On Spanning Trees and k-connectedness in Infinite Graphs

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We prove a conjecture of Širáň describing the graphs in which every spanning tree is end-faithful. This result leads to the consideration of infinite k-connected rayless graphs. We characterize these graphs in terms of tree-decompositions into finite k-connected factors.

Introduction

Let G be an infinite graph. The following assertions are equivalent for rays (1-way infinite paths) $P, Q \subset G$:

(i) there exists a ray $R \subset G$ which meets each of P and Q infinitely often;

(ii) for every finite $X \subset G$, the infinite components of $P \backslash X$ and $Q \backslash X$ lie in the same component of $G \backslash X$;

(iii) G contains infinitely many disjoint (possibly trivial) P–Q paths.

If two rays $P, Q \subset G$ satisfy (i)–(iii), we call them *end-equivalent* in G. An *end* of G is an equivalence class under this relation, and $\mathcal{E}(G)$ denotes the set of ends of G. For example, the 2-way infinite ladder has two ends, the infinite grid $\mathbb{Z} \times \mathbb{Z}$ and every infinite complete graph have one end, and the dyadic tree has $2^{\aleph_0}$ ends.

This paper is concerned with the relationship between the ends of a connected graph G and the ends of its spanning trees. If T is a spanning tree of G and P, Q are end-equivalent rays in T, then clearly P and Q are also end-equivalent in G. We therefore have a natural map $\eta : \mathcal{E}(T) \to \mathcal{E}(G)$ mapping each end of T to the end of G containing it. In general, η need be neither 1–1 nor onto. For example, the 2-way infinite ladder has a spanning tree with 4 ends (the tree consisting of its two sides together with one rung), and every infinite complete graph is spanned by a star, which has no ends at all. A spanning tree T of G for which η is 1–1 and onto is called *end-faithful*.

The concept of ends in graphs, and of end-faithful spanning trees, was introduced by Halin [4] in 1964. Halin asked whether every infinite connected graph has an end-faithful spanning tree, and proved that this is so for all countable graphs. End-faithful spanning trees have since been constructed for some classes of uncountable graphs as well (see [2] and, especially, Polat [8]), but very recent results due to Seymour and Thomas [10] and to Thomassen [12] show that some uncountable graphs have no such tree. See [3] for an up-to-date survey of results and open problems in this field.

The purpose of this paper is to solve the problem converse to Halin's, which was posed recently by Širáň [11]: *is there a simple characterization of the graphs in which every spanning tree is end-faithful*? Širáň conjectured the following, which will be the first main result of this paper:

Theorem A. *The spanning trees of a connected graph G are all end-faithful if and only if every block of G is rayless.*

The first part of this paper is devoted to a proof of this theorem, embedded in a slightly more general result (Theorem 2.1).

The fact that Širáň's conjecture is true immediately raises a further question: *what do the 2-connected rayless graphs look like*? (Interestingly, the graphs in which every block is rayless appear in a similar but unrelated role in a recent paper of Halin [6], which motivates this question further.) Moreover, if we replace 2 with a more general natural k, we obtain a problem of quite independent interest: *is there a simple structural description of the k-connected rayless graphs*?

Note that this problem too is intrinsically infinite: the raylessness condition does not bite in the finite case, and the finite k-connected graphs are clearly too varied to permit a general structural description of any detail.

In the second part of the paper, then, we shall prove what is best possible in such a case: that the uncontrollable element in the variation among the k-connected rayless graphs is confined to the finite case. More precisely, we show that an infinite graph is rayless and k-connected if and only if the 'infinite aspect' of its structure is that of an arbitrary rayless tree, while the 'finite details' of this tree are arbitrary finite k-connected graphs:

Theorem B. *An infinite graph is rayless and k-connected if and only if it has a k-connected rayless tree-decomposition into finite k-connected factors.*

(See Section 3 for precise definitions.)

Corollary. *Every finite subgraph of a rayless k-connected graph can be extended to a k-connected finite subgraph.*

In particular, we see that every rayless k-connected graph must have a finite k-connected subgraph.

1. Terminology and basic lemmas

We now run through some of the terminology and basic facts needed later. A subgraph H of G is *attached to* a connected subgraph H' of $G \backslash H$ if every vertex of H is adjacent to a vertex in H'. If H is attached to some component of $G \backslash H$, then H is *attached in* G; otherwise H is *unattached*. (Note that if $G \neq \emptyset$, then the empty graph $\emptyset \subset G$ is attached in G.)

If $P = x_1 \dots x_n$ is a path and $1 \leqslant i \leqslant j \leqslant n$, we write $\mathring{P} := x_2 \dots x_{n-1}$, $Px_i := x_1 \dots x_i$, $P\mathring{x}_i := x_1 \dots x_{i-1}$, $x_i P x_j := x_i \dots x_j$, $x_j P := x_j \dots x_n$ and $\mathring{x}_j P := x_{j+1} \dots x_n$ for subpaths of P. Analogous notation will be used for rays.

For $X, Y \subset G$, we call a path $P \subset G$ an X–Y *path* if its endvertices are in X and Y, respectively, and its interior $\mathring{P}$ lies in $G \smallsetminus (X \cup Y)$. We write $G[\,X \to Y\,]$ for the subgraph of G induced by all vertices of G that can be reached from X without passing through Y. More precisely, $G[\,X \to Y\,]$ is the subgraph of G induced by all vertices $v \in G$ for which G contains a path $x_1 \dots x_n$ satisfying $x_1 \in X$, $x_n = v$, and $x_i \notin Y$ for $i \neq n$. When the underlying graph G is fixed, we shall usually abbreviate $G[\,X \to Y\,] \cap Y$ to $Y[\,X\,]$. Thus, if X and Y are disjoint, then $Y[\,X\,]$ is the subgraph of Y induced by all terminal vertices of X–Y paths in G. On the other hand, if $Y = G$, then our definition of $Y[\,X\,]$ coincides with the conventional meaning of $G[\,X\,]$, denoting the subgraph of G induced by the vertices of X. A frequent example for the use of this notation is the following. If H is an induced subgraph of G and C is a component of $G \backslash H$, then $H[\,C\,]$ is spanned by all those vertices of H that have a neighbour in C. Then $H = H[\,C\,]$ if and only if H is attached to C in G.

For $X \subset G$ and $v \in G$, any union F of paths P_i $(i \in I)$ which begin in v, end in some vertex of X, and are disjoint except for v, will be called a v–X *fan*, with *branches* $\mathring{v}P_i$. Note that neither v nor the branches of F are required to lie outside X. If $R \subset G$ is a ray and G contains an infinite v–R fan, then v is called a *neighbour* of R in G. Note that if v is a neighbour of R, then G also contains an infinite v–R fan which covers $V(R)$: simply take any v–R fan, prune each branch after its first vertex on R, and extend the shortened branches along R to cover all its vertices. (If $v \in R$, one may have to add an extra branch.)

Similarly for $X, Y \subset G$, any union of disjoint paths, each beginning in X and ending in Y, will be called an X–Y *linkage*. Thus two rays in G are end-equivalent if and only if G contains an infinite linkage between them.

Two or more paths are *independent* if their interiors are disjoint. The *Menger number* $\mu_G(x, y)$ of two vertices $x, y \in G$ is the maximum of all cardinals κ for which there exists a κ-set of independent x–y paths in G. (It is not difficult to prove that this maximum always exists.) By Menger's theorem, the number of vertices needed to separate nonadjacent vertices x, y in G is exactly $\mu_G(x, y)$, and G is called κ-*connected* if $\mu_G(x, y) \geqslant \kappa$ for all $x, y \in G$. We shall use the infinite version of Menger's theorem (for finite κ) freely throughout the paper; see e.g. Halin [5] for a simple proof.

Another standard result we shall be using repeatedly is König's Infinity Lemma [7]:

Infinity Lemma. *Let K be a graph whose vertex set is the disjoint union of finite non-empty sets A_n, $n \in \mathbb{N}$, such that for $n > 0$ every vertex in A_n has a neighbour in A_{n-1}. Then K contains a ray $x_0 x_1 \dots$ with $x_n \in A_n$ for all $n \in \mathbb{N}$.*

Corollary 1.1. *Every infinite connected locally finite graph has a ray.* □

Lemma 1.2. *Let U and C be disjoint subgraphs of a graph G, such that C is connected, U is attached to C, and U is infinite. Then G contains either an infinite v–U fan for some $v \in C$, or an infinite R–U linkage for some ray $R \subset C$.*

Proof. We first construct a 'minimal' connected subgraph T of $G\,[\,C \to U\,]$ containing infinitely many vertices of U. Pick an ω-sequence $u_0, u_1, \ldots \in V(U)$. Let u_0' be a neighbour of u_0 in C, and set $T_0 := u_0 u_0'$. Having constructed $T_0, \ldots, T_n$ for some $n \in \mathbb{N}$, let P be a $U - (T_n \cap C)$ path beginning in u_{n+1}, and set $T_{n+1} := T_n \cup P$. Finally, set $T := \bigcup_{n \in \mathbb{N}} T_n$.

By construction, T is a tree with leaves $u_0, u_1, \ldots,$ and every vertex of T lies on a U–U path in T. Thus, if T has a vertex v of infinite degree, then $v \in C$, and T contains an infinite v–U fan.

Suppose now that T is locally finite, and let $R \subset T$ be a ray (by Corollary 1.1). Choose an ω-sequence $P_0, P_1, \ldots$ of disjoint R–U paths in T, as follows. Let P_0 be any R–U path in T. Assume that $P_0, \ldots, P_n$ have been chosen for some $n \in \mathbb{N}$. Choose $x \in R$ such that $Q_n := Rx \cup P_0 \cup \ldots \cup P_n$ is connected, and let C_n denote the component of $T - x$ containing $\mathring{x}R$. Since Q_n is a subtree of T disjoint from C_n, and since every vertex of C_n lies on a U–U path in T, we may choose P_{n+1} as an R–U path in C_n. The paths $P_0, P_1, \ldots$ form an infinite R–U linkage, as desired. □

2. The graphs in which every spanning tree is end-faithful

As our first main result, let us now prove Širáň's conjecture (Theorem A), embedded in a slightly more comprehensive characterization of the graphs in which every spanning tree is end-faithful.

Theorem 2.1. *For every connected graph G, the following assertions are equivalent:*

(a) *every spanning tree of G is end-faithful;*

(b) *every block of G is such that all its spanning trees are end-faithful;*

(c) *every block of G is rayless;*

(d) *G has no two disjoint equivalent rays, and no ray of G has a neighbour.*

Proof. (a)→(d). Suppose that every spanning tree of G is end-faithful. Then G has clearly no two disjoint equivalent rays; for the union of these rays could be extended to a spanning tree of G, which would not be end-faithful. Now suppose that R is a ray in G with a neighbour v. Choose a v–R fan $F \subset G$ that covers $V(R)$, and extend F to a spanning tree T of G. We prove that T has no ray equivalent to R, and is therefore not end-faithful. Let Q be any ray in G equivalent to R. Then Q meets R infinitely often (because G has no two disjoint equivalent rays), and hence Q meets more than two branches of F. Thus $Q \cup F$ contains a cycle. As $T = T \cup F$ is acyclic, this implies that $Q \not\subset T$.

(d)→(c). Let B be a block of G. We assume that B contains a ray R, and show that unless R has a neighbour, B contains two disjoint rays equivalent to R. We shall consider the vertices of R as ordered in the natural way, with $x < y$ if x is nearer to the initial vertex of R than y.

Let $\mathcal{C}$ be the set of components of $B \backslash R$. If $R[C]$ is infinite for some $C \in \mathcal{C}$, the assertion follows by Lemma 1.2: unless C contains a neighbour of R, there exists a ray in C (and hence disjoint from R) which is equivalent to R. We shall therefore assume that $R[C]$ is finite for every $C \in \mathcal{C}$. Regarding $C_1, C_2 \in \mathcal{C}$ as equivalent if $R[C_1] = R[C_2]$, let $\mathcal{C}' \subset \mathcal{C}$ be a set of representatives, and put $B' := B[R \cup \bigcup \mathcal{C}']$. Note that B' is still 2-connected. We may assume that

each vertex $x \in R$ is adjacent to only finitely many vertices of R, and x is contained in $R[C]$ for only finitely many $C \in \mathcal{C}'$. (1)

For if x is adjacent to infinitely many vertices of R, then x is a neighbour of R. Similarly if $\mathcal{C}'' \subset \mathcal{C}'$ is infinite, then infinitely many vertices of R are in $R[C]$ for some $C \in \mathcal{C}''$, by the choice of $\mathcal{C}'$. Thus if $x \in R[C]$ for every $C \in \mathcal{C}''$, then B' contains an infinite x–R fan, so again x is a neighbour of R.

(1) implies that, from any given vertex of R, we can only reach finitely many other vertices of R by an R–R path in B'. More generally,

$V_x := \{ v \in R \mid B'$ contains an R–R path $u \dots v$ with $u < x \}$ is finite for every $x \in \mathring{R}$. (2)

Note that since x is not a cutvertex of B', V_x contains a vertex $y > x$. In particular, max $V_x > x$.

Choose a sequence $P_1, P_2, \dots$ of paths as follows. Let y_0 be the second vertex on R. Having defined y_n for some $n \in \mathbb{N}$, put $y_{n+1} := \max V_{y_n}$, let P_{n+1} be an $R\mathring{y}_n$–R path ending in y_{n+1}, and let x_{n+1} be the initial vertex of P_{n+1}. Note that

$$x_{n+1} < y_n < y_{n+1} \qquad \text{for all } n \in \mathbb{N}. \tag{3}$$

Moreover, we have

$$y_n \leqslant x_{n+2} \qquad \text{for all } n \tag{4}$$

FIGURE 1. Finding two disjoint rays equivalent to R

(Fig. 1). For if $x_{n+2} < y_n$, then P_{n+2} is an $R\mathring{y}_n$–R path, so its endvertex y_{n+2} is in V_{y_n}. Since $y_{n+2} > y_{n+1}$ by (3), this contradicts the choice of y_{n+1} as max V_{y_n}.

Combining (3) and (4), one easily deduces that none of the R-segments $y_n R x_{n+2}$ contains any other vertices x_i or y_i. In particular, two such segments are disjoint for distinct n. Furthermore, if $n \neq m$ and $\mathring{P}_n, \mathring{P}_m \neq \emptyset$, then $\mathring{P}_n$ and $\mathring{P}_m$ lie in different components $C \in \mathcal{C}'$, by the choice of their endvertices y_n and y_m, and the fact that these are distinct. Hence, the rays

$$x_1 P_1 y_1 R x_3 P_3 y_3 R x_5 P_5 y_5 \ldots \quad \text{and} \quad x_2 P_2 y_2 R x_4 P_4 y_4 R x_6 P_6 y_6 \ldots$$

are disjoint. Since both meet R infinitely often, they are also equivalent.

(c)→(b) is trivial, because a rayless graph has no ends.

(b)→(a). Suppose that G has a spanning tree T which is not end-faithful. Assume first that two ends of T are contained in a common end of G. Then T has two disjoint rays R and Q, such that G contains an infinite R–Q linkage L. By discarding initial segments of R and Q if necessary, we may assume that $H := R \cup Q \cup L$ is 2-connected. Thus $H \subset B$ for a block B of G, and $B \cap T$ is a spanning tree of B which is not end-faithful.

Assume now that G contains a ray R which has no equivalent ray in T. For each edge e of R, let $B(e)$ be the block of G containing e. Note that if $B(e_1) = B(e_2)$, then $B(e_1) = B(e) = B(e_2)$ for every edge e between e_1 and e_2 on R. We show that $\mathcal{B} := \{ B(e) \mid e \in E(R) \}$ is finite; then R has a tail xR inside a single block B, and $B \cap T$ is a spanning tree of B which is not end-faithful.

Suppose $\mathcal{B}$ is infinite, and assume that $E(R)$ runs through the blocks $B_0, B_1, \ldots$ (in the order of R). For each $n \in \mathbb{N}$, let x_n be the first vertex on R that is in B_n. Then $B(e) = B_n$ for every edge e between x_n and x_{n+1}, so $x_n, x_{n+1} \in B_n$. Since $T \cap B_n$ is connected, it contains an x_n–x_{n+1} path P_n. These paths are independent for distinct n, so $x_1 P_1 x_2 P_2 x_3 P_3 \ldots$ is a ray in T which meets R infinitely often. This contradicts our assumption that T has no ray equivalent to R. □

In proving Širáň's conjecture, we have described the graphs in which every spanning tree is end-faithful in terms of rayless 2-connected graphs. In the remainder of this paper, we take this description a step further and characterize the rayless 2-connected graphs in terms of finite ones. The two results can then be combined into a structural characterization of the graphs in which every spanning tree is end-faithful in terms of finite 2-connected graphs. (The explicit formulation of this result should be clear and will be left to the reader.)

3. Tree-decompositions and convex subgraphs

The aim of this section is to provide the necessary background for the proof of our second main result, a characterization of the infinite rayless k-connected graphs by their tree-decompositions (Theorem 4.3). The factors in these tree-decompositions will be finite k-connected graphs, and the decomposition trees involved will be rayless and such that 'adjacent' factors overlap in at least k vertices. Although this result is easily stated (at least in an intuitive way), its proof uses a few concepts and techniques from simplicial decomposition theory as developed in [1]. In order to make this paper self-contained, everything needed is listed below; the reader who is familiar with simplicial decompositions may skip this material and go straight to Section 4.

In the following, a complete graph will often be called a *simplex*. Let G be a graph, $\sigma > 0$ an ordinal, and for each $\lambda < \sigma$ let B_λ be an induced subgraph of G. The family $F = (B_\lambda)_{\lambda<\sigma}$ is called a *simplicial tree-decomposition* of G if the following four conditions hold:

(S1) $G = \bigcup_{\lambda<\sigma} B_\lambda$;

(S2) every $G|_\mu \cap B_\mu =: S_\mu$ is a simplex, where $G|_\mu := \bigcup_{\lambda<\mu} B_\lambda \quad (0 < \mu < \sigma)$;

(S3) no S_μ contains B_μ or any other $B_\lambda \quad (0 \leqslant \lambda < \mu < \sigma)$;

(S4) each S_μ is contained in B_λ for some $\lambda < \mu \quad (\mu < \sigma)$.

Based on (S1), we shall write $\lambda(x) := \min\{\, \lambda \mid x \in B_\lambda \,\}$ for vertices $x \in G$, and $\Lambda(X) := \{\, \lambda(x) \mid x \in X \,\}$ for $X \subset G$. Note that the vertices $x \in G$ with $\lambda(x) = \mu$ are precisely the vertices of $B_\mu \backslash S_\mu$.

If F satisfies (S1) and (S4) (but not necessarily (S2) or (S3)), F is called a *tree-decomposition* of G. The factors in such a tree-decomposition may be regarded as the vertices of a tree T_F (the *decomposition tree* of F), defined inductively by joining each 'vertex' B_μ to a fixed predecessor B_λ as provided by (S4). To avoid ambiguity, this λ is chosen minimal; then S_μ is contained in B_λ but not in S_λ, so S_μ has a vertex s with $\lambda(s) = \lambda$. It is often convenient to think of the tree T_F as rooted at the vertex B_0, and of $V(T_F) = \{\, B_\lambda \mid \lambda < \sigma \,\}$ as endowed with the corresponding tree-order $<_{T_F}$. (Thus, $B <_{T_F} B'$ if B lies on the unique B_0–B' path in T_F.) Note that this partial order is compatible with the well-ordering of F: if $B_\lambda <_{T_F} B_\mu$, then $\lambda < \mu$.

We remark that the above definition of a tree-decomposition is equivalent, for finite graphs, to that introduced by Robertson and Seymour for the study of graph minors; see [1; Ch. 1, Exercise 23].

We shall need the following simple property of tree-decompositions (see [1; Ch. 1.2] for a proof):

Proposition 3.1. *If B, B', B'' are factors in a tree-decomposition F of G and B lies on the B'–B'' path in T_F, then B separates $B' \backslash B$ from $B'' \backslash B$ in G.*

A tree-decomposition or simplicial tree-decomposition $F = (B_\lambda)_{\lambda<\sigma}$ is *coherent* if S_μ is attached to $B_\mu \backslash S_\mu$ and $B_\mu \backslash S_\mu$ is connected for every $\mu < \sigma$. F will be called *k-connected* if $|S_\mu| \geqslant k$ for every $\mu > 0$, and *rayless* if T_F is rayless. For each $B \in F$, the subgraph

$$B^- := \bigcup \{ B' \in F \mid B' \leqslant_{T_F} B \}$$

of G will be called the *shadow* of B in T_F. Since $B_{\lambda(s)} <_{T_F} B_\mu$ for all $s \in S_\mu$ (induction on μ), we have $B^- = \bigcup \{ B_{\lambda(x)} \mid x \in B^- \}$ for every $B \in F$.

A subgraph $H \subset G$ is *convex* in G if H contains every induced path in G whose endvertices are in H. Examples of convex subgraphs include factors and shadows in simplicial tree-decompositions [1; Ch. 5.4]:

Proposition 3.2. *If $F = (B_\lambda)_{\lambda<\sigma}$ is a simplicial tree-decomposition of G and T is a subtree of T_F, then $\bigcup T$ is a convex subgraph of G.*

There are a number of interesting and useful equivalents of convexity, all easily proved:

Proposition 3.3. *For $H \subset G$, the following statements are equivalent:*

(i) *H is convex in G;*

(ii) *the endvertices of every H–H path in G are adjacent in H;*

(iii) *H is an induced subgraph of G and, for every vertex $x \in G \backslash H$, the subgraph $H[x] = G[x \to H] \cap H$ is a simplex;*

(iv) *if $A, B, X \subset V(H)$, then X separates A from B in H if and only if X separates A from B in G.* □

The following simple technical lemma provides a useful means for joining two convex subgraphs into one.

Lemma 3.4. *Let $G_1, G_2 \subset G$ be graphs, and suppose that $S = G_1 \cap G_2$ separates G_1 from G_2 in G.*

(i) *If G_1 and G_2 are convex in G, then so is $G_1 \cup G_2$.*

(ii) *If S is a simplex and G_i is convex in $G[G_i \to S]$, $i = 1, 2$, then $G_1 \cup G_2$ is convex in G.*

Proof. (i) is obvious from the definition of convexity.

(ii) As S is a simplex, $G[G_i \to S]$ is convex in G by Proposition 3.3. Since G_i is convex in $G[G_i \to S]$ by assumption, this implies that G_i is also convex in G. Apply (i). □

4. The structure of the rayless k-connected graphs

Given a graph G and a cardinal κ, let $[\,G\,]_\kappa$ denote the graph with vertex set $V(G)$ and edge set $E(G) \cup \{\, xy \mid \mu_G(x,y) \geqslant \kappa \,\}$. The graph $[\,G\,]_\kappa$ is usually called the *κ-closure* of G, which is justified by the following observation:

Proposition 4.1. *$[\,G\,]_\kappa$ is its own κ-closure.*

(The proof of Proposition 4.1 is not difficult; see [1; Ch. 5.3].)

Note that Proposition 4.1 implies that $\mu_{[\,G\,]_\kappa}(x,y) < \kappa$ for any two non-adjacent vertices $x, y \in [\,G\,]_\kappa$. Moreover,

Lemma 4.2. *If κ is infinite and G is rayless, then $[\,G\,]_\kappa$ is rayless.*

Proof. Suppose $[\,G\,]_\kappa$ contains a ray R. We shall choose vertices $x_n \in R$ and define paths $P_n \subset G$, for all $n \in \mathbb{N}$, such that P_n is an x_{n-1}–x_n path for each $n \geqslant 1$, and $\bigcup_{n\in\mathbb{N}} P_n$ is a ray in G.

Let x_0 be the initial vertex of R and $P_0 := \{\, x_0 \,\}$. Let $n \geqslant 1$ be given, and assume that x_i and P_i have been defined for all $i < n$. Let v be the successor of x_{n-1} on R. If $x_{n-1}v \in E(G)$, let $P_n := x_{n-1}v$ and set $x_n := v$. If $x_{n-1}v \notin E(G)$, then G contains infinitely many independent x_{n-1}–v paths. Let P be one of these paths, chosen such that $\mathring{P} \cap P_i = \emptyset$ for all $i < n$. Let x_n be the latest (farthest from x_0) vertex on R that is in $V(P)$, and set $P_n := Px_n$.

It is easily checked that $\bigcup_{n\in\mathbb{N}} P_n$ is a ray in G. □

We are now ready to prove our second main result.

Theorem 4.3. *For any graph G and $k \in \mathbb{N}$, the following two assertions are equivalent:*

(i) *G is rayless and k-connected;*

(ii) *G has a rayless and k-connected tree-decomposition into finite k-connected factors.*

Proof. (i)→(ii). Assume that G is rayless and k-connected, and let $G' := [\,G\,]_{\aleph_0}$. Clearly G' is again k-connected, and by Lemma 4.2, G' is also rayless. We shall first construct a rayless, k-connected and coherent simplicial tree-decomposition $F' = (B_\lambda)_{\lambda<\sigma}$ of G', which will then be modified to give the desired tree-decomposition F of G.

Let us choose the factors B_λ for F' in such a way that, for every $\lambda < \sigma$,

(a) *B_λ is unattached in G';*

(b) *if $xy \in E(B_\lambda) \backslash E(G)$ and $\lambda(y) = \lambda$, then $B_\lambda \cap G$ contains at least k independent x–y paths;*

(c) *$\bigcup_{\lambda'\leqslant\lambda} B_{\lambda'}$ is convex in G'.*

Let $\mu \geqslant 0$ be given, and suppose that for every $\lambda < \mu$ we have defined B_λ so as to satisfy (a)–(c). We shall seek to define B_μ in such a way that (a)–(c) hold for $\lambda = \mu$.

We first show that $G'|_\mu := \bigcup_{\lambda<\mu} B_\lambda$ is convex in G'. If $\mu = 0$, this is trivial as $G'|_\mu = \emptyset$. If μ is a successor ordinal, then $G'|_\mu$ is convex by assumption (c). Finally, if μ is a non-zero limit, then $G'|_\mu$ is the nested union of the graphs $\bigcup_{\lambda'\leqslant\lambda} B_{\lambda'}$ with $\lambda < \mu$; since these graphs are convex by (c), $G'|_\mu$ is also convex.

If $V(G')\backslash V(G'|_\mu) = \emptyset$, we put $\sigma := \mu$ and terminate the construction of F'. Note that in this case $G'|_\mu = G'$ (because, being convex, $G'|_\mu$ is induced in G'), so F' satisfies (S1).

Assume now that $V(G')\backslash V(G'|_\mu) \neq \emptyset$. Let C_μ be a component of $G'\backslash G'|_\mu$, and set

$$H_\mu := G'\,[\,C_\mu \to G'|_\mu\,]$$
$$S_\mu := H_\mu \cap G'|_\mu\,.$$

Then $S_\mu = G'|_\mu\,[\,v\,]$ for each vertex $v \in C_\mu$, so S_μ is a simplex by Proposition 3.3.(iii). Since G' is rayless and k-connected, S_μ is finite but has at least k vertices. (To be precise, the latter is true if and only if $\mu \neq 0$; note that in this case $G'|_\mu\backslash S_\mu \neq \emptyset$, since $B_0 \subset G'|_\mu$ is not attached to C_μ by (a).)

We construct B_μ in ω steps (almost all of which will later turn out to be redundant), as the union of a nested sequence $B_\mu^0 \subset B_\mu^1 \subset \ldots$ of finite supergraphs of S_μ in H_μ. With $B_\mu^0 := S_\mu$, let us assume that $B_\mu^0, \ldots, B_\mu^{n-1}$ have been defined for some $n \geqslant 1$. If B_μ^{n-1} is an attached simplex in H_μ (which is the case, for example, for $n = 1$), we pick a vertex $v \in C_\mu\backslash B_\mu^{n-1}$ such that $B_\mu^{n-1} = B_\mu^{n-1}\,[\,v\,]$, and set $B_\mu^n := B_\mu^{n-1} \cup \{\,v\,\}$. Let us further define a set $\mathcal{P}'_n := \emptyset$ for such n; this set will be needed as a 'dummy' in a recursion formula below. For the remainder of the construction of B_μ^n, we shall now assume that B_μ^{n-1} is not an attached simplex in H_μ (and in particular, that $n > 1$).

We first make B_μ^{n-1} induced in G' by adding any missing edges, putting

$$\tilde{B}_\mu^{n-1} := G'\,[\,B_\mu^{n-1}\,]\,.$$

Let us write E_μ^n for the set of edges we added; thus

$$E_\mu^n = E(\tilde{B}_\mu^{n-1}) \smallsetminus E(B_\mu^{n-1})\,.$$

Next, we let $\mathcal{P}'_n$ be any inclusion-maximal set of independent $\tilde{B}_\mu^{n-1}$–$\tilde{B}_\mu^{n-1}$ paths in H_μ whose endvertices x, y are non-adjacent in $\tilde{B}_\mu^{n-1}$. Note that for each pair xy of endvertices in $\tilde{B}_\mu^{n-1}$ there are only finitely many such paths, by the definition of G' and the remark following Proposition 4.1; since B_μ^{n-1} and hence the number of these pairs is finite, $\mathcal{P}'_n$ is also finite. Third, we let $\mathcal{P}''_n$ be another finite set of $\tilde{B}_\mu^{n-1}$–$\tilde{B}_\mu^{n-1}$ paths, this time in G itself, choosing k such paths $x \ldots y$ for each edge

$$xy \in \Big(E_\mu^n \cup \bigcup_{P \in \mathcal{P}'_{n-1}} E(P)\Big) \smallsetminus E(G)$$

in such way that all these paths are internally disjoint from each other and from every path in $\mathcal{P}'_n$. (We assume here that $\mathcal{P}'_{n-1}$ has already been defined as a set of paths in B_μ^{n-1}.) Since G contains infinitely many independent x–y paths for every such pair xy (by definition of G'), such a set $\mathcal{P}''_n$ does certainly exist. Moreover, every path of $\mathcal{P}''_n$ lies in H_μ, because it can have at most one endvertex and no interior vertex in S_μ (recall that $S_\mu = B_\mu^0 \subset B_\mu^{n-1}$). Finally, we put $\mathcal{P}_n := \mathcal{P}'_n \cup \mathcal{P}''_n$, and set

$$B_\mu^n := \tilde{B}_\mu^{n-1} \cup \bigcup \mathcal{P}_n$$

and

$$B_\mu := \bigcup_{n \in \mathbb{N}} B_\mu^n .$$

Let us prove that although we formally took infinitely many steps to construct it, B_μ is in fact finite. More precisely, let us prove that $B_\mu^{n+1} = B_\mu^n$ for all sufficiently large n. Suppose the contrary holds. Since G' is rayless and hence contains no infinite simplex, there exists an $n_0 \in \mathbb{N}$ such that B_μ^n is not an attached simplex in H_μ for any $n \geqslant n_0$. Thus $\mathcal{P}_n \neq \emptyset$ for arbitrarily large n. In fact, $\mathcal{P}_n \neq \emptyset$ for every $n > n_0$. For if $\mathcal{P}_n = \mathcal{P}'_n \cup \mathcal{P}''_n = \emptyset$, then $\mathcal{P}'_{n+1} = \emptyset$ by the maximality of $\mathcal{P}'_n$. Moreover, $B_\mu^n = \tilde{B}_\mu^{n-1}$, so B_μ^n is induced in G'. But then $E_\mu^{n+1} = \emptyset$, and hence $\mathcal{P}''_{n+1} = \emptyset$. Thus again $\mathcal{P}_{n+1} = \emptyset$. By induction, this gives $\mathcal{P}_n = \emptyset$ eventually for all n, a contradiction.

Notice that if $n > n_0$ and P is a path in $\mathcal{P}_{n+1}$, then at least one of the two endvertices of P lies in the interior of a path $Q \in \mathcal{P}_n$: if $P \in \mathcal{P}'_{n+1}$, this is a consequence of the maximality of $\mathcal{P}'_n$, while for $P \in \mathcal{P}''_{n+1}$ it follows from the definition of E_μ^{n+1}. (Recall that $\tilde{B}_\mu^{n-1} \subset B_\mu^n$ is induced in G', so any edge of $\tilde{B}_\mu^n$ that is not already an edge of B_μ^n must have one of its endvertices in $B_\mu^n \setminus \tilde{B}_\mu^{n-1} = \bigcup \{ \mathring{Q} \mid Q \in \mathcal{P}_n \}$.) Choosing a fixed such $Q = Q(P) \in \mathcal{P}_n$ for each $P \in \mathcal{P}_{n+1}$ and every $n > n_0$, let K be the graph with vertex set

$$V(K) := \bigcup_{n > n_0} \mathcal{P}_n$$

and edge set

$$E(K) := \{ PQ(P) \mid P \in \mathcal{P}_{n+1} \text{ for some } n > n_0 \} .$$

Since each of the sets $\mathcal{P}_n$ is finite, König's Infinity Lemma implies that K contains a ray $Q_1 Q_2 \ldots$ with $Q_i \in \mathcal{P}_{n_0+i}$ for every i. By construction of K, the subgraph $\bigcup_{i \in \mathbb{N}} Q_i$ of G' contains a ray, contradicting the fact that G' is rayless. This completes the proof that B_μ is finite.

Let us now check that our definition of B_μ complies with the conditions (a)–(c) for $\lambda = \mu$. For a proof of (c) note that, by construction, the endvertices x, y of any B_μ–B_μ path $P \subset H_\mu$ are adjacent in B_μ: since x and y are contained in B_μ^n for some n, the existence of P would otherwise contradict the maximality of $\mathcal{P}'_{n+1}$. By Proposition 3.3.(ii), therefore, B_μ is a convex subgraph of H_μ.

By Lemma 3.4.(ii) and our observation that $G'|_\mu$ is convex in G' (and hence in $G'\,[\,G'|_\mu \to S_\mu\,]$), this implies that $\bigcup_{\lambda' \leqslant \mu} B'_{\lambda'} = G'|_\mu \cup B_\mu$ is convex in G', as required for (c).

In order to show (a) for $\lambda = \mu$, let $n \in \mathbb{N}$ be such that $B_\mu = B_\mu^n = B_\mu^{n+1}$. Suppose that B_μ is attached in G', i.e. that $B_\mu = B_\mu\,[\,v\,]$ for some vertex $v \in G' \backslash B_\mu$. As $B_\mu \cap C_\mu \neq \emptyset$ by the construction of B_μ, clearly $v \in C_\mu$. Since B_μ is convex in H_μ, Proposition 3.3.(iii) implies that B_μ is a simplex. But then $B_\mu = B_\mu^n$ is an attached simplex in H_μ, so our construction of B_μ prescribes that $B_\mu^{n+1} = B_\mu^n \cup \{\, w \,\}$ for some vertex $w \in C_\mu \backslash B_\mu^n$, contrary to our assumption that $B_\mu^n = B_\mu^{n+1}$.

For a proof of (b), finally, notice that if $xy \in E(B_\mu) \backslash E(G)$ and $\lambda(y) = \mu$, then there exists an $n \in \mathbb{N}$ such that $xy \in E_\mu^n$ or $xy \in E(P)$ for some $P \in \mathcal{P}'_n$. The k independent x–y paths required for (b) are therefore contained in $\mathcal{P}''_n$ or in $\mathcal{P}''_{n+1}$.

To complete our construction of the family $F' = (B_\lambda)_{\lambda < \sigma}$, it remains to observe that $B_\mu \backslash G'|_\mu \neq \emptyset$ for each μ; the construction therefore terminates after no more than $|G'|$ steps.

Having noted earlier that F' satisfies (S1), we observe further that the simplex S_μ coincides with $B_\mu \cap G'|_\mu$ for each $\mu < \sigma$, so F' satisfies (S2). Moreover, as S_μ is attached, it cannot contain any B_λ by (a), so F' also satisfies (S3). Finally, it is easily checked that $S_\mu \subset B_\lambda$ for $\lambda := \max \Lambda(S_\mu)$ (observe that S_μ has a vertex in $B_\lambda \backslash S_\lambda$ and, being a simplex, is not separated by S_λ), so F' satisfies (S4). Therefore F' is a simplicial tree-decomposition of G'.

As $|S_\mu| \geqslant k$ for every $\mu > 0$, F' is k-connected. To see that F' is coherent, suppose that, for some $\mu < \sigma$, S_μ is not attached to $B_\mu \backslash S_\mu$ or $B_\mu \backslash S_\mu$ is disconnected. In either case there exists a subsimplex $S \subset S_\mu$ which separates vertices $x, y \in B_\mu \backslash S$ in B_μ. As S_μ is attached to C_μ and $B_\mu \backslash S_\mu \subset C_\mu$, S cannot separate x and y in H_μ. By Proposition 3.3.(iv), this contradicts the convexity of B_μ in H_μ noted above in the proof of (c).

To see that F' is rayless, suppose that $B_{\lambda_0} B_{\lambda_1} \ldots$ is a ray in $T_{F'}$, without loss of generality chosen such that $B_{\lambda_0} = B_0$. Then $S_{\lambda_{n+1}} \subset B_{\lambda_n}$ for each n, and $S_{\lambda_{n+1}}$ has a vertex in $B_{\lambda_n} \backslash S_{\lambda_n}$; let such a vertex v_n be chosen for each n. Now since F' is coherent, each B_{λ_n} with $n \geqslant 1$ contains a v_{n-1}–v_n path P_n whose only vertex in S_{λ_n} is v_{n-1}. The union of all these paths P_n is a ray in G', a contradiction.

We now come to the final step of the proof, the construction of a tree-decomposition of G. For each $\lambda < \sigma$, let B_λ^- be the shadow of B_λ in $T_{F'}$; thus

$$B_\lambda^- = \bigcup \{\, B \in F' \mid B \leqslant_{T_{F'}} B_\lambda \,\}\,.$$

Recall that, by Proposition 3.2, each of these B_λ^- is a convex subgraph of G'. Let us define

$$G_\lambda := B_\lambda^- \cap G$$

for each $\lambda < \sigma$, and set

$$F := (G_\lambda)_{\lambda<\sigma} .$$

We shall prove that F is a tree-decomposition of G with the desired properties.

Since F' satisfies (S1) with respect to G', clearly F satisfies (S1) with respect to G. In order to check (S4), note that if $\mu < \sigma$ is given, and $\tau(\mu) < \mu$ is such that $B_{\tau(\mu)}B_\mu \in E(T_{F'})$ (i.e., $B_{\tau(\mu)}$ is the immediate predecessor of B_μ in $T_{F'}$), then $G_\mu \cap G|_\mu = G_{\tau(\mu)}$. Thus, F is a tree-decomposition of G. (Note that F does not, in this form, satisfy (S3); however, this could easily be achieved by restricting F to those G_λ for which B_λ is a leaf in $T_{F'}$.)

To see that the factors in F are finite, recall that each B_λ^- is a finite union of finite graphs, and hence itself finite. Since $B_\lambda^- \supseteq B_0 \supseteq S_1$ for every λ, and $|S_1| \geqslant k$, any two factors $G_\lambda \in F$ have at least k vertices in common; hence F is k-connected. As for the raylessness of F, recall that S_μ, and hence $V(G_\mu \cap G|_\mu) \supseteq V(S_\mu)$, contains a vertex s with $\lambda(s) = \tau(\mu)$ (taken in F'). Thus, while $G_\mu \cap G|_\mu$ is contained in $G_{\tau(\mu)}$ (as pointed out above), $G_\mu \cap G|_\mu$ is not contained in G_λ for any $\lambda < \tau(\mu)$, so G_μ is joined to $G_{\tau(\mu)}$ when T_F is constructed. In other words, T_F is isomorphic to $T_{F'}$ under the natural isomorphism mapping G_λ to B_λ. Since $T_{F'}$ is rayless, this means that T_F too is rayless.

It remains to show that every G_λ is k-connected. Suppose not, and let $U \subset V(G_\lambda)$ be a set of fewer than k vertices separating G_λ. Let C and C' be distinct components of $G_\lambda - U$. Since G' is k-connected, there exists a C–C' path P in G' avoiding U; as B_λ^- is convex in G', we may assume that $P \subset B_\lambda^-$. Assuming further that C and C' were suitably chosen, P thus consists of a single edge xy, say with $\lambda(x) \leqslant \lambda(y)$. Then $xy \in E(B_{\lambda(y)}) \setminus E(G)$. By (b) in the construction of F', there are at least k independent x–y paths in $B_{\lambda(y)} \cap G \subset G_\lambda$. One of these paths must avoid U, contrary to our assumption that x and y are in distinct components of $G_\lambda - U$. This completes the proof that G_λ is k-connected, for every $\lambda < \sigma$.

(ii)→(i). If G has a rayless and k-connected tree-decomposition $F = (B_\lambda)_{\lambda<\sigma}$ into finite k-connected factors, then G is clearly k-connected (induction on $\mu \leqslant \sigma$ for $G|_\mu$).

Suppose G contains a ray R. As each factor in F is finite, $\Lambda(R)$ must be infinite. Let

$$U := \{\, B_\lambda \mid \lambda \in \Lambda(R) \,\} ,$$

pick a vertex $v(B_\lambda) \in R \cap (B_\lambda \backslash S_\lambda)$ from each $B_\lambda \in U$, and set

$$V := \{\, v(B) \mid B \in U \,\} .$$

Note that $v(B) \neq v(B')$ for distinct $B, B' \in U$, because $\lambda(v(B)) \neq \lambda(v(B'))$.

Let T be the infinite subtree of T_F arising from the union of all the U–U paths in T_F. As T is rayless, it has a vertex B of infinite degree (Corollary 1.1).

By the construction of T, every edge incident with B in T lies on a B–U path in T. Hence, there is an infinite subset U' of U such that B lies on the path in T_F between any two elements of U'. As B is finite, U' can be chosen such that $v(B') \notin B$ for any $B' \in U'$. By Proposition 3.1, therefore, B separates any two vertices of

$$V' := \{\, v(B') \mid B' \in U' \,\}$$

in G. Since V' is an infinite subset of $V(R)$, this contradicts the fact that B is finite.

Hence G is rayless, as claimed. □

References

[1] R. Diestel, *Graph decompositions: a study in infinite graph theory*, Oxford University Press, Oxford 1990.

[2] R. Diestel, On end-faithful spanning trees in infinite graphs, *Math. Proc. Camb. Phil. Soc.* **107** (1990), 461–473.

[3] R. Diestel, The end structure of a graph: recent results and open problems, submitted.

[4] R. Halin, Über unendliche Wege in Graphen, *Math. Ann.* **157** (1964), 125–137.

[5] R. Halin, *Graphentheorie II*, Wissenschaftliche Buchgesellschaft, Darmstadt 1981.

[6] R. Halin, Bounded graphs, in: (R. Diestel, Ed.) *Directions in infinite graph theory and combinatorics* (Annals of Discrete Mathematics), in press.

[7] D. König, *Theorie der endlichen und unendlichen Graphen*, Akademische Verlagsgesellschaft, Leipzig 1936 (reprinted: Chelsea, New York 1950).

[8] N. Polat, Topological aspects of infinite graphs, in: (G. Hahn et al., Eds.) *Cycles and rays*, NATO ASI Ser. C, Kluwer Academic Publishers, Dordrecht 1990.

[9] R. Schmidt, Ein Ordnungsbegriff für Graphen ohne unendliche Wege mit einer Anwendung auf n-fach zusammenhängende Graphen, *Archiv der Mathematik* **40** (1983), 283–288.

[10] P. D. Seymour and R. Thomas, An end-faithful counterexample, *Proceedings of the AMS*, to appear.

[11] J. Širáň, End-faithful spanning trees, paper given at the 1989 Cambridge Conference on Infinite Graph Theory and Combinatorics.

[12] C. Thomassen, Infinite connected graphs with no end-preserving spanning trees, preprint 1990.

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