained in G_m^{**} for every $m > n$. Since every flow in a finite network is non-elusive, g is an accumulation point of flows i_n satisfying $i_n(X, X') = 0$; thus $g(X, X') = 0$ holds as desired. By Theorem 4.2 and (7) we obtain $g = i$. In particular, $g = i$ is the only accumulation point of the sequence $(i(G_n^{**}))_{n \in \mathbb{N}}$, and we may write $i = \lim_n i(G_n^{**})$. $\square$

The last result motivates the following problem.

Conjecture 4.1. *If $\lim i(G_n) = \lim i(G_n^{**})$ then there is a unique flow in N satisfying Kirchhoff's cycle law.*

5   The main result: Uniqueness

As mentioned in the introduction, in order to prove the uniqueness of a non-elusive flow in N satisfying Kirchhoff's cycle law, we will consider the difference

z of two distinct hypothetical flows of this kind; we will not be able to find a cycle in z as in the finite case, but instead we will be able to find a circle. In order to obtain a contradiction, we then need to show that circles must satisfy $(K2)$ as well. This is an immediate corollary of Lemma 3.1 and the following easy property of real vectors.

Lemma 5.1. *Let R, I be two infinite-dimensional vectors with positive real values. If $\langle I, R \rangle > \langle \mathbb{1}, R \rangle$ then $\langle I^2, R \rangle > \langle I, R \rangle$.*

(where $\langle\ ,\ \rangle$ denotes the usual inner product, $\mathbb{1}$ denotes the all-ones vector and I^2 denotes the vector whose value at each coordinate is the square of the value of I at that coordinate).

Proof. For every component j, if there is a gain in this component when going from $\langle \mathbb{1}, R \rangle$ to $\langle I, R \rangle$, that is, if $I(j)R(j) \geq R(j)$, then this gain becomes even bigger when going from $\langle I, R \rangle$ to $\langle I^2, R \rangle$. On the other hand, if there is a loss in this component when going from $\langle \mathbb{1}, R \rangle$ to $\langle I, R \rangle$, then the loss becomes smaller when going from $\langle I, R \rangle$ to $\langle I^2, R \rangle$, since $I(j)R(j) < R(j)$ in that case. The assertion follows easily from these two observation. $\square$

Corollary 5.2. *Let N be a locally finite electrical network with resistances $r : E(G) \to \mathbb{R}_*^+$ such that $\sum_{e \in E(G)} r(e) < \infty$. Let i be a flow satisfying Kirchhoff's cycle law and $W(i) < \infty$. Then for every directed circle C in $|G|$ there holds $\sum_{\vec{e} \in \vec{E}(C)} v(\vec{e}) = 0$; in particular, the latter sum is well-defined.*

Proof. Lemma 3.1 yields a sparse family $\{D_n\}_{n \in \mathbb{N}}$ of finite directed cycles D_n in G such that $\vec{E}(C) = \sum_n \vec{E}(D_n)$. By $(K2)$ we have $\sum_{\vec{e} \in \vec{E}(D_n)} v(\vec{e}) = 0$ for every n. Thus it suffices to prove that the sum $\sum_n \sum_{\vec{e} \in \vec{E}(D_n)} v(\vec{e})$ is absolutely convergent. To prove this, recall that we are assuming that $\sum_{e \in E(G)} r(e) < \infty$ and $\sum_{e \in E(G)} i^2(e)r(e) < \infty$. It follows with Lemma 5.1 that $\sum_{e \in E(G)} |i(e)|r(e) =: u < \infty$. Since the family D_n is sparse, this means that $\sum_n \sum \vec{e} \in \vec{E}(D_n)|v(\vec{e})| \leq 3u < \infty$, which proves that the sum is absolutely convergent as desired. Thus $\sum_{\vec{e} \in \vec{E}(C)} v(\vec{e}) = \sum_n 0 = 0$. $\square$

The following lemma states that any infinite sequence of topological paths between two fixed vertices x, y in $|G|$ has a "converging" subsequence whose limit contains a topological x-y path. More precisely, given a sequence $E_1, E_2, \dots$ of sets, let us write

$$\liminf(E_n) := \bigcup_{i \in \mathbb{N}} \bigcap_{j > i} E_j$$

for the set of elements eventually in E_n.

Lemma 5.3 ([17])**.** *Let G be a locally finite graph, let $x, y \in V(G)$ and let $(\tau_n)_{n \in \mathbb{N}}$ be a sequence of topological x–y paths in $|G|$. Then, there is an infinite subsequence $(\tau_{a_n})_{n \in \mathbb{N}}$ of (τ_n) and a topological x–y path σ in $|G|$ such that $E(\sigma) \subseteq \liminf(E(\tau_{a_n}))$. Moreover, if no τ_n traverses any edge more than once then $E(G) \setminus E(\sigma) \subseteq \liminf(E(G) \setminus E(\tau_{a_n}))$, no edge is traversed by σ more than once, and for every finite subset F of $E(\sigma)$ there is an $m \in \mathbb{N}$ such that the linear ordering of F induced by σ coincides with that induced by τ_{a_n} for every $a_n > m$.*

We can now finish the proof of our main result.

Proof of Theorem 1.1. By Theorem 4.2 there exists at least one such flow, so it only remains to prove the uniqueness of i. Suppose, to the contrary, there are two non-elusive flows $i \neq f$ of finite energy in N that both satisfy Kirchhoff's cycle law, and put $z := i - f$.

Since $i \neq f$, there is an oriented edge $\vec{e_0}$ with $z(\vec{e_0}) > 0$. Let $G_1 \subseteq G_2 \subseteq G_3 \ldots$ be a sequence of finite subgraphs of G such that $\bigcup_n G_n = G$ and $e_0 \in E(G_1)$.

Since both i, f are non-elusive, so is z. Thus z induces a circulation z_n on G_n^{**} for every n (G_n^{**} is defined as in Section 4). Since z_n a circulation in a finite network, there must exist an oriented cycle C_n in G_n^{**} such that $\vec{e_0} \in \vec{E}(C_n)$ and $z(\vec{e}) > 0$ for every $\vec{e} \in E(C_n)$. As z_n is induced from the circulation z of G that, clearly, satisfies Kirchhoff's cycle law, C_n cannot be a cycle of G; thus C_n contains a contracted vertex of G_n^{**}. We are now going to make use of the sequence $(C_i)_{i \in \mathbb{N}}$ in order to construct an oriented circle C in $|G|$ with $z(\vec{e}) > 0$ for every $\vec{e} \in E(C)$. For this, let P_n be the (oriented) path $C - e_0$.

Applying Lemma 5.3 to G, letting τ_n be a topological path that traverses P_n in a straight manner, we obtain a topological path σ, and we can concatenate σ with e_0 to obtain a closed topological path σ', which we may assume traverses e_0 in the same direction as the C_n do. By the second sentence of Lemma 5.3, and since no τ_n visits a vertex more than once, σ also visits no vertex more than once. Moreover, we may assume that every edge $e = xy \in E(\sigma')$ is traversed by σ in the direction in which z flows, in other words, $z(x, y) > 0$ if σ visits x before y. Indeed, recall that each P_n traverses its edges in the direction in which z flows. To make sure that σ traverses its edges in the same direction as the P_n do, we can, before applying Lemma 5.3 to obtain σ, subdivide each edge $e = xy$ of G by a dummy vertex z into two edges xz, zy. Now the second sentence of Lemma 5.3 implies that if xz, zy are traversed by σ then they are traversed in the same order as in infinitely many of the P_n, which means that σ traverses e in the same direction as these P_n, which is the direction in which z flows along e.

By Lemma 3.2, there is a circle C that shortcuts σ'. This circle clearly violates Corollary 5.2, since for one of its orientations $\vec{C}$ there holds $z(\vec{e}) > 0$ for every $\vec{e} \in E(\vec{C})$. This contradiction completes the proof. ☐

Theorem 1.1 is best possible in the sense that we can not drop any of its requirements. Indeed, to see why the condition of being non-elusive is necessary, recall that the network of Figure 1 has several flows satisfying Kirchhoff's cycle law no matter how we choose its resistances.

To see that the condition $W(i) < \infty$ is necessary, consider the network N of Figure 3. We will construct a non-trivial non-elusive p–q flow f of intensity 0 in N, i.e. a circulation, that satisfies Kirchhoff's cycle law. Let $f(p, q) = 1$, and let f also send a flow of value 1 along the two edges incident with p, q, so that $(K1)$ is satisfied at both p, q. We can now assign a flow $f(e)$ to the perpendicular edge e forming a 4-cycle C with those three edges so that C complies with $(K2)$. Then, we can assign a flow to each of the two edges incident with e so that the endvertices of e comply with $(K1)$. Continuing like this, we obtain a function f that satisfies both Kirchhoff laws and is non-elusive.
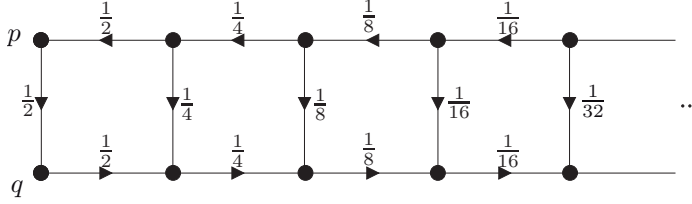
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Figure 3: A network that has a non-constant non-elusive circulation satisfying Kirchhoff's cycle law. The numbers on the edges denote their resistances.

In view of Theorem 2.2 it is tempting to conjecture that the requirement $\sum r(e) < \infty$ in Theorem 1.1 can be replaced by the weaker requirement $|G|_r \approx |G|$. However, in [10] we show a counterexample to this conjecture.

6   Finite total conductance

It is known that networks of finite total conductance have unique electrical currents. We give a proof of this fact here for the sake of completeness.

Theorem 6.1. *Let $N = (G, r, p, q, I)$ be countable network with $\sum_{e \in E(G)} 1/r(e) < \infty$. Then there is a unique flow of finite energy satisfying Kirchhoff's cycle law. This flow is non-elusive.*

Proof. Suppose there are distinct flows f, g in N both having finite energy and satisfying Kirchhoff's cycle law and consider their difference $z := f - g$. It is not hard to check that the circulation z has finite energy too, and it clearly also satisfies Kirchhoff's cycle law.

Pick an edge f such that $z(f) > 0$. Since $\sum_{e \in E(G)} 1/r(e) < \infty$ and $\sum_{e \in E(G)} z^2(e)r(e) < \infty$, there is a finite set of edges $F \subset E(G)$ such that

$$\sum_{e \in E(G) \setminus F} |z(e)| < |z(f)|. \tag{8}$$

Let U be the set of vertices incident with an edge in F, and let G^* be the graph obtained from G after identifying all vertices in $V(G) \setminus U$ into a single vertex v^*, keeping parallel edges if any arise. Note that z, considered as a function from $E(G) = E(G^*)$ to $\mathbb{R}$, is also a circulation in G^*; the sum $\sum \{z(e) \mid$ e is incident with $v^*\}$ is well-defined even if v^* has infinite degree by (8), and it equals zero since the set U is finite and every vertex in it satisfies $(K1)$. We may assume that the graph G^* has only finitely many edges, for if $x, y \in V(G^*)$ are joined by an infinite set of edges B, then we may replace all these edges by a single edge carrying the flow $\sum_{e \in B} z(e)$.

We claim that z must traverse a cycle in G^* that does not visit v^*. Indeed, since z is a circulation in the finite graph G^*, it must traverse some cycle C_0. If C_0 visits v^*, then we can subtract from z a constant circulation along C_0 to obtain a new circulation z_1 that does not traverse C_0. By (8) we have $z_1(f) > 0$ since no edges incident with v^* lie in $E(G) \setminus F$. If z_1 still traverses a cycle C_1 visiting v^* we can subtract it to obtain z_2, and so on. Continuing like this we can, after finitely many steps, reach a circulation z_k that traverses no cycle incident with v^* and satisfies $z_k(f) > 0$. But then, z_k must traverse a cycle

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C not incident with v^*, and by the construction of z_k there follows that z also traverses C. But C is also a cycle in the original graph G, which contradicts the fact that z satisfies Kirchhoff's cycle law.

Thus we have proved the uniqueness of a flow i of finite energy satisfying Kirchhoff's cycle law. To prove that i is non-elusive, it would now suffice to show the existence of a non-elusive flow f in N having finite energy and satisfying Kirchhoff's cycle law since uniqueness implies $f = i$. We already proved the existence of such a flow f in Theorem 4.2 in the case that G is locally finite. The interested reader will be able to check that our second proof of Theorem 4.2 also works in the non-locally-finite case if $\sum_{e \in E(G)} 1/r(e) < \infty$ —although not necessarily otherwise, see Section 7. $\qquad \square$

7   Non-locally-finite networks

In this section we discuss how the results of this paper behave with respect to non-locally-finite graphs.

Let us start with an example. The graph G of Figure 4 consists of two edges $e = pu, f = vq$ and an infinite family $\{P_n\}_{n \in \mathbb{N}}$ of independent u–v paths of length two. Let us suppose that all edges in the P_n have the same resistance, although this makes little difference. It is straightforward to check that the corresponding network does not have a flow of minimum energy (with a fixed intensity I), and in fact no flow that satisfies Kirchhoff's cycle law.
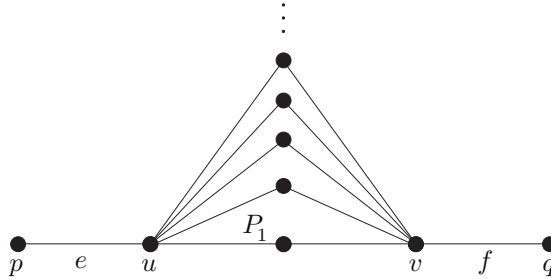

Figure 4: A simple example showing that non-locally-finite networks do not necessarily have flows satisfying Kirchhoff's cycle law.

This seems to suggest that the results of this paper cannot be extended to non-locally-finite graphs, but nevertheless there is a way around this: call a function $f : V^2 \to \mathbb{R}$ a *relaxed flow* from p to q if it satisfies $f(x, y) = -f(y, x)$ for every x, y, $f(x, y) = 0$ if $xy \notin E(G)$, and moreover it is non-elusive (defined as in Section 2). Note that if G is locally finite then every relaxed flow is a non-elusive flow, since applying the non-elusiveness condition to the set of edges incident with a given vertex x yields Kirchhoff's node law for x. It is worth mentioning that this definition effectively imposes Kirchhoff's node law to the points of $|G|$ rather than the vertices of G.

The interested reader will be able to show that Theorem 1.1 extends to non-locally-finite graphs to assert the existence and uniqueness of a relaxed flow of finite energy if the total resistance is finite (and thus the graph countable). Indeed, all intermediate results we used are proved (either here or in the respective

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sources) for countable graphs as well, and the rest of the proof of Theorem 2.2 easily extends to countable graphs.

8 Relationship to stochastic processes

There is a well studied and fruitful relationship between electrical networks and random walks. Every electrical network gives rise to a random walk, in which the transition probabilities are proportional to the conductances of the corresponding edges and, conversely, every reversible irreducible Markov chain can be represented by an electrical network [21].

It turns out that in a finite network the probability that random walk starting at a vertex x reaches a fixed vertex p before having visited another fixed vertex q equals the potential of vertex x induced by a voltage source imposing potential 1 at p and potential 0 at q. It is natural to ask if this relationship carries over to the networks of the current paper, in other words, if the unique flow provided by Theorem 1.1 coincides with the flow induced by the corresponding random walk on the same graph. This is however not the case: in the graph of Figure 1, if the sum of the resistances is finite then the probability that random walk starting at a vertex $x \neq p$ to the left of p visits p before having visited q is less than 1, since such a random walk is transient and so the probability that p is never visited is positive. It follows that the corresponding flow fails to be non-elusive.

It would be interesting though to try to define a continuous random process, i.e. a brownian motion, on $|G|_r$ in which the particle continues moving after reaching a boundary point of $|G|_r$; for such a process it might be the case that the corresponding flow is indeed non-elusive. To support this idea, we now show that given a network N and a flow f of finite energy in N satisfying Kirchhoff's cycle law, it is possible to extend the potential function $P : V(G) \to \mathbb{R}$ from the vertices of G to all of $|G|_r$ in a natural way. Recall that given a flow i satisfying Kirchhoff's cycle law, one usually defines P by fixing $P(q) = 0$ and then letting $P(u) = \sum_{\vec{e} \in \vec{E}(R)} v(\vec{e})$, where R is any path from q to u; Kirchhoff's cycle law implies that $P(u)$ does not depend on the choice of R but only on its endpoints. The following lemma allows us to extend P to the boundary $\partial^r G$ of $|G|_r$ whenever we have a flow of finite energy satisfying Kirchhoff's cycle law. Note that for every such boundary point x there is a ray of finite total resistance converging to x in $|G|_r$, see [15, Lemma 4.1].

Lemma 8.1. *Let f be a flow of finite energy satisfying Kirchhoff's cycle law in a locally finite network $N = (G, r, p, q, I)$, let $x \in \partial^r G$, and let R be a ray of finite total resistance starting at q and converging to x. Then $\sum_{\vec{e} \in \vec{E}(R)} f(\vec{e}) r(\vec{e})$ is well defined and does not depend on R.*

Proof (sketch). Well-definedness follows easily by Lemma 5.1. To show independence from R, let R, S be two such rays. Since, easily, R and S belong to the same end of G, it is possible to find an infinite sequence $(P_i)_{i \in \mathbb{N}}$ of disjoint R, S paths such that $\sum_{e \in \bigcup_i P_i} r(e) < \infty$, the endpoint of P_i in R comes after the endpoint of P_{i-1} for every i, and similarly for the endpoints in S. The sum of potential differences $\sum_{e \in \bigcup_i P_i} f(e) r(e)$ is absolutely convergent by Lemma 5.1. Adding potential differences along all finite cycles formed by R, S and a pair of subsequent P_i, and using the fact that these cycles satisfy $(K2)$, we obtain $\sum_{\vec{e} \in \vec{E}(R)} f(\vec{e}) r(\vec{e}) - \sum_{\vec{e} \in \vec{E}(S)} f(\vec{e}) r(\vec{e}) = 0$. □

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Thus we can extend P to the boundary $\partial^r G$ by letting $P(x) = \sum_{\vec{e} \in \vec{E}(R)} f(\vec{e}) r(\vec{e})$ for some such ray R. The interested reader will be able to check that the function $P : |G|_r \to \mathbb{R}$ defined this way is continuous provided f has finite energy.

Conversely, one may ask whether given any continuous potential function $P : \partial^r G \to \mathbb{R}$ on the boundary of $|G|_r$ there is a flow in G inducing P. This is an instance of the well-known discrete Dirichlet Problem. Let us state it more precisely:

Conjecture 8.1. *Let G be a locally finite graph and fix a function $r : E(G) \to \mathbb{R}^+$ such that $|G|_r$ is compact. Then, for every continuous function $P : \partial^r G \to \mathbb{R}$ there is a circulation in G that satisfies Kirchhoff's cycle law and induces (in the above sense) P.*

A typical instance of the discrete Dirichlet Problem presupposes a locally finite graph G with a fixed compactification, usually the Freudenthal compactification $|G|$ or the hyperbolic compactification, and asks whether every continuous function on the boundary can be induced by a circulation in G; see for example [3, 20, 27]. In most cases, all edges of G are considered with resistance 1. In comparison, Conjecture 8.1 allows more flexibility as far as the choice of resistances and the choice of a compactification is concerned, but demands that these two choices are compatible with each other.

An interesting aspect of Conjecture 8.1 is the fact that every compact metric space is isometric to $\partial^r G$ for some appropriate choice of G and r, see [15, Section 4].

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