The Structure of TK_a-free Graphs

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A well-known theorem of Halin implies that the graphs not containing a TK_a, where a is any regular uncountable cardinal, can be decomposed into induced subgraphs of order $< a$ which are arranged in a tree-like fashion. We formalize this observation by introducing the new concept of a generalized tree-decomposition, which is shown to extend in a natural way the familiar tree-decompositions of finite graphs. We then prove that the TK_a-free graphs can be characterized purely in terms of these decompositions: a graph is TK_a-free if and only if it admits a generalized tree-decomposition into subgraphs of order $< a$ such that every branch of the corresponding decomposition tree has length $< a$.

1. Introduction

A number of important results in infinite graph theory are based on the following decomposition theorem of Halin [6]:

Theorem 1.1. *Any graph not containing a subdivided complete graph of regular uncountable order a can be extended, by adding edges, to a graph which admits a simplicial decomposition into factors of order $< a$.*

A *simplicial decomposition* of a graph G is a family of induced subgraphs, making up the whole of G, which are related to one another by a number of simple conditions. These conditions ensure, among other things, that the subgraphs constituting the decomposition are arranged in a tree-like fashion; in particular, if G is finite then their vertex sets define a *tree-decomposition* of G in the sense of Robertson and Seymour [9].

Conversely, the tree-decomposition induced by a finite simplicial decomposition extracts that part of its information which concerns the intersection pattern of its factors; for example, the deletion of edges (which does not alter this pattern) leaves a tree-decomposition intact, while it may destroy some of the other properties of the original simplicial decomposition. In all cases where the tree-like intersection pattern of the factors in a finite decomposition is all that matters, it is therefore convenient to work with the more general tree-decompositions rather than with simplicial decompositions.

If G is infinite, on the other hand, one may no longer be able to associate a (graph theoretical) tree with a simplicial decomposition of G. However, the factors in a simplicial decomposition F always give rise to a (more general) *order theoretical tree* T_F (Halin [7]). Basically, such a generalized decomposition tree is a rooted tree in which the branches are allowed to contain limit points in the 'outward' direction. As in the finite case, it would be convenient to have a concept which captures this 'tree'-shape of an infinite simplicial decomposition, without actually imposing such a decomposition along with its other, less relevant, conditions.

In this paper we introduce the concept of *generalized tree-decompositions* to serve this purpose. These decompositions generalize finite tree-decompositions in a natural way, and like those they remain unaffected by any deletion of edges from the decomposed graph. Halin's theorem, then, translates to the following: for any regular uncountable ordinal a, each $\mathrm{T}K_a$*-free* graph (that is, each graph not containing any subdivided complete graph of order a as a subgraph) admits a generalized tree-decomposition into factors of order $< a$.

The main result of this paper is that this theorem can be strengthened in a natural way that gives it a direct converse, and thus turns it into a characterization of the $\mathrm{T}K_a$-free graphs purely in terms of their generalized tree-decompositions:

Theorem 1.2. *If G is a graph and $a > \aleph_0$ is a regular cardinal, then the following assertions are equivalent:*

(i) $G \not\supseteq \mathrm{T}K_a$;

(ii) *G has no complete minor of order a;*

(iii) *G admits a generalized tree-decomposition F such that every factor in F and every chain in T_F has order $< a$.*

(The additional equivalence between (i) and (ii) is due to Jung [8].) Robertson, Seymour and Thomas [11] have recently extended an appropriate weakening of Theorem 1.2 to singular a and to the case of $a = \aleph_0$, thereby completing the classification of $\mathrm{T}K_a$-free graphs by their tree structures (for infinite a). Their proof, which also reobtains a version of Theorem 1.2 independently, does not explicitly build on Halin's theorem but is otherwise similar to ours. For an impressive overview of this and other results on infinite excluded minors or subdivisions see Robertson, Seymour and Thomas [10].

The paper is organized as follows. Section 3, which follows a brief section on notation, introduces the concepts of simplicial and generalized tree-decompositions. The latter are studied to some degree; mainly to prepare the ground for the proof of Theorem 1.2, but also to exhibit some of the properties of generalized tree-decompositions that make them such an interesting tool for structural characterization.

Section 4 sketches a proof of Theorem 1.1, which is in fact a strengthened version of Halin's original result. (The original version looks exactly the same, but is based on a slightly weaker concept of simplicial decomposition.) The proof is included not only for the reader's convenience, but also because it will be built on in Section 6. There the foundations are laid for an application to a problem on *ends* in infinite graphs, which has appeared in [3].

Section 5 contains the proof of Theorem 1.2.

2. Basic concepts and notation

Let G be a graph. For subgraphs $A, B \subset G$ we call a path $P \subset G$ an A–B path if its endvertices are in A and B, respectively, and its interior $\mathring{P}$ lies in $G \smallsetminus (A \cup B)$. We write $G[A \to B]$ for the subgraph of G induced by all vertices of G that can be reached from A without passing through B. More precisely, $G[A \to B]$ is the subgraph of G induced by all vertices $v \in G$ for which G contains a path $x_1 \ldots x_n$ satisfying $x_1 \in A$, $x_n = v$, and $x_i \notin B$ for $i \neq n$. When the underlying graph G is fixed, we shall usually abbreviate $G[A \to B] \cap B$ to $B[A]$. Thus, if A and B are disjoint, then $B[A]$ is the subgraph of B induced by all terminal vertices of A–B paths in G. If $B = G$, on the other hand, the above definition of $B[A]$ coincides with the conventional meaning of $G[A]$, denoting the subgraph of G induced by the vertices of A.

Two paths are *independent* if their interiors are disjoint. The *Menger number* $\mu_G(x, y)$ of two vertices $x, y \in G$ is the maximum of all cardinals a for which there exists a set $\mathcal{P}(x, y)$ of pairwise independent x–y paths in G with $|\mathcal{P}(x, y)| = a$. (It is not difficult to see that this maximum exists.)

A subgraph $H \subset G$ is *convex* in G if H contains every induced path in G whose endvertices are in H. There are a number of interesting and useful equivalents of convexity, all easily proved:

Proposition 2.1. *For $H \subset G$, the following statements are equivalent:*

(i) *H is convex in G;*

(ii) *the endvertices of every H–H path in G are adjacent in H;*

(iii) *H is an induced subgraph of G and, for every vertex $x \in G \backslash H$, the subgraph $H[x] = G[x \to H] \cap H$ is complete;*

(iv) *if $A, B, X \subset V(H)$, then X separates A from B in H if and only if X separates A from B in G.* □

The following simple technical lemma provides a useful means for joining two convex subgraphs into one.

Lemma 2.2. *Let $G_1, G_2 \subset G$ be graphs, and suppose that $S = G_1 \cap G_2$ separates G_1 from G_2 in G.*

(i) *If G_1 and G_2 are convex in G, then so is $G_1 \cup G_2$.*

(ii) *If S is a simplex and G_i is convex in $G[G_i \to S]$, $i = 1, 2$, then $G_1 \cup G_2$ is convex in G.*

Proof. (i) is obvious from the definition of convexity.

(ii) As S is a simplex, $G[G_i \to S]$ is convex in G. Since G_i is convex in $G[G_i \to S]$ by assumption, this implies that G_i is also convex in G. Apply (i). □

3. Simplicial decompositions and tree-decompositions

Let G be a graph, $\sigma > 0$ an ordinal, and for each $\lambda < \sigma$ let B_λ be an induced subgraph of G. The family $F = (B_\lambda)_{\lambda<\sigma}$ is called a *simplicial decomposition* of G if the following three conditions hold:

(S1) $G = \bigcup_{\lambda<\sigma} B_\lambda$;

(S2) every $G|_\mu \cap B_\mu =: S_\mu$ is a (possibly empty) complete graph, where $G|_\mu := \bigcup_{\lambda<\mu} B_\lambda \quad (0 < \mu < \sigma)$;

(S3) no S_μ contains B_μ or any other $B_\lambda \quad (0 \leqslant \lambda < \mu < \sigma)$.

F is called a *simplicial tree-decomposition* if, in addition to (S1)–(S3),

(S4) each S_μ is contained in B_λ for some $\lambda < \mu \quad (\mu < \sigma)$.

If F satisfies (S1) and (S4) (but not necessarily (S2) or (S3)), F is called a *tree-decomposition* of G. The factors in such a tree-decomposition may be regarded as the vertices of a tree T_F (the *decomposition tree* of F), defined inductively by joining each 'vertex' B_μ to a 'predecessor' B_λ as provided by (S4) (with λ minimal).

For finite graphs this definition of a tree-decomposition is equivalent to the one recently introduced by Robertson and Seymour [9] (although more specific in that it fixes an enumeration of the tree's vertices), and it is general enough to include all finite simplicial decompositions as well: if σ is finite, then (S4) follows automatically from (S2) (see [2]). In the infinite case, however, a simplicial decomposition is not necessarily also a tree-decomposition: an infinite complete subgraph, used as S_μ for a factor B_μ, may be contained in the union of earlier factors without lying in any one of them.

In order to be able to describe the structure of a graph as imposed by a general simplicial decomposition, we therefore introduce the following concept: call the family $F = (B_\lambda)_{\lambda<\sigma}$ a *generalized tree-decomposition* of G if it satisfies (S1) and

(S5) $x \in G|_\mu \backslash S_\mu,\ y \in B_\mu \backslash S_\mu \ \Rightarrow\ \nexists \lambda < \sigma : x, y \in B_\lambda \quad$ (for every $\mu < \sigma$).

As is easily seen, (S5) is a weakening of condition (S2) as well as one of (S4). Thus every simplicial decomposition, as well as every tree-decomposition, is also a generalized tree-decomposition. We remark that in all these decompositions the graphs $G|_\mu = \bigcup_{\lambda<\mu} B_\lambda$ are induced subgraphs of G.

If $F = (B_\lambda)_{\lambda<\sigma}$ is a generalized tree-decomposition of G, and if $x \in V(G)$ and $H \subset G$, we denote by $\lambda(x)$ the minimal λ for which $x \in B_\lambda$, and set $\Lambda(H) := \{\, \lambda(x) \mid x \in V(H) \,\}$. Thus, the vertices x with $\lambda(x) = \mu$ are precisely those in $B_\mu \backslash S_\mu$. F will be called *coherent* if, for each $\mu < \sigma$, $B_\mu \backslash S_\mu$ is connected and every vertex of S_μ has a neighbour in $B_\mu \backslash S_\mu$.

As is customary in the field of simplicial decompositions, we shall usually refer to a complete graph as a *simplex*, and call the graphs S_μ in (S2) the simplices *of attachment* in F.

$H \subset G$ is called *attached to* $H' \subset G\backslash H$ if every vertex of H is adjacent to a vertex in H'. Thus, if F is coherent, then each S_μ is attached to $B_\mu\backslash S_\mu$. More generally we shall say that H is *attached* (*in* G) if H is attached to some component of $G\backslash H$; otherwise H is *unattached* (*in* G). One of the basic properties of factors in simplicial decompositions is that they are unattached subgraphs [1].

Let us call a partially ordered set T an *order-theoretical forest* if all its subsets of the form $\{ t' \mid t' \leqslant t \}$, $t \in T$, are chains; if, moreover, for every $t_1, t_2 \in T$ there exists a $t \in T$ with $t \leqslant t_1, t_2$, then T will be called an *order-theoretical tree*. Note that every rooted (graph-theoretical) tree T is such an order-theoretical tree with respect to the following natural partial order on its vertices: if r is the root of T and $x, y \in V(T)$, set $x \leqslant y$ if x is on the unique r–y path in T.

The order-theoretical forests of interest to us will be *well-founded*, which means that all their non-empty subsets have minimal elements. Observe that

Proposition 3.1. *A non-empty well-founded order-theoretical forest is an order-theoretical tree if and only if it has a unique minimal element.* □

With every generalized tree-decomposition $F = (B_\lambda)_{\lambda<\sigma}$ we can associate an order-theoretical tree T_F, its *(generalized) decomposition tree*. Indeed, let

$$T_F := \{ B_\lambda \mid \lambda < \sigma \}$$

and define a partial ordering $\leqslant$ on T_F recursively as follows. Let $\mu < \sigma$ be given, and suppose that $\leqslant$ has been defined for all pairs $(B_\lambda, B_{\lambda'})$ with $\lambda, \lambda' < \mu$. For each $\lambda < \mu$, set

$$B_\lambda < B_\mu \qquad \text{if } B_\lambda \leqslant B_{\lambda(s)} \text{ for some } s \in S_\mu\,;$$

otherwise let B_λ and B_μ be incomparable. Notice that this definition is such that $B_\lambda < B_{\lambda'}$ implies $\lambda < \lambda'$, so $\leqslant$ is antisymmetric; the transitivity of $\leqslant$ is clear by induction on μ.

Proposition 3.2. *If $F = (B_\lambda)_{\lambda<\sigma}$ is a generalized tree-decomposition of a graph G, then T_F is a well-founded order-theoretical tree.*

Proof. As $B_\lambda < B_{\lambda'}$ implies $\lambda < \lambda'$ and the set $\{ \lambda \mid \lambda < \sigma \}$ is well-ordered, T_F is clearly well-founded. As further $B_0 \leqslant B_\lambda$ for every $\lambda < \sigma$ (induction on λ), all we have to show is that T_F is an order-theoretical forest: that B_λ and $B_{\lambda'}$ are comparable whenever $\mu < \sigma$ and $B_\lambda, B_{\lambda'} < B_\mu$.

Let us apply induction on μ. By definition of $\leqslant$, there exist $s, s' \in S_\mu$ such that $B_\lambda \leqslant B_{\lambda(s)}$ and $B_{\lambda'} \leqslant B_{\lambda(s')}$; we shall assume that $\lambda(s) \leqslant \lambda(s')$. We prove that $B_\lambda \leqslant B_{\lambda(s')}$; the assertion then follows by the induction hypothesis applied to $\lambda(s')$. The validity of $B_\lambda \leqslant B_{\lambda(s')}$ is trivial if $\lambda(s) = \lambda(s')$, so let us assume that $\lambda(s) < \lambda(s')$. Then $s \in G|_{\lambda(s')}$, and therefore $s \in S_{\lambda(s')}$ (by (S5), $s' \in B_{\lambda(s')} \backslash S_{\lambda(s')}$ and $s, s' \in B_\mu$). Hence $B_\lambda \leqslant B_{\lambda(s)} < B_{\lambda(s')}$ by definition of $\leqslant$. □

One of the distinguishing features of an ordinary tree-decomposition F is the fact that the intersection of any two factors $B, B'' \in F$ is contained in every factor B' that lies on the B–B'' path in the decomposition tree. In the context of generalized tree-decompositions this continues to hold if the 'path' between B and B'' is a chain, that is, if B and B'' are comparable:[1] if $B, B', B'' \in T_F$ are such that $B \leqslant B' \leqslant B''$, then $B'' \cap B \subset B'$.

Let us prove this fact in the following slightly stronger version:

Proposition 3.3. *Let $F = (B_\lambda)_{\lambda<\sigma}$ be a generalized tree-decomposition of G, and let $B_\lambda, B_\mu, B_\nu \in T_F$ be such that $\lambda \leqslant \mu$ and $B_\mu \leqslant B_\nu$. Then $B_\nu \cap B_\lambda \subset B_\mu$.*

Proof. Suppose the assertion fails, and let B_λ, B_μ, B_ν form a counterexample such that ν is minimal. Clearly $\lambda < \mu$ and $B_\mu < B_\nu$. Since B_μ is an induced subgraph of G, our assumption of $B_\nu \cap B_\lambda \not\subset B_\mu$ means that $V(B_\nu) \cap V(B_\lambda) \smallsetminus V(B_\mu) \neq \emptyset$. By definition of T_F, there exists a vertex $s \in S_\nu$ for which $B_\mu \leqslant B_{\lambda(s)}$. From $\lambda(s) < \nu$ and the minimality of ν we have $B_{\lambda(s)} \cap B_\lambda \subset B_\mu$. Therefore

$$B_\nu \cap B_\lambda \smallsetminus B_{\lambda(s)} = (B_\nu \cap B_\lambda) \smallsetminus (B_{\lambda(s)} \cap B_\lambda) \supseteq B_\nu \cap B_\lambda \smallsetminus B_\mu \neq \emptyset\,.$$

As $B_\lambda \backslash B_{\lambda(s)} \subset G|_{\lambda(s)} \backslash S_{\lambda(s)}$ (by $\lambda < \mu \leqslant \lambda(s)$) and $s \in B_{\lambda(s)} \backslash S_{\lambda(s)}$, B_ν therefore meets $G|_{\lambda(s)} \backslash S_{\lambda(s)}$ as well as $B_{\lambda(s)} \backslash S_{\lambda(s)}$, a contradiction to (S5). □

Corollary 3.4. *If $F = (B_\lambda)_{\lambda<\sigma}$ is a generalized tree-decomposition of G, and if $B_\lambda, B_\mu \in T_F$ are such that $B_\lambda < B_\mu$, then*

$$S_\mu \subset \bigcup \{\, B \mid B_\lambda \leqslant B < B_\mu \,\}\,.$$

Proof. By Propositions 3.2 and 3.3 it suffices to show that

$$S_\mu \subset \bigcup \{\, B \mid B < B_\mu \,\}\,.$$

Since, for each $s \in S_\mu$, we have $B_{\lambda(s)} < B_\mu$ by definition of T_F, clearly $V(S_\mu) \subset \bigcup \{\, V(B) \mid B < B_\mu \,\}$. Now let ss' be an edge of S_μ, with $\lambda(s) \leqslant \lambda(s')$ say. Then $s \in B_{\lambda(s')}$ by (S5), and hence $ss' \in E(B_{\lambda(s')})$ because $B_{\lambda(s')}$ is an induced subgraph of G. □

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[1] If B and B'' are incomparable, a suitable translation of the finite condition might be that whenever $x \in B \cap B''$ and T is a subtree of T_F containing B and B'' (a subtree is defined below), then $x \in B'$ for some (and hence for every sufficiently great) $B' \in T$. However, we do not need this generalization in the present context.

Whenever T and T' are order-theoretical trees and T' is an induced subposet of T, let us call T' an (order-theoretical) *subtree* of T if the following holds for any $t, t', t'' \in T$:

$$t, t'' \in T', \quad t < t' < t'' \qquad \Rightarrow \qquad t' \in T'.$$

Equipped with the concept of a generalized decomposition tree we can easily express a fact which accounts for much of the usefulness of simplicial and related decompositions:

Theorem 3.5. *Let $F = (B_\lambda)_{\lambda<\sigma}$ be a decomposition of G satisfying* (S1) *and* (S2), *and let T be an order-theoretical subtree of T_F. Then*

$$G(T) := \bigcup T$$

is a convex subgraph of G. In particular, every B_μ and every $G|_\mu$ is convex in G.

Proof. The convexity of every B_μ and $G|_\mu$ is known; see e.g. [1, Prop. 1.1].[2] For general T we apply induction on

$$\tau := \sup\{\,\lambda \mid B_\lambda \in T\,\}.$$

If $\tau = 0$, then either $T = \emptyset$ or $G(T) = B_0$; in both cases we are done. Let now $\tau > 0$, and assume that the assertion holds for smaller values of τ. For each $\mu \leqslant \tau$ consider

$$T_\mu := \{\,B_\lambda \in T \mid \lambda \leqslant \mu\,\}.$$

As is easily checked, every T_μ is a subtree of T, and for $\mu < \tau$ the graphs $G(T_\mu)$ are convex in G by the induction hypothesis. Since these graphs are nested by inclusion, their union

$$H := \bigcup_{\mu<\tau} G(T_\mu) = \bigcup\{\,B_\lambda \in T \mid \lambda \neq \tau\,\}$$

is again convex in G. Now if $B_\tau \notin T$, then $H = G(T)$ and we are done. But otherwise $G(T) = H \cup B_\tau$, so the convexity of $G(T)$ will follow from that of H and B_τ as soon as we have shown that $H \cap B_\tau$ separates H from B_τ in G (Lemma 2.2.(i)).

Assuming that $H \neq \emptyset$ (as otherwise $G(T) = B_\tau$), pick B_ρ from some T_μ, $\mu < \tau$. Since T is an order-theoretical tree, there exists $B_\lambda \in T$ with $B_\lambda \leqslant B_\rho, B_\tau$. Then $\lambda \leqslant \rho \leqslant \mu < \tau$, so $B_\lambda \neq B_\tau$ and hence $B_\lambda < B_\tau$.

[2] That proposition is expressed for simplicial decompositions, i.e. formally assumes (S3) as well as (S1) and (S2). However, (S3) is not used in its proof.

By the subtree condition the graphs $B \in T_F$ with $B_\lambda < B < B_\tau$ are all in T. Therefore

$$H \supseteq \bigcup \{ B \in T_F \mid B_\lambda \leqslant B < B_\tau \} \supseteq S_\tau$$

by Corollary 3.4, and hence $S_\tau = H \cap B_\tau$ (as $H \subset G|_\tau$). Since S_τ separates $G|_\tau \supseteq H$ from B_τ in $G|_{\tau+1}$ and hence in G (by Proposition 2.1.(iv) and the already established convexity of $G|_{\tau+1}$ in G), this completes the proof. □

One of the most basic properties of a tree is that any two vertices can be separated by removing a single edge or vertex of the path between them. If F is a tree-decomposition of a graph G, this separation property of the tree T_F carries over to G: if B, B', B'' are factors in F such that B' separates B from B'' in T_F, then $V(B')$ separates the vertices of B from those of B'' in G.

This observation can be extended to generalized tree-decompositions in various ways. One of them—by no means the most general but all we shall need for the proof of Theorem 1.2—is the following.

Lemma 3.6. *Let $F = (B_\lambda)_{\lambda<\sigma}$ be a generalized tree-decomposition of G, and let $x, y \in V(G)$ be such that $x \in G|_\mu \backslash S_\mu$ and $y \in B_\mu \backslash S_\mu$ for some $\mu < \sigma$. Then S_μ separates x from y in G.*

Proof. Let G' be obtained from G by making all the S_λ's complete, i.e., let $G' := (V(G), E')$ where

$$E' := E(G) \cup \{ ss' \mid \exists\, \lambda < \sigma : \ s, s' \in V(S_\lambda) \}.$$

For each $\lambda < \sigma$ put $B'_\lambda := G'[B_\lambda]$ and $S'_\lambda := G'[S_\lambda]$. As $G' = \bigcup_{\lambda<\sigma} B'_\lambda$ and every S'_λ is a simplex, the family $F' := (B'_\lambda)_{\lambda<\sigma}$ will satisfy (S1) and (S2), provided that our definition for S'_λ is compatible with the one assumed in (S2), i.e., provided that $S'_\lambda = B'_\lambda \cap G'|_\lambda$ for every $\lambda < \sigma$. In order to show this, notice first that clearly

$$V(S'_\lambda) = V(S_\lambda) = V(B_\lambda \cap G|_\lambda) = V(B'_\lambda \cap G'|_\lambda).$$

Now let $ss' \in E(S'_\lambda)$ be given. Then $ss' \in E(B'_\lambda)$ by definition of B'_λ. To show that ss' is also in $E(G'|_\lambda)$, assume that $\lambda(s) \leqslant \lambda(s')$ $(< \lambda)$. Then $s' \in B_{\lambda(s')} \backslash S_{\lambda(s')}$, so, as $s, s' \in B_\lambda$ and F satisfies (S5), s cannot be in $G|_{\lambda(s')} \backslash S_{\lambda(s')}$. Hence s, as well as s', is in $B_{\lambda(s')}$. Since $B'_{\lambda(s')}$ is induced in G', this means that $ss' \in E(B'_{\lambda(s')}) \subset E(G'|_\lambda)$. F' therefore satisfies (S1) and (S2) as claimed.

To complete the proof, let now x and y be given as stated. $V(S_\mu)$ clearly separates x and y in $G'|_{\mu+1}$. Since $G'|_{\mu+1}$ is convex in G' by Theorem 3.5, this implies by Proposition 2.1 that $V(S_\mu)$ still separates x and y in G'. Therefore S_μ separates x and y in G. □

We conclude this section on the basic properties of generalized tree-decompositions with the observation that the 'branches' in a decomposition tree cannot be much longer than they are wide:

Proposition 3.7. *Let a be a regular cardinal, with initial ordinal α, and let F be such that $|B_\mu| < a$ for all $\mu < \sigma$. Then the following holds:*

(i) *the order type of any chain in T_F is at most α;*

(ii) *$\bigcup\{\, B \in T_F \mid B \leqslant B_\mu \,\}$ $(\subset G)$ has order $< a$, for every $\mu < \sigma$.*

Proof. (i) Suppose the contrary, and let $\mu < \sigma$ be minimal such that

$$\mathcal{C} := \{\, B \in T_F \mid B \leqslant B_\mu \,\}$$

has order a. Then $|\mathcal{C} \setminus \{\, B_\mu \,\}| = a$, and

$$\mathcal{C} \setminus \{\, B_\mu \,\} = \bigcup_{s \in S_\mu} \{\, B \in T_F \mid B \leqslant B_{\lambda(s)} \,\}$$

by definition of T_F. Thus $\mathcal{C} \setminus \{\, B_\mu \,\}$ is a union of $< a$ sets of order $< a$ (by the choice of μ), which contradicts the regularity of a.

(ii) is immediate from (i) and the regularity of a. □

4. Decompositions into small factors

In this section we sketch a proof Theorem 1.1. Those parts of the theorem which go beyond Halin's original results are proved in full (Lemma 4.2 and Theorem 4.3), while the rest is outlined so as to convey the main ideas. For a more thorough treatment of this material see [2, Ch. 5].

In order to understand the conception of the proof of Theorem 1.1, one has to take into account that it was not originally intended to solve the problem of describing the structure of TK_a-free graphs. The problem it was intended to solve—and which it did solve except for the case of $a = \aleph_0$—was to decide which graphs have a simplicial decomposition into factors smaller than a given infinite cardinal a. The TK_a-free graphs, as we shall see, were merely incidental to interpreting the solution (Theorem 4.3).

Suppose we are trying to construct a simplicial decomposition F of some graph G, so that every factor in F has order less than some given cardinal a. The main difficulty we face in keeping the factors small is that they have to be convex subgraphs of G; cf. Theorem 3.5.

Now suppose we have a subgraph $D \subset G$ which is a candidate for being a factor in our decomposition, but which is not convex. By Proposition 2.1.(ii), this means that D has non-adjacent vertices x, y which are the endvertices of some D–D path in G. An obvious way of trying to make D convex therefore is to incorporate, for every such pair x, y, some maximal set $\mathcal{P}(x, y)$ of independent D–D paths $x \dots y$ into D, and then to iterate this procedure as long as necessary.

And indeed, it is not difficult to show that the supergraph H of D obtained in this way must be convex. Moreover, H will have order $< a$ provided that we never add a or more paths at a time, and that a is large enough to allow for a countable number of iterations. (To see that countably many extensions are enough, notice that the endvertices x, y of any H–H path $P \subset G$ will be present after finitely many steps, so P should have been added in the next step unless x and y were adjacent.) More precisely, H will have order $< a$ provided that $\mu_G(x, y) < a$ for any non-adjacent vertices x, y for which paths have to be added, and that a is uncountable:

Lemma 4.1. [5] *Let G be a graph, $a > \aleph_0$ a regular cardinal, and suppose that $\mu_G(x, y) < a$ for any two non-adjacent vertices x, y of G. Then, for every $D \subset G$ with $|D| < a$, there exists a convex subgraph H of G such that $D \subset H$ and $|H| < a$.*

Lemma 4.1 will be our principal tool in constructing, factor by factor, a simplicial decomposition into factors of order $< a$. Yet simplicial factors are not only convex but also unattached (see § 3). The convex graphs H provided by Lemma 4.1, however, may well be attached in G: consider, for example, the case where G is a simplex of order a and $D = H$ is a proper subsimplex of G.

In order to sharpen Lemma 4.1 in such a way that it guarantees the existence of small convex and unattached supergraphs, we have to ban large complete subgraphs:

Lemma 4.2. *Let G be a graph, $a > \aleph_0$ a regular cardinal, and suppose that $\mu_G(x, y) < a$ for any two non-adjacent vertices x, y of G. Suppose further that G has no complete subgraph of order a. Then, for every $D \subset G$ with $|D| < a$, there exists a convex and unattached subgraph H of G such that $D \subset H$ and $|H| < a$.*

Proof. By Lemma 4.1, G has a convex subgraph H' of order $< a$ that contains D. Suppose H' is attached in G. Then H' is a simplex, by Proposition 2.1.(iii). Let $D' \subset G$ be a maximal simplex containing H'; note that $|D'| < a$ by assumption. If $D' = G$, then $H := D'$ is as desired. Otherwise pick $v \in V(G) \backslash V(D')$, and let H be a convex subgraph of G such that $D' \cup \{ v \} \subset H$ and $|H| < a$ (again by Lemma 4.1). H is unattached in G, because otherwise H would be a simplex contradicting the maximality of D'. □

Using Lemma 4.2, let us now prove Halin's basic existence theorem for simplicial decompositions into small factors [6].[3]

Theorem 4.3. *Let G be a graph and a a regular cardinal, such that $|G| \geqslant a > \aleph_0$. Suppose that $G \not\supseteq K_a$, and that $\mu_G(x, y) < a$ for any two non-adjacent vertices x, y of G. Let σ be the initial ordinal of $|G|$. Then G admits a coherent simplicial decomposition $F = (B_\lambda)_{\lambda<\sigma}$ with $|B_\lambda| < a$ for all $\lambda < \sigma$.*

Proof. Let $V(G)$ be well-ordered as $(v_\rho)_{\rho<\sigma}$. We define the factors B_λ of F recursively for $\lambda < \sigma$, so that the following two conditions are satisfied for each λ:

(i) B_λ is unattached in G, and $|B_\lambda| < a$;

(ii) $\bigcup_{\lambda' \leqslant \lambda} B_{\lambda'}$ is convex in G.

Let μ be given, $0 \leqslant \mu < \sigma$, and suppose that for every $\lambda < \mu$ we have defined B_λ so as to satisfy (i) and (ii). We shall define B_μ in such a way that (i) and (ii) hold for $\lambda = \mu$.

We first show that $G|_\mu := \bigcup_{\lambda<\mu} B_\lambda$ is convex in G. If $\mu = 0$, this is trivial as $G|_\mu = \emptyset$. If μ is a successor ordinal, then $G|_\mu$ is convex by assumption (ii). Finally, if μ is a limit ordinal, then $G|_\mu$ is the nested union of the graphs $\bigcup_{\lambda' \leqslant \lambda} B_{\lambda'}$ with $\lambda < \mu$; since these graphs are convex by (ii), $G|_\mu$ is also convex.

Since $\big|G|_\mu\big| < a \leqslant |G|$ by (i) and the regularity of a, we have $G \backslash G|_\mu \neq \emptyset$. Let $\rho(\mu) := \min \{ \rho \mid v_\rho \notin G|_\mu \}$, and put

$$\begin{aligned} G_\mu &:= G\,[\,v_{\rho(\mu)} \to G|_\mu\,]\,, \\ C_\mu &:= G_\mu \setminus G|_\mu\,, \\ S_\mu &:= G_\mu \cap G|_\mu\,. \end{aligned}$$

Then $S_\mu = G|_\mu\,[\,v_{\rho(\mu)}\,]$, so S_μ is a simplex (Proposition 2.1.(iii)) and hence of order $< a$.

Applying Lemma 4.2 to the graph G_μ, we choose as B_μ a convex and unattached subgraph of G_μ such that $S_\mu \cup \{ v_{\rho(\mu)} \} \subset B_\mu$ and $|B_\mu| < a$. Then (i) holds for $\lambda = \mu$. Furthermore, $S_\mu = G|_\mu \cap B_\mu$, as required by (S2). By Lemma 2.2.(ii), $G|_\mu \cup B_\mu$ is convex in G, establishing (ii) for $\lambda = \mu$. Finally, as S_μ is attached, it cannot contain any factor B_λ for $\lambda < \mu$, since these are unattached by assumption (i). Therefore our choice of B_μ satisfies (S3).

It remains to show that $\bigcup_{\lambda<\sigma} B_\lambda = G$ (condition (S1)), and that F is coherent. To see (S1), notice that $v_\lambda \in G|_{\lambda+1}$ for every $\lambda < \sigma$, which follows by our choice of B_λ and an easy induction on λ. Thus

$$V(G) \subset \bigcup_{\lambda<\sigma} V(G|_{\lambda+1}) = \bigcup_{\lambda<\sigma} V(B_\lambda)\,.$$

[3] Halin's original version of this theorem was for decompositions satisfying (S1) and (S2), but not necessarily (S3).

As every $G|_{\lambda+1}$ is convex and hence induced in G, this implies (S1).

To see that F is coherent, suppose that, for some $\lambda < \sigma$, S_λ is not attached to $B_\lambda \backslash S_\lambda$ or $B_\lambda \backslash S_\lambda$ is disconnected. In either case there exists a subsimplex $S \subset S_\lambda$ which separates vertices $x, y \in B_\lambda \backslash S$ in B_λ. As S_λ is attached to C_λ and $B_\lambda \backslash S_\lambda \subset C_\lambda$, S cannot separate x and y in G_λ. This contradicts the convexity of B_λ in G_λ, by Proposition 2.1.(iv). □

The applicability of Theorem 4.3 is likely to be hampered by an obvious shortcoming: its awkward condition on the Menger numbers $\mu_G(x,y)$. However, there is a strikingly simple and, with hindsight, ingenious way of dealing with this problem, again due to Halin.

Define the *a-closure* $[G]_a$ of G to be the graph with vertex set $V(G)$ and edge set $E(G) \cup \{ xy \mid \mu_G(x,y) \geqslant a \}$. The term '$a$-closure' is justified by the following observation:

Proposition 4.4. *If G is a graph and a is a cardinal, then $[G]_a$ is its own a-closure.*

(The proof of Proposition 4.4 is not difficult; see e.g. [2].)

By Proposition 4.4 it is clear that the a-closure $[G]_a$ of a graph G is such that $\mu_{[G]_a}(x,y) < a$ for any two non-adjacent vertices x and y. If $|G| \geqslant a > \aleph_0$ and a is regular, we can therefore apply Theorem 4.3 to $[G]_a$ rather than to G, provided that $[G]_a \not\supseteq K_a$. The next question is therefore how the latter can be ensured by imposing an additional constraint on G.

The following lemma of Halin [6] answers this question in a very satisfactory way: it implies that if G contains no *subdivided* K_a then $[G]_a \not\supseteq K_a$.

Proposition 4.5. *If G is a graph and a is an infinite cardinal, then*

$$[G]_a \supseteq \mathrm{T}K_a \quad \Leftrightarrow \quad [G]_a \supseteq K_a \quad \Leftrightarrow \quad G \supseteq \mathrm{T}K_a \,.$$

Notice that the assertions of Theorem 4.3 and Propositions 4.4–5 together amount to a proof of Theorem 1.1: if $G \not\supseteq \mathrm{T}K_a$, then $[G]_a \not\supseteq K_a$ and $\mu_{[G]_a}(x,y) < a$ for any non-adjacent $x, y \in [G]_a$, so $[G]_a$ admits a simplicial decomposition F into factors of order $< a$. Moreover, F induces a generalized tree-decomposition of G into factors of order $< a$—recall that F satisfies (S5) as a consequence of (S2), and notice that the validity of (S5) remains unaffected by the deletion of edges.

Conversely, it is clear that in general not every graph with a generalized tree-decomposition into factors of order $< a$ will be $\mathrm{T}K_a$-free. For example, a K_a has itself such a decomposition: well-order its vertices by the initial ordinal of a, and consider the decomposition induced by the initial segments of this well-ordering.

In the next section we shall prove that this simple example describes essentially the only way in which a $\mathrm{T}K_a$ can arise in a graph with a generalized tree-decomposition into 'small' factors, namely, corresponding to a 'long' chain in T_F: all graphs without a $\mathrm{T}K_a$ will be shown to admit a generalized tree-decomposition F in which every factor and every chain $\mathcal{C} \subset T_F$ has order $< a$, and, conversely, any graph G with such a decomposition will be seen to satisfy $G \not\supseteq \mathrm{T}K_a$ (Theorem 1.2).

5. The structure of $\mathbf{T}K_a$-free graphs

We begin our proof of Theorem 1.2 with two lemmas.

Lemma 5.1. *Let $a > \aleph_0$ be a regular cardinal, $\alpha = \{\,\beta \mid \beta < \alpha\,\}$ its initial ordinal, and $f\colon \alpha \to \alpha$ any map satisfying*

$$f(\beta) < \beta \tag{1}$$

for all $\beta > 0$, and

$$\beta < \gamma \quad\Rightarrow\quad f(\beta) \leqslant f(\gamma) \tag{2}$$

for all $\beta, \gamma < \alpha$. Then $\sup\{\, f(\beta) \mid \beta < \alpha\,\} < \alpha$.

Proof. Suppose the assertion fails. Then, for each $\gamma < \alpha$, the set $\{\,\beta < \alpha \mid f(\beta) > \gamma\,\}$ is non-empty; let

$$\gamma^+ := \min\{\,\beta < \alpha \mid f(\beta) > \gamma\,\}.$$

By (1), clearly $\gamma^+ > \gamma$ for every $\gamma < \alpha$. (For the case of $\gamma = 0$ notice that $f(0) \leqslant f(1) < 1$ by (2), so $f(0) = 0$.) Set $\gamma_1 := 0$, define $\gamma_{n+1} := \gamma_n^+$ recursively for all $n \in \mathbb{N}$, and let

$$\gamma^* := \sup\{\,\gamma_n \mid n \in \mathbb{N}\,\}.$$

As $\gamma_n < \alpha$ for every n, and as a is regular, we have $\gamma^* < \alpha$, so f is still defined for γ^*. Since $f(\gamma^*) < \gamma^*$ by (1), there exists an $n \in \mathbb{N}$ such that $f(\gamma^*) < \gamma_n$. Then

$$f(\gamma^*) < \gamma_n < f(\gamma_{n+1})$$

by definition of γ_{n+1}, while at the same time

$$\gamma_{n+1} < \gamma^*.$$

This contradicts assumption (2). □

Lemma 5.2. *Let $a > \aleph_0$ be a regular cardinal, and let $F = (B_\lambda)_{\lambda<\sigma}$ be a simplicial decomposition of a graph G such that T_F contains a chain of order a. Then $G \supseteq K_a$.*

Proof. Let α be the initial ordinal of a. Choose a chain $\mathcal{C} \subset T_F$ of order type α such that $\mathcal{C}$ is a subtree of T_F (i.e., such that $B < B' < B''$ and $B, B'' \in \mathcal{C}$ imply $B' \in \mathcal{C}$), and let

$$\Lambda := \{ \lambda < \sigma \mid B_\lambda \in \mathcal{C} \}.$$

Note that the natural well-ordering of Λ mirrors that of $\mathcal{C}$, in that $\lambda \leqslant \mu \Leftrightarrow B_\lambda \leqslant B_\mu$ for any $\lambda, \mu \in \Lambda$. In particular, Λ has order type α.

We shall find a K_a in $\bigcup \mathcal{C}$, proceeding in two steps. First, we use Lemma 5.1 to show that in every tail of $\mathcal{C}$ there appear new vertices which stay in every subsequent $B \in \mathcal{C}$. In the second step we construct a K_a from these vertices, using the fact that any two of them are adjacent because they are both in the simplex of attachment of every later $B \in \mathcal{C}$.

For the first step of the proof let us show that

$$(*) \qquad \forall \lambda \in \Lambda \colon \exists v \in V(G),\ \lambda \leqslant \lambda(v) \in \Lambda \colon v \in \bigcap_{\lambda(v) \leqslant \mu \in \Lambda} B_\mu\,.$$

Let $\lambda \in \Lambda$ be given, and put

$$\Lambda' := \{ \lambda' \in \Lambda \mid \lambda \leqslant \lambda' \}.$$

Note that $|\Lambda'| = a$, so Λ', like Λ, has order type α. Let us define a map $f \colon \Lambda' \to \Lambda'$ by setting

$$f(\mu) := \min \left(\Lambda(B_\mu) \cap \Lambda' \right)$$

for each $\mu \in \Lambda'$. Notice that while $f(\lambda) = \lambda$, we have

$$\lambda \leqslant f(\mu) < \mu \tag{1}$$

for all $\mu \in \Lambda'$ with $\lambda < \mu$: since $B_\lambda < B_\mu$, there exists a vertex $s \in S_\mu$ such that $B_\lambda \leqslant B_{\lambda(s)}$ (and hence $\lambda \leqslant \lambda(s)$), so $f(\mu) \leqslant \lambda(s) < \mu$. (Note that $\lambda(s)$ is in Λ, and hence in Λ', because $B_\lambda, B_\mu \in \mathcal{C}$ and $\mathcal{C}$ is a subtree of T_F.) Moreover, if $\mu, \nu \in \Lambda'$ satisfy $\mu < \nu$, and if $x \in B_\nu$ is such that $\lambda(x) \leqslant \mu$, then $x \in B_\mu$ by Proposition 3.3. Thus

$$\mu < \nu \quad \Rightarrow \quad f(\mu) \leqslant f(\nu) \tag{2}$$

for all $\mu, \nu \in \Lambda'$. Since Λ' has order type α, (1) and (2) imply by Lemma 5.1 that $\{ f(\mu) \mid \mu \in \Lambda' \}$ is bounded in Λ': there exists a $\mu^* \in \Lambda'$ such that $f(\mu) \leqslant \mu^*$ for all $\mu \in \Lambda'$. Set

$$\Lambda'' := \{ \mu \in \Lambda \mid \mu^* < \mu \}.$$

Note again that $|\Lambda''| = a$, so Λ'' too has order type α.

For each $\mu \in \Lambda''$, pick a vertex x from B_μ with $\lambda(x) = f(\mu)$. Since $B_\mu \cap B_{f(\mu)} \subset B_{\mu^*}$ by Proposition 3.3, x must be in B_{μ^*}. Thus,

$$\forall\, \mu \in \Lambda'' : \exists\, x \in B_\mu \cap B_{\mu^*} : \lambda(x) = f(\mu)\,. \tag{3}$$

Let

$$S := B_{\mu^*} \cap \bigcup_{\mu \in \Lambda''} B_\mu\,.$$

Notice that S is a simplex: if $s \in B_{\mu^*} \cap B_\mu$ and $t \in B_{\mu^*} \cap B_\nu$ for $\mu, \nu \in \Lambda''$, $\mu \leqslant \nu$, then $t \in B_\mu$ by Proposition 3.3; thus $s, t \in B_\mu$ and hence $s, t \in S_\mu$ (because $\lambda(s), \lambda(t) \leqslant \mu^* < \mu$), so s and t are adjacent. Our proof would thus be complete if $|S| = a$; we shall therefore assume that $|S| < a$.

Since any vertex $x \in B_{\mu^*}$ which satisfies (3) for some $\mu \in \Lambda''$ is a vertex of S, our assumption of $|S| < a$ means that B_{μ^*} contains a vertex x which satisfies (3) for a different values of μ, and hence for arbitrarily large $\mu \in \Lambda''$. Choose v for $(*)$ to be such a vertex x.

Now if $\mu \in \Lambda$ with $\lambda(v) \leqslant \mu$ is given, then there exists a $\nu \in \Lambda''$, $\mu \leqslant \nu$, such that v satisfies (3) for ν. Then $v \in B_\nu \cap B_{\lambda(v)} \subset B_\mu$ by Proposition 3.3. This completes the proof of $(*)$.

To complete the proof of the Lemma, let us now use $(*)$ to find a simplex of order a in G. For each $\beta < \alpha$, choose a vertex $s_\beta \in G$ with $\lambda(s_\beta) \in \Lambda$, as follows. Suppose s_γ has been chosen for every $\gamma < \beta$. In order to choose s_β, take the supremum of $\{\, \lambda(s_\gamma) \mid \gamma < \beta \,\}$ in Λ, let λ be the successor in Λ of this supremum, and let s_β be the corresponding vertex v provided by $(*)$. (The said supremum exists, in Λ, because $|\Lambda| = a$ is regular and β, as well as each of the sets $\{\, \zeta \in \Lambda \mid \zeta < \lambda(s_\gamma) \,\}$, has order $< a$.) It remains to show that $\{\, s_\beta \mid \beta < \alpha \,\}$ spans a simplex in G. Let $\delta, \gamma < \alpha$ be given, and pick $\beta < \alpha$ with $\delta, \gamma < \beta$. Then $\lambda(s_\delta), \lambda(s_\gamma) < \lambda(s_\beta)$ by choice of s_β. Therefore $s_\delta, s_\gamma \in B_{\lambda(s_\beta)}$ by $(*)$, and hence $s_\delta, s_\gamma \in S_{\lambda(s_\beta)}$, so s_δ and s_γ are adjacent in G. □

Equipped with Lemmas 5.1 and 5.2, we can now prove Theorem 1.2 without difficulty. In order to make the result a little stronger, we restate its two main implications separately:

Theorem 5.3. *Let G be a graph and $a > \aleph_0$ a regular cardinal.*

- (i) *If $G \not\supseteq \mathrm{T}K_a$, then G admits a generalized tree-decomposition $F = (B_\lambda)_{\lambda<\sigma}$ satisfying* (S3)*, such that every B_λ and every chain in T_F has order $< a$. If $|G| \geqslant a$, then F can be chosen in such a way that σ is the initial ordinal of $|G|$.*
- (ii) *If G admits a generalized tree-decomposition $F = (B_\lambda)_{\lambda<\sigma}$ such that every B_λ and every chain in T_F has order $< a$, then $G \not\supseteq \mathrm{T}K_a$.*

Proof. (i) If $|G| < a$, the assertion is trivial; we shall therefore assume that $|G| \geqslant a$. Let $G' := [G]_a$ be the a-closure of G. By our assumption of $G \not\supseteq \mathrm{T}K_a$ and Proposition 4.5, G' contains no K_a. Since $\mu_{G'}(x, y) < a$ for any two non-adjacent vertices $x, y \in G'$ by Proposition 4.4, Theorem 4.3 implies that G' admits a simplicial decomposition $F' = (B'_\lambda)_{\lambda<\sigma}$ satisfying (S3), where $|B'_\lambda| < a$ for all $\lambda < \sigma$ and σ is the initial ordinal of $|G'| = |G|$. By Lemma 5.2 any chain in $T_{F'}$ has order $< a$. The family $F = (B'_\lambda \cap G)_{\lambda<\sigma}$ thus is a generalized tree-decomposition of G with the desired properties.

(ii) Suppose $G \supseteq T \simeq \mathrm{T}K_a$, and let

$$\Lambda := \{ \lambda(t) \mid t \text{ is a branch vertex of } T \}.$$

As $|B_\lambda| < a$ for every $\lambda < \sigma$, clearly $|\Lambda| = a$ by the regularity of a. We shall prove that $\{ B_\lambda \mid \lambda \in \Lambda \}$ is a chain in T_F.

Let distinct $\mu, \nu \in \Lambda$ be given, with $\mu < \nu$ say. In order to show that B_μ and B_ν are comparable in T_F, we prove that $\mu = \lambda(s)$ for some $s \in S_\nu$. Choose s to be any branch vertex of T with $\lambda(s) = \mu$, and let t be a branch vertex of T with $\lambda(t) = \nu$. Then $t \in B_\nu \backslash S_\nu$, and $s \in G|_\nu$. Now if $s \in G|_\nu \backslash S_\nu$, then S_ν separates s from t in G by Lemma 3.6, contradicting the fact that S_ν has order $< a$. Therefore $s \in S_\nu$ as claimed. □

6. A refined decomposition theorem

By placing the emphasis in Theorem 1.2 just on the size of the factors and their relative position within G, we lost some possibly valuable information on $G' = [G]_a$ (see the proof of Theorem 5.3) which might still be exploited for G: the fact that the endvertices of any edge in $E(G') \backslash E(G)$ are joined in G by at least a independent paths. For example, we might be trying to prove that G has a subgraph with a certain property $\mathcal{P}$, and it is easier to find such a subgraph H in G'. Now if $\mathcal{P}$ is invariant under subdivision, i.e. under the replacement of edges with independent paths, we may be able to transform H back into a subgraph of G still satisfying $\mathcal{P}$, by replacing the foreign edges with suitable paths in G. The fact that for each edge there are $\geqslant a$ paths to choose from helps to select them in such a way that they are pairwise disjoint and avoid the rest of H. However, it may be necessary to keep the interiors of different replacement paths well apart, for example in distinct components of the subgraphs $G \backslash G|_\mu$, or inside different factors.

In the remainder of this paper we shall follow this line. The theorem we obtain will be used, in a subsequent paper, for the proof of a result concerning the existence of end-faithful spanning trees in infinite graphs [3].

Lemma 6.1. *Let a and b be regular cardinals with $a > b \geqslant \aleph_0$. Let G be a graph not containing any complete subgraph of order b, and such that $\mu_G(x,y) < a$ for any two non-adjacent vertices x, y of G. Let further $E \subset E(G)$. Then every graph $D \subset G$ of order $< a$ has a convex and unattached supergraph H in G, $|H| < a$, which satisfies the following condition:*

(∗) *For every two vertices x, y of H, the number of components C of $G \backslash H$ for which $G[C \to H]$ contains an H–H path P with endvertices x, y and $E(P) \subset E$, is either 0 or at least a.*

Proof. Let β be the initial ordinal of b. For every $\lambda \leqslant \beta$ we shall define a convex and unattached supergraph H_λ of D, such that $|H_\lambda| < a$ for all $\lambda \leqslant \beta$, and $H_\lambda \subset H_\mu$ whenever $\lambda < \mu \leqslant \beta$. H_β will be our desired graph H.

Let H_0 be the convex and unattached supergraph of D provided by Lemma 4.2. Suppose that H_λ has been defined as stated for every $\lambda < \mu$, where $0 < \mu \leqslant \beta$. If μ is a limit ordinal, we set $H_\mu := \bigcup_{\lambda<\mu} H_\lambda$; then H_μ is convex in G because every H_λ is convex, H_μ is unattached because $H_\mu \supseteq H_0$ and H_0 is unattached, and $|H_\mu| < a$, because $|H_\lambda| < a$ for every $\lambda < \mu$ and $|\mu| \leqslant |\beta| = b < a$.

Suppose now that μ is a successor ordinal, say of λ. For distinct vertices x, y of H_λ, let $\mathcal{C}_\lambda(x,y)$ denote the set of all components C of $G \backslash H_\lambda$ that are relevant for condition (∗)—i.e., for which $G[C \to H_\lambda]$ contains an H_λ–H_λ path P with endvertices x, y and $E(P) \subset E$. Let E_λ be the set of all pairs $\{x, y\} \subset V(H_\lambda)$ for which $\mathcal{C}_\lambda(x,y)$ is non-empty but of order $< a$. Notice that since H_λ is convex, any graph of the form $H_\lambda[C]$, where C is a component of $G \backslash H_\lambda$, must be a simplex, and hence of order $< b$. Thus $E_\lambda \subset E(H_\lambda)$, and for every $xy \in E_\lambda$ and each $C \in \mathcal{C}_\lambda(x,y)$ Lemma 4.1 provides us with a convex subgraph $H_\lambda(C) \subset G[C \to H_\lambda]$ of order $< a$, so that $H_\lambda(C)$ contains $H_\lambda[C]$ and at least one vertex of C.

Set

$$H_\mu := H_\lambda \cup \bigcup_{xy \in E_\lambda} \; \bigcup_{C \in \mathcal{C}_\lambda(x,y)} H_\lambda(C).$$

H_μ is convex in G, because H_λ and every $H_\lambda(C)$ is convex, and $H_\lambda[C] = H_\lambda \cap H_\lambda(C)$ is a simplex which separates H_λ from $H_\lambda(C)$ in G (see Lemma 2.2.(ii)). H_μ is unattached in G, because $H_\mu \supseteq H_0$ and H_0 is unattached. Finally, $|H_\mu| < a$, because $|E_\lambda| \leqslant |H_\lambda|^2 < a$, $|\mathcal{C}_\lambda(x,y)| < a$ for $xy \in E_\lambda$, and $|H_\lambda(C)| < a$ for $C \in \mathcal{C}_\lambda(x,y)$.

Let us finally set $H := H_\beta$.

It remains to prove that H satisfies (∗). Suppose (∗) fails, and let $x, y \in H$ be such that $1 \leqslant |\mathcal{C}_\beta(x,y)| < a$. Pick $C \in \mathcal{C}_\beta(x,y)$. We shall prove that $C \in \mathcal{C}_\gamma(x,y)$ for some $\gamma < \beta$, deduce that $|\mathcal{C}_\gamma(x,y)| \geqslant a$ (as otherwise $H_{\gamma+1}$ ought to contain a vertex from C), and use this to derive a contradiction to the fact that $|\mathcal{C}_\beta(x,y)| < a$ and $|H| < a$.

Let $S := H[C]$. Clearly $x, y \in S$. For vertices $s \in S$, write $\lambda(s) := \min\{\lambda \mid s \in H_\lambda\}$, and put $\Lambda := \{\lambda(s) \mid s \in S\}$. As β is the initial ordinal of b and hence a limit ordinal, we have $\lambda(s) < \beta$ for every $s \in S$, and therefore $|\lambda| < b$ for every $\lambda \in \Lambda$. Moreover, $|\Lambda| \leqslant |S| < b$, because S is a simplex. Therefore $\gamma := \bigcup \Lambda < \beta$ (by the regularity of b), and $S \subset H_\gamma$.

We now show that $C \in \mathcal{C}_\gamma(x,y)$. Clearly $C \subset G \backslash H \subset G \backslash H_\gamma$; let C' be the component of $G \backslash H_\gamma$ containing C. Then $C' \in \mathcal{C}_\gamma(x,y)$, so all we have to show is that $C' \backslash C = \emptyset$. But if $C' \backslash C \neq \emptyset$, then G contains a C–$(C' \backslash C)$ edge, because C' is connected. Therefore $(C' \backslash C) \cap H[C] \neq \emptyset$, which contradicts the fact that $H[C] = S \subset H_\gamma$. Thus $C' \backslash C = \emptyset$, as required.

To show that $|\mathcal{C}_\gamma(x,y)| \geqslant a$, recall that if $|\mathcal{C}_\gamma(x,y)| < a$, then $xy \in E_\gamma$, and consequently $H_{\gamma+1} \cap C \neq \emptyset$. But then also $H \cap C \neq \emptyset$, a contradiction. Now as $|\mathcal{C}_\gamma(x,y)| \geqslant a$ but $|\mathcal{C}_\beta(x,y)| < a$, we have $|\mathcal{C}_\gamma(x,y) \setminus \mathcal{C}_\beta(x,y)| \geqslant a$, while $H \cap C_\gamma \neq \emptyset$ for every $C_\gamma \in \mathcal{C}_\gamma(x,y) \setminus \mathcal{C}_\beta(x,y)$. This contradicts the fact that $|H| < a$. □

Theorem 6.2. *Let a and b be regular cardinals with $a > b \geqslant \aleph_0$, and let G be a graph of order $\geqslant a$ which does not contain any $\mathrm{T}K_b$. Let σ be the initial ordinal of $|G|$. Then the a-closure $[G]_a =: G'$ of G admits a coherent simplicial decomposition $F = (B_\lambda)_{\lambda<\sigma}$ into factors of order $< a$ and with simplices of attachment of order $< b$, which has the following property: for every $\mu < \sigma$ and every edge $xy \in E_\mu := \big(E(B_\mu) \backslash E(S_\mu)\big) \setminus E(G)$, there are at least a ordinals ν, with $S_\nu \subset B_\mu$, such that B_ν contains an S_ν–S_ν path P with endvertices x, y and $E(P) \subset E(G)$.*

Proof. Our proof is along the lines of that of Theorem 4.3; the main difference will be that we use Lemma 6.1 rather than Lemma 4.2 to choose the factors for F, and that our choice of the graph D in the lemma has to ensure that each factor B_ν contains the correct S_ν–S_ν paths. For the application of Lemma 6.1, note that $G' \not\supseteq K_b$ by Proposition 4.5, and that $\mu_{G'}(x,y) < a$ for non-adjacent $x, y \in G'$ by Proposition 4.4.

Let $V(G')$ be well-ordered as $(v_\rho)_{\rho<\sigma}$. In our construction of F, we shall use the following abbreviations:

$$G'|_\lambda := \bigcup_{\lambda'<\lambda} B_{\lambda'}$$
$$E_\lambda := \big(E(B_\lambda) \backslash E(G'|_\lambda)\big) \setminus E(G)\,.$$

Let μ be given, $0 \leqslant \mu < \sigma$. Suppose that for every $\lambda < \mu$ we have already defined an induced subgraph B_λ of G' and, for every $e = xy \in E_\lambda$, a set $\mathcal{P}(e)$ of $G'|_{\lambda+1}$–$G'|_{\lambda+1}$ paths $P \subset G$ with endvertices x, y, such that

(i) B_λ is unattached in G';

(ii) $G'|_{\lambda+1}$ is convex in G';

(iii) if C is a component of $G' \setminus G'|_{\lambda+1}$ and $e \in E_\lambda$, then at most one $P \in \mathcal{P}(e)$ meets C.

We shall define B_μ and $\mathcal{P}(e)$ (for all $e \in E_\mu$) such that (i)–(iii) hold for $\lambda = \mu$.

As in the proof of Theorem 4.3, note that $G'|_\mu$ is convex. Let $\rho(\mu) := \min\{\,\rho \mid v_\rho \notin G'|_\mu\,\}$, and put

$$\begin{aligned} G'_\mu &:= G'\,[\,v_{\rho(\mu)} \to G'|_\mu\,]\,,\\ C_\mu &:= G'_\mu \setminus G'|_\mu\,,\\ S_\mu &:= G'_\mu \cap G'|_\mu\,. \end{aligned}$$

Then $S_\mu = G'|_\mu\,[\,v_{\rho(\mu)}\,]$, so S_μ is a simplex (Proposition 2.1.(iii)), and hence of order $< b$.

Notice that if $\Lambda(S_\mu)$ has a maximum λ, then C_μ is already a component of $G'\backslash G'|_{\lambda+1}$ (even if $\lambda+1 < \mu$). Then the graph

$$Q_\mu := \bigcup\{\,P \in \mathcal{P}(e) \mid e \in E_\lambda \text{ and } P \cap C_\mu \neq \emptyset\,\}$$

has order $< a$, by (iii) and the fact that $|S_\mu| < a$. Applying Lemma 6.1 to the graph G'_μ, let B_μ be the graph H that extends $D \subset G'_\mu$ with respect to $E := E(G'_\mu) \cap E(G)$, where

$$D := \begin{cases} S_\mu \cup \{\,v_{\rho(\mu)}\,\} \cup Q_\mu, & \text{if } \lambda = \max \Lambda(S_\mu);\\ S_\mu \cup \{\,v_{\rho(\mu)}\,\}, & \text{if } \Lambda(S_\mu) \text{ has no maximum.} \end{cases}$$

(It is clear that D has order $< a$.) Then $|B_\mu| < a$, and B_μ is convex and unattached in G'_μ. Since $B_\mu\,[\,v\,] = S_\mu \subsetneqq B_\mu$ for every $v \in G'\backslash G'_\mu$, B_μ is also unattached in G', establishing (i) for $\lambda = \mu$. By Lemma 2.2.(ii), $G'|_{\mu+1} = G'|_\mu \cup B_\mu$ is convex in G', giving (ii).

Before we define $\mathcal{P}(e)$ for $e \in E_\mu$ and show (iii) for $\lambda = \mu$, let us note the following consequence of the fact that S_μ separates B_μ from $G'|_\mu$ in G'. Let $P = x \ldots y$ be any B_μ–B_μ path in G', and assume that $y \in B_\mu\backslash S_\mu$. Then $y \in C_\mu$ and $\mathring{P} \cap S_\mu = \emptyset$, so $\mathring{P} \subset C_\mu$. Thus $\mathring{P} \subset G'_\mu\backslash G'|_{\mu+1}$ and

$$G'\,[\,P \to G'|_{\mu+1}\,] = G'\,[\,P \to B_\mu\,] = G'_\mu\,[\,P \to B_\mu\,]$$

$$G'\,[\,P \to G'|_{\mu+1}\,] \cap G'|_{\mu+1} = G'\,[\,P \to B_\mu\,] \cap B_\mu = G'_\mu\,[\,P \to B_\mu\,] \cap B_\mu\,, \qquad (1)$$

and hence

$$G'\,[\,P \to G'|_{\mu+1}\,]\backslash G'|_{\mu+1} = G'\,[\,P \to B_\mu\,]\backslash B_\mu = G'_\mu\,[\,P \to B_\mu\,]\backslash B_\mu \qquad (2)$$

(see Fig. 1). Another way of stating (2) is to say that whenever $C \subset G'$ contains the interior of a B_μ–B_μ path one of whose endvertices is in $B_\mu\backslash S_\mu$, then C is a component of $G'\backslash G'|_{\mu+1}$ if and only if C is a component of $G'\backslash B_\mu$, and if and only if C is a component of $G'_\mu\backslash B_\mu$.

For the definition of $\mathcal{P}(e)$, let now $e = xy \in E_\mu$ be given. As $e \notin E(G'|_\mu)$ by definition of E_μ, we may assume that $y \in B_\mu\backslash S_\mu$. Since $e \in E(G')\backslash E(G)$

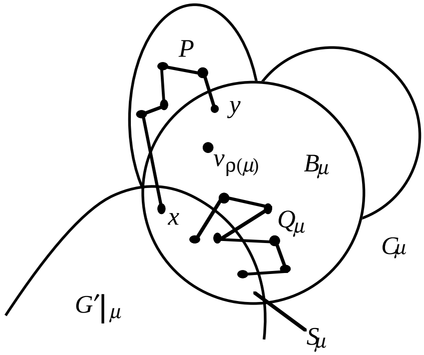

FIGURE 1.

and G' is the a-closure of G, there exists a set $\mathcal{P}$ of at least a independent x–y paths $P \subset G$. Since $|B_\mu| < a$, not all of these paths can have a vertex of B_μ in their interior, so there exists at least one (in fact, at least a) component(s) C of $G' \backslash B_\mu$ such that $\mathring{P} \subset C$. Then $C = G'[P \to B_\mu] \backslash B_\mu \underset{(2)}{=} G'_\mu[P \to B_\mu] \backslash B_\mu$, so C is also a component of $G'_\mu \backslash B_\mu$. Since B_μ was chosen by Lemma 6.1, this implies that there exists a set $\mathcal{C}$ of at least a components C' of $G'_\mu \backslash B_\mu$ for which $G'_\mu[C' \to B_\mu]$ contains a B_μ–B_μ path P' with endvertices x, y and $E(P') \subset E(G)$, and every such C' is also a component of $G' \backslash G'|_{\mu+1}$ (again by (2)). For each $C' \in \mathcal{C}$ pick one such path P', and let $\mathcal{P}(xy)$ be the set of all these paths. Clearly $|\mathcal{P}(xy)| \geqslant a$, and (iii) holds for $\lambda = \mu$.

The proof that $F = (B_\lambda)_{\lambda<\sigma}$ is a coherent simplicial decomposition of G' is the same as for Theorem 4.3. It thus remains to show that for every $\mu < \sigma$ and every edge $xy \in E_\mu$ there are at least a ordinals ν, with $S_\nu \subset B_\mu$, such that B_ν contains an S_ν–S_ν path P with endvertices x, y and $E(P) \subset E(G)$. Let $\mu < \sigma$ and $e = xy \in E_\mu$ be given, assuming again that $y \in B_\mu \backslash S_\mu$. By definition of $\mathcal{P}(e)$, every $\mathring{P}$ for $P \in \mathcal{P}(e)$ is contained in some component $C(P)$ of $G' \backslash G'|_{\mu+1}$, and these components are distinct for different paths P. Consider any fixed $P \in \mathcal{P}(e)$, and put $C(P) =: C$ and $\min \Lambda(C) =: \nu$. Then $C = C_\nu$, so C is still a component of $G' \backslash G'|_\nu$. Furthermore,

$$\begin{aligned} S_\nu &= G'_\nu \cap G'|_\nu \\ &= G'[C \to G'|_\nu] \cap G'|_\nu \\ &= G'[C \to G'|_{\mu+1}] \cap G'|_{\mu+1} \\ &= G'[P \to G'|_{\mu+1}] \cap G'|_{\mu+1} \\ &\underset{(1)}{=} G'[P \to B_\mu] \cap B_\mu \\ &\subset B_\mu . \end{aligned}$$

As $y \in S_\nu$ and $y \in B_\mu \backslash S_\mu$, this further implies that $\mu = \lambda(y) = \max \Lambda(S_\nu)$, so $P \subset Q_\mu \subset B_\nu$ by the choice of B_ν. Finally, since $C(P) \cap C(P') = \emptyset$ for $P \neq$

P', the values of ν differ for distinct $P \in \mathcal{P}(e)$. As $|\mathcal{P}(e)| \geqslant a$, this completes the proof. □

We finish this paper with a corollary of Theorem 6.2, which plays a central role in the proof of a result concerning the *end-faithful spanning tree* problem for infinite graphs [3].

If $F = (B_\lambda)_{\lambda<\sigma}$ is a simplicial decomposition of a graph G in which every S_μ is finite, let us denote by $\tau(\mu)$ the largest element of $\Lambda(S_\mu)$, for each $\mu < \sigma$. Since any $s \in S_\mu$ with $\lambda(s) = \tau(\mu)$ is in $B_{\tau(\mu)} \backslash S_{\tau(\mu)}$, S_μ must be contained in $B_{\tau(\mu)}$ by (S5). Hence, F satisfies (S4) and is therefore a simplicial tree-decomposition.

Corollary 6.3. *Let G be an uncountable graph not containing any subdivided infinite simplex. Then the $\aleph_1$-closure G' of G admits a coherent simplicial tree-decomposition $F = (B_\lambda)_{\lambda<\sigma}$ into countable factors and with finite simplices of attachment, which has the following property: for every $\mu < \sigma$ and every edge $xy \in \big(E(B_\mu)\backslash E(S_\mu)\big) \smallsetminus E(G)$, there are uncountably many ordinals ν, with $\tau(\nu) = \mu$, such that B_ν contains an S_ν–S_ν path P with endvertices x, y and $E(P) \subset E(G)$.*

Proof. Choose F by Theorem 6.2, putting $a := \aleph_1$ and $b := \aleph_0$. As every S_μ is finite, F is a simplicial tree-decomposition. The assertion that $\tau(\nu) = \mu$ follows from the fact that $S_\nu \subset B_\mu$ by Theorem 6.2, while $S_\nu \not\subset S_\mu$, because $x, y \in S_\nu$ and $xy \notin E(S_\mu)$ by assumption. □

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