SIMPLICIAL TREE-DECOMPOSITIONS OF INFINITE GRAPHS III

The Uniqueness of Prime Decompositions

Reinhard Diestel
Department of Pure Mathematics
University of Cambridge
16 Mill Lane
Cambridge, England

Abstract. In this last of three papers on simplicial tree-decompositions of graphs we investigate the extent to which prime factors in such decompositions are unique, or depend on the decomposition chosen. A simple example shows that a prime decomposition of a graph may have superfluous factors, the omission of which leaves a set of factors that can be rearranged into another decomposition of the same graph. As our main result we show that this possibility is the only way in which prime decompositions can vary: we prove that all prime decompositions of a countable graph without such superfluous members have the same set of factors. We also obtain a characterization theorem which identifies these factors among similar subgraphs by their position within the graph considered, independently of their role in any decomposition.

The material presented in this paper rests on concepts and results developed in [4]; in particular, the reader will benefit from familiarity with [4, Theorem 3.2]. The paper does not depend on [5], but its results are complementary and in that sense related to those obtained in [5].

Let G be a graph, $\sigma > 0$ an ordinal, and let B_λ be an induced subgraph of G for every $\lambda < \sigma$. The family $(B_\lambda)_{\lambda<\sigma}$ is called a *simplicial tree-decomposition* of G if

(S1) $G = \bigcup_{\lambda<\sigma} B_\lambda$;

(S2) $\big(\bigcup_{\lambda<\mu} B_\lambda\big) \cap B_\mu =: S_\mu$ is a complete graph for each μ $\quad (0 < \mu < \sigma)$;

(S3) no S_μ contains B_μ or any other B_λ $\quad (0 \le \lambda < \mu < \sigma)$;

(S4) each S_μ is contained in B_λ for some $\lambda < \mu$ $\quad (\mu < \sigma)$.

A graph is called *prime* if it has no such decomposition into more than one factor. A graph is prime if and only if it contains no separating simplex [4, Corollary 1.2]. A simplicial tree-decomposition in which all factors are prime will be called a simplicial tree-decomposition *into primes*, or a *prime decomposition*.

1. The Uniqueness Problem

The purpose of this paper is to explore to what extent simplicial tree-decompositions into primes are unique.

The extreme case of such uniqueness would be that any two prime decompositions of a graph consisted of the same set of factors, and these were necessarily arranged in the same order. The latter, however, is almost never true: even in cases when the factors in any two prime decompositions of a certain graph are the same, their order is likely to vary greatly. We shall not consider the order aspect of uniqueness in this paper.

A third aspect of uniqueness in prime decompositions, in a sense a weakening of the unattainable uniqueness of the order of factors, is the question of uniqueness for simplices of attachment. And, remarkably, we do have complete uniqueness here, even in cases where the set of factors is not unique: a simplex S contained in a graph G is a simplex of attachment in any prime decomposition of G if and only if S is a minimal relative separator in G, i.e. an induced subgraph minimally separating some two vertices of G (see Proposition 3).

It thus remains to investigate to what extent the factors in a prime decomposition of a graph vary with the decomposition chosen, and this will be the subject of this paper.

As an example, let us consider the graph H^2 introduced in [4]. H^2 consists of an infinite simplex $S = S[s_1, s_2, \dots]$, independent vertices $x_1, x_2, \dots$ joined to S by the edges $x_i s_j$ for $i, j \in \mathbb{N}$, $i \ge j$, and another vertex q joined to all s_i with odd i [4, Figure 4]. The maximally prime subgraphs of H^2 (and therefore its potential prime factors, cf. [4, Theorem 1.10]) are $B'_i := H^2[x_i, s_1, \dots, s_i]$ $(i \in \mathbb{N})$, S and $B'' := H^2[q \to S]$.

Let us first look at the graph $H^2\backslash\{\, q\,\}$. This graph has two different prime decompositions,

$$F_1 := \big(S, B_1', B_2', \ldots\big)$$

and

$$F_2 := \big(B_1', B_2', \ldots\big).$$

Notice that after dropping S from F_1 and thus obtaining F_2, we cannot omit any more factors: unlike S in F_1, none of the B_i''s in F_2 is covered by the remaining factors in F_2, so each factor in F_2 is indispensable for F_2 to cover the graph $H^2\backslash\{\, q\,\}$.

Let us now consider the graph H^2 itself. H^2 admits the prime decomposition

$$F_3 := \big(S, B_1', B_2', \ldots, B''\big).$$

Again, S is covered by the other factors of F_3. But this time S must not be omitted: the family

$$F_4 := \big(B_1', B_2', \ldots, B''\big)$$

satisfies (S1)–(S3), but it violates (S4), because $B'' \cap \bigcup_{i=1,2,\ldots} B_i' = S\,[\, s_1, s_3, \ldots]$ is not contained in any of the factors B_i'. Thus S is indispensable as a factor in F_3, not because it is needed to cover H^2, but because it serves as an 'interface' between B'' and the other factors in F_3.

This will be made more precise in the next section. We shall prove that the above two reasons for being indispensable as a factor are the only possible ones, and that any prime decomposition from which all 'dispensable' factors have been removed, consists of a unique set of factors.

2. Reduced Decompositions

Let G be a graph, $F = (B_\lambda)_{\lambda<\sigma}$ a simplicial tree-decomposition of G, and $\mu < \sigma$. We shall call B_μ *dispensable in* F if G has a simplicial tree-decomposition $F' = (B_\lambda')_{\lambda<\sigma'}$ that satisfies $\{\, B_\lambda' \mid \lambda < \sigma' \,\} \subset \{\, B_\lambda \mid \mu \neq \lambda < \sigma \,\}$; otherwise B_μ is *indispensable in* F. F will be called a *reduced* simplicial tree-decomposition if every B_μ, $\mu < \sigma$, is indispensable in F.

Let us call a subgraph H of G *weakly attached in* G if every vertex of H has a neighbour in $G\backslash H$. If H is not weakly attached in G, we shall call H *strongly unattached in* G. It is clear that attached subgraphs are also weakly attached, and that strongly unattached subgraphs are unattached.

Suppose finally that H is an induced subgraph of G, and that $\mathcal{C}$ is the set of all components of $G \backslash H$. H will be called an *interface* in G if H is weakly attached in G and

$$\exists\, C, C' \in \mathcal{C} : \forall\, C'' \in \mathcal{C} :\ H\,[\,C\,] \cup H\,[\,C'\,] \not\subset H\,[\,C''\,]$$

(recall that $H\,[\,C\,] = G\,[\,C \to H\,] \cap H$). Notice that interfaces are by definition unattached but never strongly unattached.

Before we use these terms to tackle our uniqueness problem, let us note that as far as non-complete or finite factors are concerned, all prime decompositions of a given graph G agree anyhow: a non-complete or finite subgraph of G is a prime factor in any given decomposition iff it is maximally prime in G [4, Theorems 1.10/1.11]. (Thus in particular, all prime decompositions into finite graphs are reduced.)

For reduced decompositions, we are now able to supply the missing characterization for infinite complete subgraphs: the following theorem implies that a simplex $S \subset G$ is a factor in any reduced prime decomposition of G if and only if S is strongly unattached or an interface in G.

Theorem 1. *Let G be a countable graph, $F = (B_\lambda)_{\lambda<\sigma}$ a simplicial tree-decomposition of G into primes, and $S \subset G$ a simplex.*

(i) *If S is strongly unattached or an interface in G, then S is a factor in F.*

(ii) *If S is an indispensable factor in F, then S is strongly unattached or an interface in G.*

Theorem 1 is the main result of this paper. We reserve its proof for Section 3, and first take a look at its consequences.

Corollary 2. *Let G be a countable graph, $F = (B_\lambda)_{\lambda<\sigma}$ a simplicial tree-decomposition of G into primes, and $\mu < \sigma$. B_μ is indispensable in F if and only if B_μ is either strongly unattached or an interface in G.*

Proof. Suppose first that B_μ is strongly unattached or an interface in G. If B_μ is a simplex, then any prime decomposition $F' = (B'_\lambda)_{\lambda<\sigma'}$ of G with

$$\{\, B'_\lambda \mid \lambda < \sigma' \,\} \subset \{\, B_\lambda \mid \mu \neq \lambda < \sigma \,\}$$

violates Theorem 1 (i), so B_μ is indispensable in F. If B_μ is not a simplex, then B_μ is indispensable in F by [4, Theorems 1.10/1.11].

Suppose now that B_μ is indispensable in F. If B_μ is a simplex, it is strongly unattached or an interface by Theorem 1 (ii); it cannot be both, because an interface is by definition never strongly unattached. Assume now that B_μ is not a simplex, let $v, v' \in V(B_\mu)$ be non-adjacent, and suppose that B_μ is weakly attached in G. Then $G \backslash B_\mu$ has components

C, C' with $v \in B_\mu[C]$ and $v' \in B_\mu[C']$. But no component C'' of $G \backslash B_\mu$ can be such that $B_\mu[C] \cup B_\mu[C'] \subset B_\mu[C'']$ and hence $v, v' \in B_\mu[C'']$: for since B_μ is convex, this would imply that $vv' \in E(B_\mu)$, contradicting our assumption that v, v' are non-adjacent. Hence B_μ is an interface in G. $\square$

We already mentioned that the simplices of attachment in simplicial tree-decompositions into primes are uniquely determined and coincide with the simplices that are minimal relative separators. Indeed, a simplex $S \subset G$ that is a minimal relative separator in G is among the simplices of attachment in *any* simplicial decomposition of G into primes [3]. The converse, however, is only true for simplicial tree-decompositions:

Proposition 3. *Let F be a simplicial tree-decomposition of a graph G into primes, and let $S \subset G$ be a simplex. Then S is a simplex of attachment in F if and only if S is a minimal relative separator in G.*

Proof. Let $F = (B_\lambda)_{\lambda < \sigma}$, and suppose that $S = S_\mu$. Then $S \subset B_\lambda$ for some $\lambda < \mu$ (S4). Since B_μ and B_λ are prime, S is attached to $B_\lambda \backslash S$ as well as to $B_\mu \backslash S$ [4, Corollary 1.3], so S is a minimal relative separator in G. $\square$

Let us now sum up the uniqueness properties of reduced prime decompositions in a single theorem.

Theorem 4. *Let G be a countable graph, B an induced subgraph of G, and $S \subset G$ a simplex. Suppose that G has a reduced simplicial tree-decomposition F into primes.*

(i) *The following statements are equivalent:*

  (a) *B is maximally prime in G and either strongly unattached or an interface;*

  (b) *B is minimally convex in G and either strongly unattached or an interface;*

  (c) *B is a factor in F.*

(ii) *S is a simplex of attachment in F if and only if S is a minimal relative separator in G.*

Proof. Part (ii) of the theorem follows from Proposition 3. In part (i), the implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) can be read out of Corollary 2 and [4, Theorem 1.10]. The reverse implications, (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c), follow from Theorem 1 (i) (if B is a simplex) or from [4, Theorem 1.11] (if B is not a simplex). $\square$

Corollary 5. *If a countable graph has a reduced simplicial tree-decomposition into primes, then its factors and simplices of attachment in any such decomposition are uniquely determined.* $\square$

In fact, Corollary 5 is still valid for uncountable graphs, whose indispensable prime factors can be characterized in way similar to—though less attractive than—Theorem 1 and Corollary 2, without reference to any fixed prime decomposition. If G is an arbitrary graph and $S \subset G$ a simplex, let us call a family $(C_n)_{n \in \mathbb{N}}$ of components of $G \backslash S$ *comprehensive* if for every component C of $G \backslash S$ there is some $n \in \mathbb{N}$ such that $S\,[\,C\,] \subset S\,[\,C_n\,]$. Moreover, let us say that S is the *yield* of such a family if $S = \bigcup_{n \in \mathbb{N}} S\,[\,C_n\,]$ and $S\,[\,C_n\,] \subset S\,[\,C_{n+1}\,]$ for every n.

Theorem 6. *Let G be a graph of arbitrary cardinality, $S \subset G$ a simplex, and F a simplicial tree-decomposition of G into primes. Then S is an indispensable factor in F if and only if S is not the yield of a comprehensive family of components of $G \backslash S$.*

The proof of Theorem 6 is very similar to that of Theorem 1, and left to the reader. (In fact, the proof is even simpler than that of Theorem 1, in which a comprehensive family of components of $G \backslash S$ first has to be found, using the simpler but also weaker concept of an interface.)

If a 'comprehensive family' is defined in a slightly more complicated way, Theorem 6 can even be adapted to simplicial decompositions that are not necessarily tree-decompositions. Thus for general simplicial decompositions, too, the factors in reduced prime decompositions (defined analogously) are uniquely determined. Details will be given in [3].

The immediate question arising from these results is whether every graph that has some simplicial (tree-) decomposition into primes also has a reduced such decomposition—in which case prime decompositions could in practice be taken reduced as a matter of course. However, this is not the case: Section 4 contains an example of a graph that has a simplicial tree-decomposition into primes but no reduced prime decomposition.

It is an open problem to determine which graphs have a reduced simplicial tree-decomposition into primes.

3. Proof of Theorem 1

Let G, F and S be given as stated in Theorem 1. To prove assertion (i), let us suppose that S is strongly unattached or an interface in G. Since any interface is unattached, S is unattached and hence a maximal simplex in G. We can therefore apply [4, Theorem 3.2].

Suppose $\Lambda(S)$ is infinite. Then [4, Theorem 3.2 (iv)] applies; let Λ and $(C_\lambda)_{\lambda \in \Lambda}$ be given as stated there. Since

$$S = \bigcup_{\lambda \in \Lambda} S|_\lambda = \bigcup_{\lambda \in \Lambda} S|_{\lambda^+} = \bigcup_{\lambda \in \Lambda} S\,[\,C_\lambda\,]\,,$$

S is not strongly unattached in G. But [4, Theorem 3.2] also implies that $\Lambda(S\,[\,C\,])$ is finite for every component C of $G \backslash S$, because S is unattached, and hence $S\,[\,C\,] \subsetneqq S$. Thus if C, C' are components of $G\backslash S$, then $S[C] \cup S[C'] \subset S|_\lambda \subset S|_{\lambda^+} = S[C_\lambda]$ for some $\lambda \in \Lambda$. Hence S is not an interface either, contrary to our assumption.

Therefore $\Lambda(S)$ is finite. Since S is unattached, S is a factor in F by [4, Theorem 3.2 (i)].

To prove part (ii) of the theorem, we now assume that S is a factor in F, say $S = B_\mu$. Let us suppose that S is neither strongly unattached nor an interface in G, and prove that S is dispensable in F.

Let $\mathcal{C}$ denote the set of all components of $G\backslash S$. Then every B_λ, $\lambda \neq \mu$, meets exactly one such component $C \in \mathcal{C}$; for as $B_\lambda \not\subset B_\mu = S$ by (S3), we have $B_\lambda \backslash S \neq \emptyset$, and if $B_\lambda \cap C \neq \emptyset$ then $B_\lambda \subset G[C \to S]$, because B_λ is prime and therefore not separated by any subsimplex of S. We shall use this fact repeatedly later on.

Since the definition of a factor's dispensability in a given decomposition does not depend on the order of factors in that decomposition, we may assume by [4, Theorem 3.1] that $\sigma \leq \omega$. Then μ is finite, and

$$\mathcal{C}_0 := \{ C \in \mathcal{C} \mid C|_\mu \neq \emptyset \}$$

is also finite. Since S is by assumption not an interface, this means that there exists a component C_0 of $G\backslash S$ satisfying

$$S[C] \subset S[C_0], \qquad \forall\, C \in \mathcal{C}_0. \tag{1}$$

Let $C^1, C^2, \ldots$ be a fixed enumeration of $\mathcal{C}$. Define a sequence $C_1, C_2, \ldots$ of components of $G\backslash S$ by selecting as C_n $(n = 1, 2, \ldots)$ any $C \in \mathcal{C}$ satisfying

$$S[C_{n-1}] \cup S[C^{k(n)}] \subset S[C],$$

where $k(n)$ denotes the minimal k for which $S[C^k] \not\subset S[C_{n-1}]$ (again using our assumption that S is not an interface). We shall use the notation $\mathcal{C}_1 := \{ C_1, C_2, \ldots \}$ and $\mathcal{C}_2 := \mathcal{C}\backslash(\mathcal{C}_0 \cup \mathcal{C}_1)$. Clearly

$$S[C_n] \subsetneqq S[C_m] \qquad \text{if } 0 \leq n < m, \tag{2}$$

and

$$\forall\, C \in \mathcal{C} : \exists\, n \in \mathbb{N} : S[C] \subset S[C_n]. \tag{3}$$

Since S is by assumption not strongly unattached in G, every vertex of S is contained in $S[C]$ for some $C \in \mathcal{C}$, so (3) implies that

$$S = \bigcup_{n=1,2,\ldots} S[C_n]. \tag{4}$$

On the other hand, S is unattached in G by [4, Theorem 1.10], so $S \neq S[C_n]$ for all n. By (2/4), the sequence $C_1, C_2, \ldots$ must therefore be infinite. Finally, (1), (2) and the definition of $\mathcal{C}_0$ imply that

$$C_n|_\mu = \emptyset \qquad \forall\, n \in \mathbb{N}\,. \tag{5}$$

For each $n \in \mathbb{N}$ set $B_{(n)} := B_{\lambda(C_n)}$, and let

$$F' := (B_\lambda)_{\lambda<\mu} \oplus (B_{(n)})_{n=1,2,\ldots} \oplus (B_\lambda)_{\substack{\mu<\lambda<\sigma \\ B_\lambda \neq B_{(n)}\,\forall\, n=1,2,\ldots}}$$

where $\oplus$ denotes the concatenation of well-ordered families.

Let us take a closer look at this definition. F' is a well-ordered family whose members are precisely the members of F other than $S = B_\mu$, arranged in a slightly different order. We shall prove that F' is a simplicial tree-decomposition of G, thus showing that S is dispensable in F. In its first part, F' coincides with F (up to $G|_\mu$). In the middle part of F', which 'replaces' the factor B_μ, the simplex S is built up by a sequence of contributions from the factors $B_{(n)}$; recall that $B_{(n)} = B_{\lambda(C_n)}$, so the simplex of attachment of $B_{(n)}$ in F is precisely $S[C_n]$ (consider the side $(C_n, S[C_n])$ and apply [4, Corollary 1.7 (iii)]). The third part of F' consists of the remaining factors of F, in their original order.

Our main concern in proving that F' is indeed a simplicial tree-decomposition, is to verify that every 'attachment graph' in F' is a simplex (S2) and contained in some earlier factor (S4). We shall give a brief outline of the proof at this point, which for the reader familiar with simplicial decompositions may be as illuminating as the subsequent more rigorous proof.

For the first part of F', (S2) and (S4) are obvious. The first factor of the second part of F', $B_{(1)}$, satisfies (S2) and (S4) because, roughly speaking, its relationship to the factors preceding it in F' is the same as the relationship of $S = B_\mu$ to the factors preceding B_μ in F (by (1/2)), and F satisfies (S2) and (S4) by assumption.

The subsequent factors $B_{(n)}$ in the middle part of F' will be seen to satisfy (S2) and (S4) because the contribution of each $B_{(n)}$ to the construction of S is the entire segment $S[C_n]$ [4, Corollary 1.7 (iii)], and these segments $S[C_n]$ form a nested sequence by (2).

Checking (S2) and (S4) for the factors B_λ in the third part of F' will depend on the nature of the component C of $G\backslash S$ that contains $B_\lambda\backslash S$. If C is in $\mathcal{C}_0$, the attachment graph of B_λ in F' will be the same as in F, because the transition from F to F' leaves the subgraphs $G[C \to S]$, $C \in \mathcal{C}_0$, essentially unaffected (again by (1)). If $C \in \mathcal{C}_1$, say $C = C_n$, the attachment of B_λ in F' is again the same as in F. For in both families B_λ is preceded by $B_{(n)} = B_{\lambda(C)}$, which covers all vertices of $G\backslash C$ that could possibly be contained in B_λ, namely those of $S[C]$. If $C \in \mathcal{C}_2$ finally, then C appears in F' after the completion of S. By (3), there exists $n \in \mathbb{N}$ with $S[C] \subset S[C_n]$. For the construction of $G[C \to S]$ in F', $B_{(n)}$ can therefore assume the role played by B_μ in F. Thus again the attachment graph of B_λ is the same in F' as it is in F, and it is contained in the same earlier factor or, in the case of $\lambda = \lambda(C)$, in $B_{(n)}$.

We now give a more detailed proof that F' is a simplicial tree-decomposition of G. Recall that

$$B_{(n)} \subset G\,[\,C_n \to S\,], \qquad \forall\, n \in \mathbb{N}\,, \tag{6}$$

because $B_{(n)} \cap C_n \neq \emptyset$ by definition of $B_{(n)}$, and $B_{(n)}$ is prime. As immediate consequences of (6) and (5/6), respectively, we have

$$B_{(n)} \cap B_{(m)} \subset S \quad \text{whenever } n, m \in \mathbb{N},\ n \neq m\,, \tag{7}$$

and

$$B_{(n)}|_\mu \subset S, \qquad \forall\, n \in \mathbb{N}\,. \tag{8}$$

Let us first check that F' satisfies (S1), i.e. that F' covers S. Let $B_{(n)}$, $n \in \mathbb{N}$, be an arbitrary factor from the middle part of F'. Since $B_{(n)} \cap C_n \neq \emptyset$, (5) implies that $B_{(n)} \not\subset G|_\mu$. The index of $B_{(n)}$ in F must therefore be greater than μ, say $B_{(n)} = B_\tau$, $\tau > \mu$. Thus $S \subset G|_\tau$, so in particular $S\,[\,C_n\,] \subset G|_\tau$. Therefore $S_\tau = S\,[\,C_n\,]$ by [4, Corollary 1.7 (iii)]; recall that $\tau = \lambda(C_n)$ by definition of $B_{(n)}$.

Restating this fact without reference to F, we obtain

$$B_{(n)} \cap S = S\,[\,C_n\,], \qquad \forall\, n \in \mathbb{N}\,. \tag{9}$$

In combination with (4), (9) implies that F' covers S; thus F' satisfies (S1).

We now prove that the factors in F' satisfy (S2) and (S4). This is clear for the factors in $(B_\lambda)_{\lambda<\mu}$ (because F satisfies (S2) and (S4)), so let us turn to the middle part of F'. For $n \in \mathbb{N}$, we shall denote $\bigcup_{\lambda<\mu} B_\lambda \oplus \bigcup_{i<n} B_{(i)}$ by $G|_{(n)}$ and $B_{(n)} \cap G|_{(n)}$ by $S_{(n)}$.

Let us first look at $B_{(1)}$. We shall verify (S2) and (S4) for $B_{(1)}$ by showing that $S_{(1)}$ equals S_μ. Since $S_{(1)} = B_{(1)}\big|_\mu \subset S$ by (8), we have $S_{(1)} \subset S \cap G|_\mu = S_\mu$. To see the reverse inclusion, recall first that any factor B_λ, $\lambda < \mu$, satisfies $B_\lambda \subset G\,[\,C \to S\,]$ for some $C \in \mathcal{C}_0$, and hence $B_\lambda \cap S \subset S\,[\,C\,] \subset S\,[\,C_0\,]$ by (1). Therefore

$$G|_\mu \cap S = \bigcup_{\lambda<\mu} (B_\lambda \cap S) \subset S\,[\,C_0\,]\,, \tag{10}$$

and hence

$$\begin{aligned} S_\mu &= G|_\mu \cap S \\ &\subset S\,[\,C_0\,] \cap G|_\mu \\ &\subset S\,[\,C_1\,] \cap G|_\mu \\ &\subset B_{(1)} \cap G|_\mu \\ &= S_{(1)} \end{aligned}$$

by (2) and (9). Thus $S_{(1)} = S_\mu$ as claimed, so $B_{(1)}$ satisfies (S2) and (S4).

Let us now check (S2) and (S4) for the remaining factors of the form $B_{(n)}$. Let $n \geq 2$. Then

$$\begin{aligned} S_{(n)} &= B_{(n)} \cap G|_{(n)} \\ &= (B_{(n)} \cap G|_{\mu}) \cup \bigcup_{i=1}^{n-1} \big(B_{(n)} \cap B_{(i)}\big) \\ &= (B_{(n)} \cap S \cap G|_{\mu}) \cup \bigcup_{i=1}^{n-1} \big(B_{(n)} \cap S \cap B_{(i)}\big) && \text{(by (8/7))} \\ &= (S\,[\,C_n\,] \cap G|_{\mu}) \cup \bigcup_{i=1}^{n-1} \big(S\,[\,C_n\,] \cap S\,[\,C_i\,]\big) && \text{(by (9))} \\ &= (S\,[\,C_n\,] \cap G|_{\mu}) \cup S\,[\,C_{n-1}\,] && \text{(by (2))} \\ &= S\,[\,C_{n-1}\,] && \text{(by (10/2))} \\ &= B_{(n-1)} \cap S && \text{(by (9))}. \end{aligned}$$

Therefore $S_{(n)}$ is a simplex (S2) contained in an earlier factor of F' (S4).

It remains to show (S2) and (S4) for the factors in the third part of F', i.e. for the factors B_ν with $\mu < \nu < \sigma$ and $B_\nu \neq B_{(n)}$ for all $n \in \mathbb{N}$. Let such B_ν be given, and let C be the component of $G \backslash S$ for which $B_\nu \subset G\,[\,C \to S\,]$. Put

$$F|_\nu := (B_\lambda)_{\lambda<\nu}\,,$$

and let $F'|_\nu$ be the corresponding subfamily of F', i.e.,

$$F'|_\nu := (B_\lambda)_{\lambda<\mu} \oplus \big(B_{(n)}\big)_{n=1,2,\ldots} \oplus (B_\lambda)_{\substack{\mu<\lambda<\nu \\ B_\lambda \neq B_{(n)}\ \forall\, n=1,2,\ldots}}\,.$$

Let us further define

$$H|'_\nu := H \cap \bigcup_{B \in F'|_\nu} B$$

for subgraphs H of G, and set

$$S'_\nu := B_\nu|'_\nu.$$

Thus S'_ν is the 'attachment graph' of B_ν in F', and we want to show that S'_ν is a simplex contained in some $B \in F'|_\nu$.

Note first that every $B \subset G$ satisfies

$$B \in F|_\nu \Leftrightarrow B \in F'|_\nu \qquad \text{if } B \cap C \neq \emptyset\,. \tag{11}$$

Indeed, by the construction of F' we have $B \in F|_\nu \;\Rightarrow\; B \in F'|_\nu$ unless $B = S$, but $S \cap C = \emptyset$. And conversely, we have $B \in F'|_\nu \;\Rightarrow\; B \in F|_\nu$ unless $B = B_{(n)}$ for some $n \in \mathbb{N}$;

but $B_{(n)} \cap C$ is non-empty only if n is such that $C = C_n$, in which case $B = B_{(n)} = B_{\lambda(C_n)}$, which is in $F|_\nu$ by $B_\nu \cap C \neq \emptyset$, the definition of $\lambda(C_n)$, and the obvious fact that $B \neq B_\nu$.

Moreover,

$$G\,[\,C \to S\,]\big|_\nu = G\,[\,C \to S\,]\big|'_\nu. \tag{12}$$

To verify (12), notice that any vertex or edge of $G\,[\,C \to S\,]\big|_\nu$ or $G\,[\,C \to S\,]\big|'_\nu$ that is not contained in $S\,[\,C\,]$ belongs to some $B \in F|_\nu$ (or $B \in F'|_\nu$, respectively) with $B \cap C \neq \emptyset$, and is therefore in $G\,[\,C \to S\,]\big|_\nu \cap G\,[\,C \to S\,]\big|'_\nu$ by (11). But $S\,[\,C\,]$ is a subgraph of $G\,[\,C \to S\,]\big|_\nu$ (because $S \subset G|_\nu$) as well as of $G\,[\,C \to S\,]\big|'_\nu$ (by (3/9)), so (12) follows.

Using (12) and the fact that $B_\nu \subset G\,[\,C \to S\,]$, it is easy to determine S'_ν:

$$\begin{aligned} S'_\nu &= B_\nu \cap G|'_\nu \\ &= B_\nu \cap G\,[\,C \to S\,]\big|'_\nu \\ &= B_\nu \cap G\,[\,C \to S\,]\big|_\nu \\ &= B_\nu \cap G|_\nu \\ &= S_\nu\,. \end{aligned}$$

Thus S'_ν is a simplex because S_ν is a simplex, giving (S2). But also (S4) is now obvious: if $S'_\nu \subset S$, then $S'_\nu \subset S\,[\,C\,] \subset B_{(n)}$ for every sufficiently large $n \in \mathbb{N}$ (3/9). But if $S'_\nu \not\subset S$ then $S'_\nu \cap C \neq \emptyset$, so any $B \in F|_\nu$ with $S'_\nu = S_\nu \subset B$ will also be in $F'|_\nu$ by (11). Such B exists, because F satisfies (S4).

It remains to check that F' satisfies (S3). As we have seen, every simplex of attachment in F' is also a simplex of attachment in F or a subgraph of $S = B_\mu$. Moreover, every factor in F' is also a factor in F, and no factor $B_\lambda \in F$, $\lambda \neq \mu$, can be contained in B_μ or in any simplex of attachment of F, because F satisfies (S3). We may therefore deduce that no factor in F' can be contained in any simplex of attachment of F', i.e. F' satisfies (S3).

This completes the proof of Theorem 1.

4. An Example

We now give an example of a graph which admits a simplicial tree-decomposition into primes, but has no reduced decomposition. Our example is the graph T_0 introduced in [5]: its vertices are the finite 0-1 sequences (including the empty sequence), and two vertices $(a_0, \dots, a_\mu)$ and $(b_0, \dots, b_\nu)$ are joined by an edge whenever $\mu < \nu$ and $a_\lambda = b_\lambda$ for $\lambda = 0, \dots, \mu$. For every $\alpha = (a_0, a_1, \dots) \in \{0,1\}^\omega$, the subgraph

$$S_\alpha := T_0\big[\{(a_\lambda)_{\lambda<\mu} \mid \mu < \omega\}\big]$$

of T_0 spanned by all the finite initial segments of α is a simplex, and it is easily seen that the graphs S_α, $\alpha \in \{0,1\}^\omega$, are precisely the maximally prime subgraphs of T_0. Moreover, it is not difficult to arrange them into simplicial tree-decompositions of T_0. In fact, if $F = (S_{\alpha_\lambda})_{\lambda<\sigma}$ is any maximal (with respect to extension) well-ordered family of S_α's satisfying $S_{\alpha_\mu} \setminus \bigcup_{\lambda<\mu} S_{\alpha_\lambda} \neq \emptyset$ for all $\mu < \sigma$, then F is a simplicial tree-decomposition of T_0 into primes.

However, T_0 has no reduced prime decomposition. For any factor in such a decomposition must be maximally prime in T_0 and therefore of the form S_α, with $\alpha = (a_\lambda)_{\lambda<\omega}$ say. Then the graphs $S_\alpha[C] = T_0[C \to S_\alpha] \cap S_\alpha$ (for components C of $G \setminus S_\alpha$) are precisely the nested simplices $S_\alpha[\{a_\lambda \mid \lambda < \mu\}]$, $\mu < \omega$. Therefore S_α is neither strongly unattached nor an interface, contrary to Theorem 1.

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