Graph-theoretical versus topological ends of graphs

Reinhard Diestel & Daniela Kühn

Abstract

We compare the notions of an end that exist in the graph-theoretical and, independently, in the topological literature. These notions conflict except for locally finite graphs, and we show how each can be expressed in the context of the other. We find that the topological ends of a graph are precisely the undominated of its graph-theoretical ends, and that graph theoretical ends have a simple topological description generalizing the definition of a topological end.

1 Introduction, and overview of results

In 1931, Freudenthal [5] introduced ends of certain topological spaces X as points at infinity for compactification purposes. Essentially, those ends are defined 'from above' as descending sequences $U_1 \supseteq U_2 \supseteq \dots$ of connected open sets with compact frontiers ∂U_i, such that $\bigcap_i \overline{U}_i = \emptyset$. In 1964, Halin [7] independently introduced ends of infinite graphs. These are defined 'from below' as equivalence classes of 1-way infinite paths in the graph, two such paths being equivalent if no finite set of vertices separates them.

For locally finite graphs these two definitions agree: there is a natural bijection between their topological ends and their graph-theoretical ends. (And for locally finite graphs with only finitely many such ends these are equivalent to Nash-Williams's *wings* [14], which can be viewed as an edge-separation analogue to the 'directions' we shall define below.) This correspondence is well known and has become a standard tool in the study of locally finite graphs, especially of Cayley graphs of finitely generated groups. See eg. [10, 11, 13, 17] for references.

For graphs with vertices of infinite degree, however, the two notions of an end differ, and it is the purpose of this note to clarify their relationship. This has become relevant in the context of our papers [3, 4], where we found that some ends of arbitrary infinite graphs behaved better than others. It now turns out that these are precisely their topological ends.

We shall bridge the gap between the topological and the graph-theoretical notion of an end in two ways. We prove that, when a graph G is viewed as a 1-complex, its topological ends correspond naturally to those of its graph-theoretical ends ω that are not *dominated*, ie. for which there is no vertex sending an infinite fan to a ray in ω. (Note that such a dominating vertex would have infinite degree.) We thus have an injection $\mathfrak{e} \mapsto \omega_{\mathfrak{e}}$ from the topological ends $\mathfrak{e}$ of G onto its undominated graph-theoretical ends.

Conversely, we show that arbitrary graph-theoretical ends ω of a graph G can still be expressed topologically, as follows. Let f_ω be the function that assigns to every compact set $Z \subseteq G$ (these are essentially the finite subgraphs of G) the unique connected component U of $G \setminus Z$ that contains a tail of every ray in ω. Then the 'directions' in which f_ω points are compatible in the sense that $f_\omega(Z) \supseteq f_\omega(Z')$ whenever $Z \subseteq Z'$. We prove that *every* function $f : Z \mapsto U$ from the set of compact subsets Z of G to the components of $G \setminus Z$ with this compatibility property arises from an end ω in this way, and thus that $\omega \mapsto f_\omega$ is a bijection between the ends of G and all those functions f.

Topological ends $\mathfrak{e}$ in more general spaces X can be expressed similarly. Indeed, given a defining sequence $U_1 \supseteq U_2 \supseteq \ldots$ for $\mathfrak{e}$, it is easy to see that the function $f(\partial U_i) \mapsto U_i$ (for all i) extends uniquely to a function $f : Z \mapsto U$ (which we denote as $f_{\mathfrak{e}}$) mapping all compact $Z \subseteq X$ compatibly to components of $G \setminus Z$, and different ends $\mathfrak{e}$ yield different functions $f_{\mathfrak{e}}$.

When X is a graph then all these maps are compatible as expected. More precisely, the composition of $\mathfrak{e} \mapsto \omega_{\mathfrak{e}}$ and $\omega \mapsto f_\omega$ commutes with the injection $\mathfrak{e} \mapsto f_{\mathfrak{e}}$, so $f_{\omega_{\mathfrak{e}}} = f_{\mathfrak{e}}$.

2 Ends in graphs

In this section, we briefly review the standard definition of ends in graphs, define the 'direction' functions f mentioned in the introduction (so far, just for graphs), and establish the injection $\omega \mapsto f_\omega$ mapping ends to directions.

Ends of graphs were introduced by Halin [7], and his definition (given below) has been adopted by most writers in graph theory. (For notable exceptions see Jung & Niemeyer [11]; Hahn, Laviolette and Širán [6]; and Cartwright, Soardi & Woess [1]. Overviews of some existing concepts and their relationships have been given by Hien [8] and by Krön [12].)

A 1-way infinite path is a *ray*. The subrays of rays are their *tails*. Two rays R, R' in a graph G are *equivalent* if no finite set of vertices separates them in G, i.e. if G contains infinitely many disjoint R–R' paths. This is an equivalence relation on the set of rays in G; its equivalence classes are the (graph-theoretical) *ends* of G. The set of ends of G is denoted by $\Omega(G)$, and given a ray $R \subseteq G$ we write $\omega(R)$ for the end of G containing R.

We say that a vertex x of G *dominates* an end ω of G if G contains an infinite x–R fan for some ray $R \in \omega$ (ie. an infinite set of x–R paths, disjoint except in x), or equivalently, if x cannot be separated by finitely many vertices from some (and hence any) ray in ω. We write $\Omega'(G)$ for the set of all undominated ends of G.

When S is a finite set of vertices, then $G - S$ has a unique component containing a tail of some (and hence every) ray of a given end ω; we say that ω *belongs* to that component.

We will need the following standard lemma; its proof is included for completeness.

Lemma 2.1 *Let U be an infinite set of vertices in a connected graph G. If $|U|$ is regular, then G contains either a ray R with $|U|$ disjoint U–R paths (possibly trivial) or a subdivided star with $|U|$ leaves in U. (Note that if U is uncountable then the latter holds.)*

Proof. Let T be a minimal subtree of G that contains all the vertices of U. Note that every edge of T lies on a U–U path in T.

Suppose first that T contains a vertex t whose degree in T is $|U|$. Since every T-edge at t lies on a U–U path, T contains a subdivided star with centre t and $|U|$ leaves in U.

Suppose now that T contains no such vertex t. Then $|U| = |T| = \aleph_0$ and T is locally finite, so T contains a ray R. As every edge of T lies on a U–U path in T, it is easy to find infinitely many disjoint R–U paths in T inductively. □

Let $\mathcal{S} = \mathcal{S}(G)$ be the set of all finite sets of vertices in a given graph G. Let us call a map f with domain $\mathcal{S}$ a *direction* of G if f maps every $S \in \mathcal{S}$ to a component of $G - S$ and $f(S) \supseteq f(S')$ whenever $S \subseteq S'$. Note that for $S_1, S_2 \in \mathcal{S}$ we have both $f(S_1 \cup S_2) \subseteq f(S_1)$ and $f(S_1 \cup S_2) \subseteq f(S_2)$; in particular,

$$f(S_1) \cap f(S_2) \neq \emptyset. \tag{1}$$

We denote the set of directions in G by $\mathcal{D}(G)$.

The following relationship between ends and directions was observed by Robertson, Seymour and Thomas [16], who considered directions in a criminology context (calling them $\aleph_0$-havens). However, they apparently missed the one non-trivial aspect of the proof (surjectivity), so we prove the result in full.

Theorem 2.2 *Let G be an infinite graph. For every end ω of G there is a unique direction f_ω of G such that, for every $S \in \mathcal{S}$, $f_\omega(S)$ is the component of $G - S$ to which ω belongs. The map $\omega \mapsto f_\omega$ is a bijection between $\Omega(G)$ and $\mathcal{D}(G)$.*

Proof. It is straightforward to check that the function f_ω assigning to every $S \in \mathcal{S}$ the component of $G-S$ to which ω belongs is indeed a direction of G, and that the map $\omega \mapsto f_\omega$ is injective. We show that this map is surjective, ie. that for every direction $f \in \mathcal{D}(G)$ there is an end $\omega \in \Omega(G)$ such that $f = f_\omega$.

Given $S \in \mathcal{S}$, let us write $\widehat{S} := V(f(S)) \cup N(f(S))$ for the set of vertices in $f(S)$ and their neighbours in S. Thus $\widehat{S} \supseteq \widehat{S'}$ whenever $S \subseteq S'$. Put

$$S^* := \bigcap_{S \in \mathcal{S}} \widehat{S}.$$

Suppose first that S^* is infinite (and hence meets $f(S)$ for every $S \in \mathcal{S}$). Then it is easy to construct a ray R through infinitely many vertices in S^* (considering as S initial segments of the ray being constructed), and it is easily checked that $f_{\omega(R)} = f$.

Suppose now that S^* is finite. Replacing G by $f(S^*)$, we may assume that $S^* = \emptyset$ and G is connected. (Formally, we define a direction f' in $G' := G - S^*$ by $f'(S) := f(S^* \cup S)$; then a ray found for G' and f' also works for G and f.) We can now find an infinite sequence $S_1, S_2, \dots$ of non-empty sets in $\mathcal{S}$ such that $f(S_i)$ contains both S_{i+1} and $f(S_{i+1})$, for all i. (Then S_{i+1} separates S_i from $f(S_{i+1})$ in G.) Indeed, having constructed S_i we can find for every $s \in S_i$ an $S_s \in \mathcal{S}$ such that $s \notin \widehat{S_s}$ (because $s \notin S^*$), and take as S_{i+1} the set of vertices in $S := \bigcup_{s \in S_i} S_s$ that have a neighbour in $f(S)$. Then $f(S_{i+1}) = f(S)$, because $S_{i+1} \subseteq S$ implies $f(S_{i+1}) \supseteq f(S)$ but $f(S)$ is already a component of $G - S_{i+1}$. Together with $\widehat{S} \subseteq \widehat{S_s} \not\ni s$ this implies $s \notin \widehat{S} = \widehat{S_{i+1}} \supseteq S_{i+1}$. Thus $G[\widehat{S_{i+1}}]$ is a connected subgraph of $G - S_i$ containing both $f(S_{i+1})$ and S_{i+1}, so by (1) the component of $G - S_i$ containing it must be $f(S_i)$.

Since all the S_i are disjoint, the descending sequence $\widehat{S_1} \supseteq \widehat{S_2} \supseteq \dots$ has an empty overall intersection: every vertex in $\widehat{S_i}$ has distance at least $i-1$ from S_1 (because every S_1–$\widehat{S_i}$ path has to pass through all the disjoint sets $S_2, \dots, S_{i-1}$), so no vertex can lie in $\widehat{S_i}$ for every i. For every i pick a vertex $u_i \in f(S_i)$, and note that for $U = \{u_i \mid i \in \mathbb{N}\}$ Lemma 2.1 must return a ray R: the centre of any subidivided star with infinitely many leaves in U would lie outside $\widehat{S_i}$ for some i, and all its paths to leaves u_j with $j \geq i$ would have to pass through the finite set $S_i \subseteq \widehat{S_i}$ a contradiction. By the same argument, the existence of infinitely many disjoint U–R paths (cf. Lemma 2.1) implies that every $f(S_i)$ contains a tail of R.

To show that $f_{\omega(R)} = f$, consider any $S \in \mathcal{S}$. Choose i large enough that $S \cap f(S_i) = \emptyset$. Then $f(S_i)$ is a connected subgraph of $G - S$, and by (1) the component of $G - S$ containing it must be $f(S)$. So $f(S)$ contains a tail of R, giving $f_{\omega(R)}(S) = f(S)$ as desired. □

3 Ends in topological spaces

In this section, we review the standard definition of ends in general topological spaces. As with graphs in Section 2, we define *directions* in such spaces, and show that ends canonically induce directions. (However, not all directions arise from ends.)

Ends in topological spaces have been considered in a variety of contexts and with a corresponding variety of definitions. We adopt the original definition proposed by Freudenthal [5], which appears to be the fundamental concept adopted also by more recent standard works such as Hughes & Ranicki [9].

Let X be a locally connected Hausdorff space. Given a subset $Y \subseteq X$, we write $\overline{Y}$ for the closure of Y, and $\partial Y := \overline{Y} \cap \overline{X \setminus Y}$ for its frontier. In order to define the (topological) ends of X, we consider infinite sequences $U_1 \supseteq U_2 \supseteq \dots$ of non-empty connected open subsets of X such that each ∂U_i is compact and $\bigcap_{i \geq 1} \overline{U_i} = \emptyset$. Note that then no $\overline{U_i}$ can be compact. We say that two such sequences $U_1 \supseteq U_2 \supseteq \dots$ and $U'_1 \supseteq U'_2 \supseteq \dots$ are *equivalent* if for every i there exist j, k such that $U_i \supseteq U'_j$ and $U'_i \supseteq U_k$. The equivalence classes of those sequences are the *topological ends* of X, and the set of all topological ends of X is denoted by $\mathfrak{E}(X)$. If $U_1 \supseteq U_2 \supseteq \dots$ is a sequence contained in a topological end $\mathfrak{e}$, we say that $U_1 \supseteq U_2 \supseteq \dots$ *represents* $\mathfrak{e}$.

We remark that, given any sequence $U_1 \supseteq U_2 \supseteq \dots$ of sets in X with $\bigcap_{i \geq 1} \overline{U_i} = \emptyset$, the above assumptions about the U_i are equivalent to just requiring them to be components of $X \setminus X_i$ with compact X_i. Indeed, any connected open set U is a component of $X \setminus \partial U$, so the U_i above are indeed components of $X - X_i$ with compact $X_i := \partial U_i$. Conversely, for every compact $Z \subseteq X$ the frontier of any component of $X \setminus Z$ is a closed subset of Z and hence compact. Thus if each U_i is a component of $X \setminus X_i$ with compact X_i, it is a non-empty connected open set with compact frontier.

A proof of the following easy lemma can be found in [5, Satz 3].

Lemma 3.1 *If $\mathfrak{e}$ and $\mathfrak{e}'$ are distinct topological ends of X, and if $U_1 \supseteq U_2 \supseteq \dots$ and $U'_1 \supseteq U'_2 \supseteq \dots$ are sequences representing $\mathfrak{e}$ and $\mathfrak{e}'$ respectively, then $U_i \cap U'_i = \emptyset$ for some i.* □

Next, let us define directions for arbitrary Hausdorff spaces X. A function g with domain the set of compact subsets $Z \subseteq X$ is a *direction* of X if $g(Z)$ is a connected component of $X \setminus Z$ and $g(Z) \supseteq g(Z')$ whenever $Z \subseteq Z'$. Since connected components are non-empty, this implies that $g(Z) \neq \emptyset$ for all Z. (In particular, if X admits a direction it is not itself compact.) As the union of two compact sets is again compact, all such functions satisfy (1)—just as for graphs. We denote the set of directions of X by $\mathfrak{D}(X)$.

Topological ends canonically define directions, as follows. Given $\mathfrak{e} \in \mathfrak{E}(X)$, with representing sequence $U_1 \supseteq U_2 \supseteq \dots$, say, put $g(\partial U_i) := U_i$ for all i. This map extends uniquely to a direction $g_{\mathfrak{e}}$ of X: given any compact $Z \subseteq X$, there exists an i such that $Z \cap \overline{U}_j = \emptyset$ for all $j \geq i$ (because $\bigcap_i \overline{U}_i = \emptyset$ and Z is compact), and we let $g_{\mathfrak{e}}(Z)$ be the connected component of $X \setminus Z$ containing U_i. It is easily checked that $g_{\mathfrak{e}}$ is indeed a direction, and that it is the only direction g of X satisfying $g(\partial U_i) = U_i$ for all i.

Lemma 3.2 *The map $\mathfrak{E}(X) \to \mathfrak{D}(X)$ defined by $\mathfrak{e} \mapsto g_{\mathfrak{e}}$ is injective.* □

Proof. Suppose that $g_{\mathfrak{e}} = g_{\mathfrak{e}'} =: g$ for $\mathfrak{e} \neq \mathfrak{e}'$. Choose i as in Lemma 3.1. Then $g(\partial U_i \cup \partial U'_i) \subseteq U_i \cap U'_i = \emptyset$, a contradiction. □

We shall see in Section 4 that the map of Lemma 3.2 need not be surjective, not even for graphs.

4 Ends of graphs as 1-complexes

We now come to compare the concepts of ends and directions in graphs with the corresponding topological concepts. We shall find that while the concepts of directions correspond canonically, the notions of an end do not. But we shall see that the topological ends of a graph do correspond canonically to a subset of its graph-theoretical ends, those that are not dominated.

For this purpose, we consider a graph G as a 1-complex with the usual identification topology. Thus, every edge is homeomorphic to the real interval $[0, 1]$, and the basic open neighbourhoods of a vertex x are the unions of half-open intervals $[x, z)$, one from every edge $[x, y]$ at x. Note that our graphs may have vertices of infinite degree, in which case this topology differs from the subspace topology of G in any graph-theoretical 'embedding' of G in Euclidean 3-space (say). We remark that a subset of G is connected if and only if it is path-connected, which is not difficult to show.

The following lemma is also straightforward to verify.

Lemma 4.1 *A subset $Z \subseteq G$ is compact if and only if Z is closed and contains only finitely many vertices and inner points from only finitely many edges.* $\square$

Given a subgraph H of G, we must distinguish between the (graph-theoretical) components of the graph $G - H$ and the connected components of the topological space $G \setminus H$. (As usual, $G - H$ is the subgraph of G induced by all the vertices outside H, while $G \setminus H$ is a difference of point sets.) However, there is a simple relationship between the two:

Lemma 4.2 *If $H \subseteq G$ is a subgraph and U is a connected component of $G \setminus H$, then either U is the interior of an edge of G with endvertices in H, or the graph $G - H$ has a component C such that U is the union of C with the interiors of all the C–H edges of G.* $\square$

Lemmas 4.1 and 4.2 imply that our topological and our graph-theoretical definitions of directions are compatible in the following obvious sense. Let us say that $f' \in \mathfrak{D}(G)$ *extends* $f \in \mathcal{D}(G)$ if $f(S) \subseteq f'(S)$ for every $S \in \mathcal{S}$.

Corollary 4.3 *Every direction $f \in \mathcal{D}(G)$ of a graph G extends uniquely to a direction $f' \in \mathfrak{D}(G)$ of the Hausdorff space G, and the map $f \mapsto f'$ defined in this way is a bijection between $\mathcal{D}(G)$ and $\mathfrak{D}(G)$.*

Proof. Given a compact set $Z \subseteq G$, pick a finite subgraph $H \subseteq G$ containing Z, and let $f'(Z)$ be the unique connected component of $G \setminus Z$ containing $f(V(H))$. It is easy to check that f' is well defined and indeed a direction of the space G, and that the map $f \mapsto f'$ is injective.

To show that $f \mapsto f'$ is surjective, let $f' \in \mathfrak{D}(G)$ be given. To define f, consider a finite set S of vertices and put $H := G[S]$. By Lemma 4.2, $f'(H)$ contains a component C of $G - S$, and we put $f(S) := C$. It is again easy to check that f is a well-defined function in $\mathcal{D}(G)$ whose unique extension in $\mathfrak{D}(G)$ is f'. $\square$

Here are two more consequences of Lemma 4.1:

Lemma 4.4 *Let $\mathfrak{e} \in \mathfrak{E}(G)$, and let $U_1 \supseteq U_2 \supseteq \dots$ be a sequence representing $\mathfrak{e}$. Then G does not contain a subdivided star S such that every U_i contains a leaf of S.*

Proof. Suppose there is a subdivided star S as stated, with centre s say. Since $\bigcap \overline{U}_i = \emptyset$, there is an i such that $s \notin \overline{U}_i$. But U_i contains infinitely many leaves of S (because every U_j with $j > i$ contains a leaf, which again lies in only finitely many $\overline{U}_j$). So the compact set ∂U_i meets infinitely many of the subdivided edges of S, contrary to Lemma 4.1. $\square$

Lemma 4.5 *Let $R \subseteq G$ be a ray, and let $U \subseteq G$ be a connected set with compact frontier. If G contains infinitely many disjoint paths between R and vertices in U, then U contains a tail of R.* $\square$

Our next lemma shows how topological ends define graph-theoretical ends:

Lemma 4.6 *For every topological end $\mathfrak{e} \in \mathfrak{E}(G)$ there exists a unique end $\omega_{\mathfrak{e}} \in \Omega(G)$ such that for every sequence $U_1 \supseteq U_2 \supseteq \ldots$ representing $\mathfrak{e}$ each U_i contains a ray from $\omega_{\mathfrak{e}}$.*

Proof. Let $U_1 \supseteq U_2 \supseteq \ldots$ be a fixed sequence representing $\mathfrak{e}$. None of the U_i is just an interval on an edge of G, because $\overline{U}_i$ is not compact. As U_i is connected, it therefore contains a vertex u_i. Since $\bigcap \overline{U}_i = \emptyset$, we can choose these u_i distinct. By Lemmas 2.1 and 4.4, the component of G containing U_1 contains a ray R and disjoint paths from R to infinitely many u_i. By Lemma 4.5, every U_i contains a tail of R. And if $U_1' \supseteq U_2' \supseteq \ldots$ is any other sequence representing $\mathfrak{e}$, then every U_i' contains some U_j and hence a tail of R. Let $\omega_{\mathfrak{e}} := \omega(R)$.

To show the uniqueness of $\omega_{\mathfrak{e}}$ let $\tau \neq \omega_{\mathfrak{e}}$ be another end of G, and let $S \in \mathcal{S}$ separate $\omega_{\mathfrak{e}}$ from τ. Since $\bigcap \overline{U}_i = \emptyset$, there is an i such that $S \cap U_i = \emptyset$. As U_i is connected, but no connected component of $G \setminus S$ contains both a ray from $\omega_{\mathfrak{e}}$ and a ray from τ, it follows that U_i cannot contain a ray from τ. $\square$

Let $\sigma : \mathfrak{E}(G) \to \Omega(G)$ denote the map defined by $\sigma : \mathfrak{e} \mapsto \omega_{\mathfrak{e}}$. This map blends naturally with the maps between ends and directions we have defined so far. Indeed, its composition with our maps $\omega \mapsto f_\omega$ from Section 2 and $f \mapsto f'$ from Corollary 4.3 is easily seen to commute with our map $\mathfrak{e} \mapsto g_{\mathfrak{e}}$ from Section 3 (Fig. 1):

Proposition 4.7 *For every topological end $\mathfrak{e}$ of G we have $f'_{\omega_{\mathfrak{e}}} = g_{\mathfrak{e}}$.* $\square$

$$
\begin{tikzcd}
\Omega \ni \omega_{\mathfrak{e}} \arrow[r, leftrightarrow, "\text{bij}", "(2.2)"'] & \mathcal{D} \ni f_{\omega(\mathfrak{e})} \arrow[r, leftrightarrow, "\text{bij}", "(4.3)"'] & \mathfrak{D} \ni g_{\mathfrak{e}} = f'_{\omega(\mathfrak{e})} \\
& \mathfrak{e} \in \mathfrak{E} \arrow[ul, "\text{inj}"', "(4.8)"] \arrow[ur, "\text{inj}", "(3.2)"'] &
\end{tikzcd}
$$

Figure 1: The topological ends of G induce corresponding directions and graph-theoretical ends

Our aim now is to show that σ is a bijection between $\mathfrak{E}(G)$ and $\Omega'(G)$, the set of undominated ends of G.

Lemma 4.8 *The function $\sigma : \mathfrak{e} \mapsto \omega_{\mathfrak{e}}$ is injective.*

Proof. Let $\mathfrak{e} \neq \mathfrak{e}' \in \mathfrak{E}(G)$, with representing sequences $U_1 \supseteq U_2 \supseteq \ldots$ and $U'_1 \supseteq U'_2 \supseteq \ldots$ say. By Lemma 3.1 there is an i such that $U_i \cap U'_i = \emptyset$. By definition, $\omega_{\mathfrak{e}}$ has a ray R in U_i, and $\omega_{\mathfrak{e}'}$ has a ray R' in U'_i. By Lemma 4.5 there cannot be infinitely many disjoint R–R' paths in G, so $\omega_{\mathfrak{e}} \neq \omega_{\mathfrak{e}'}$. $\square$

Next, we show that σ sends $\mathfrak{E}(G)$ to $\Omega'(G)$:

Lemma 4.9 *No end of the form $\omega_{\mathfrak{e}}$ is dominated.*

Proof. Let $\omega \in \Omega(G)$ be an end dominated by some vertex $x \in G$. Suppose that $\omega = \omega_{\mathfrak{e}}$ for some $\mathfrak{e} \in \mathfrak{E}(G)$. Let $U_1 \supseteq U_2 \supseteq \ldots$ be a sequence representing $\mathfrak{e}$, and choose i large enough that $x \notin \overline{U}_i$. By definition of $\omega_{\mathfrak{e}}$ (in Lemma 4.6), U_i contains a ray $R \in \omega$. As x dominates ω, the compact set ∂U_i meets infinitely many disjoint x–R paths, contradicting Lemma 4.1. $\square$

It remains to show that σ sends $\mathfrak{E}(G)$ onto $\Omega'(G)$. Let us call an end ω *fat* if it contains a family of uncountably many disjoint rays.

Lemma 4.10 *Every fat end of G is dominated.*

Proof. Let $\omega \in \Omega(G)$ be fat. Pick a vertex from each of some uncountably many disjoint rays in ω. By Lemma 2.1, G contains a subdivided star S with uncountably many leaves among the vertices picked. Choose a countably infinite subset of these leaves, and use the equivalence of the rays in ω to extend the corresponding paths from S disjointly to some fixed ray $R \in \omega$. Then the centre of S sends an infinite fan to R, and hence dominates ω. $\square$

Theorem 4.11 *In every infinite graph G, the function $\sigma : \mathfrak{E}(G) \to \Omega(G)$ is injective with image $\Omega'(G)$. Thus, σ maps the topological ends of G bijectively to its undominated ends.*

Proof. It only remains to show that for every undominated end ω there is a topological end $\mathfrak{e}$ such that $\omega = \omega_{\mathfrak{e}}$. Let $\mathcal{R} = \{R^1, R^2, \dots\}$ be a maximal set of disjoint rays in ω; recall that, by Lemma 4.10, any such set is countable. Put $X := \bigcup \mathcal{R}$. For every i choose a sequence $P_1^i \subseteq P_2^i \subseteq \dots$ of finite initial subpaths of R^i whose union is R^i, and put

$$X_i := \bigcup_{j \le i} P_i^j \quad \text{and} \quad Y_i := X - X_i\,.$$

Note that $X_1 \subset X_2 \subset \dots$ and $Y_1 \supset Y_2 \supset \dots$. Let C_i be the unique component of $G - X_i$ that contains a ray from ω, and let U_i be the unique connected component of $G \setminus X_i$ that contains a ray from ω. Then $Y_i \subseteq C_i \subseteq U_i$, and by Lemma 4.2

$$U_i \text{ is the union of } C_i \text{ with the interiors of all the } X_i\text{–}C_i \text{ edges.} \qquad (*)$$

In particular, U_i is open and $\partial U_i \subseteq V(X_i)$. Hence ∂U_i is compact. Our aim is to show that $\bigcap_{i \ge 1} \overline{U}_i = \emptyset$; then $U_1 \supseteq U_2 \supseteq \dots$ represents a topological end $\mathfrak{e}$ of G, and clearly $\omega_{\mathfrak{e}} = \omega$.

By $(*)$, $\widehat{C}_i := \overline{U}_i$ is the subgraph of G consisting of C_i together with all X_i–C_i edges. Hence if $\bigcap_i \overline{U}_i \neq \emptyset$, then G has a vertex z that lies in every $\widehat{C}_i$. Choose i large enough that $z \notin Y_i$. For every $j \ge i$ pick a z–Y_j path $Q_j = z \dots y_j$ in $\widehat{C}_j$. Then the only vertices of Q_j that are contained in X are y_j and possibly z. Moreover, each y_j lies on only finitely many paths Q_k. Let $H \subseteq G$ be the union of some infinite set of Q_j with $y_j \notin Q_k$ for all $k \neq j$. Then the only vertices of $H - z$ in X are these y_j, and they all have degree 1 in H. Now apply Lemma 2.1 to H with the set U of all these y_j. If the lemma returns a ray R, then $R \in \omega$ (because R is joined by infinitely many disjoint paths to vertices on X), and R has no vertex in X other than possibly z. So a tail of R violates the maximality of $\mathcal{R}$. If the lemma returns a star, then the centre of this star dominates ω, contrary to the choice of ω. □

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With a correction in Section 3 (third paragraph, first line) made in 2012, and another in the statement of Lemma 2.1 made in 2017.

Reinhard Diestel & Daniela Kühn
Mathematisches Seminar
Universität Hamburg
Bundesstraße 55
D - 20146 Hamburg
Germany