Orthogonality and minimality
in the homology of locally finite graphs

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October 25, 2013

Abstract
Given a finite set $E$, a subset $D \subseteq E$ (viewed as a function $E \to \mathbb{F}_2$) is orthogonal to a given subspace $F$ of the $\mathbb{F}_2$-vector space of functions $E \to \mathbb{F}_2$ as soon as $D$ is orthogonal to every $\leq$-minimal element of $F$. This fails in general when $E$ is infinite.

However, we prove the above statement for the six subspaces $F$ of the edge space of any 3-connected locally finite graph that are relevant to its homology: the topological, algebraic, and finite cycle and cut spaces. This solves a problem of [5].

1 Introduction
Let $G$ be a 2-connected locally finite graph, and let $E = \mathcal{E}(G)$ be its edge space over $\mathbb{F}_2$. We think of the elements of $E$ as sets of edges, possibly infinite. Two sets of edges are orthogonal if their intersection has (finite and) even cardinality. A set $D \in \mathcal{E}$ is orthogonal to a subspace $F \subseteq \mathcal{E}$ if it is orthogonal to every $F \in F$. See [4, 5] for any definitions not given below.

The topological cycle space $C_{\text{top}}(G)$ of $G$ is the subspace of $\mathcal{E}(G)$ generated (via thin sums, possibly infinite) by the circuits of $G$, the edge sets of the topological circles in the Freudenthal compactification $|G|$ of $G$. This space $C_{\text{top}}(G)$ contains precisely the elements of $\mathcal{E}$ that are orthogonal to $B_{\text{fin}}(G)$, the finite-cut space of $G$ [4]. The algebraic cycle space $C_{\text{alg}}(G)$ of $G$ is the subspace of $\mathcal{E}$ consisting of the edge sets inducing even degrees at all the vertices. It contains precisely the elements of $\mathcal{E}$ that are orthogonal to the skew cut space $B_{\text{skew}}(G)$ [3], the subspace of $\mathcal{E}$ consisting of all the cuts of $G$ with one side finite. The finite-cycle space $C_{\text{fin}}(G)$ is the subspace of $\mathcal{E}$ generated (via finite sums) by the finite circuits of $G$. This space $C_{\text{fin}}(G)$ contains precisely the elements of $\mathcal{E}$ that are orthogonal to $B(G)$, the cut space of $G$ [4, 5]. Thus,

$$C_{\text{top}} = B_{\text{fin}}^\perp, \quad C_{\text{alg}} = B_{\text{skew}}^\perp, \quad C_{\text{fin}} = B^\perp.$$  

Conversely,

$$C_{\text{top}}^\perp = B_{\text{fin}}, \quad C_{\text{alg}}^\perp = B_{\text{skew}}, \quad C_{\text{fin}}^\perp = B.$$  

Thus, for any of the six spaces $F$ just mentioned, we have $F^\perp \perp F = F$.  

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Proofs of most of the above six identities were first published by Casteels and Richter [3], in a more general setting. Any remaining proofs can be found in [5], except for the inclusion $C_{\text{alg}}^+ \supseteq \mathcal{B}_{\text{skew}}$, which is easy.

The six subspaces of $\mathcal{E}$ mentioned above are the the ones most relevant to the homology of locally finite infinite graphs. See [5], Diestel and Sprüssel [6], and Georgakopoulos [7, 8]. Our aim in this note is to facilitate orthogonality proofs for these spaces by showing that, whenever $\mathcal{F}$ is one of them, a set $D$ of edges is orthogonal to $\mathcal{F}$ as soon as it is orthogonal to the minimal nonzero elements of $\mathcal{F}$.

This is easy when $\mathcal{F}$ is $\mathcal{C}_{\text{fin}}$ or $\mathcal{B}_{\text{fin}}$ or $\mathcal{B}_{\text{skew}}$:

**Proposition 1.** Let $\mathcal{F}$ be a subspace of $\mathcal{E}$ all whose elements are finite sets of edges. Then $\mathcal{F}$ is generated (via finite sums) by its $\subseteq$-minimal nonzero elements.

**Proof.** For a contradiction suppose that some $F \in \mathcal{F}$ is not a finite sum of finitely many minimal nonzero elements of $\mathcal{F}$. Choose $F$ with $|F|$ minimal. As $F$ is not minimal itself, by assumption, it properly contains a minimal nonzero element $F'$ of $\mathcal{F}$. As $F$ is finite, $F + F' = F \setminus F' \in \mathcal{F}$ has fewer elements than $F$, so there is a finite family $(M_i)_{i \leq n}$ of minimal nonzero elements of $\mathcal{F}$ with $\sum_{i \leq n} M_i = F + F'$. This contradicts our assumption, as $F' + \sum_{i \leq n} M_i = F$. \(\square\)

**Corollary 2.** If $\mathcal{F} \in \{\mathcal{C}_{\text{fin}}, \mathcal{B}_{\text{fin}}, \mathcal{B}_{\text{skew}}\}$, a set $D$ of edges is orthogonal to $\mathcal{F}$ as soon as $D$ is orthogonal to all the minimal nonzero elements of $\mathcal{F}$.

When $\mathcal{F} \in \{\mathcal{C}_{\text{top}}, \mathcal{C}_{\text{alg}}, \mathcal{B}\}$, the statement of Corollary 2 is generally false for graphs that are not 3-connected. Here are some examples.

For $\mathcal{F} = \mathcal{B}$, let $G$ be the graph obtained from the $\mathbb{N} \times \mathbb{Z}$ grid by doubling every edge between two vertices of degree 3 and subdividing all the new edges. The set $D$ of the edges that lie in a $K^3$ of $G$ is orthogonal to every bond $F$ of $G$; their intersection $D \cap F$ is finite and even. But $D$ is not orthogonal to every element of $\mathcal{F} = \mathcal{B}$, since it meets some cuts that are not bonds infinitely.

For $\mathcal{F} = \mathcal{C}_{\text{top}}$, let $B$ be an infinite bond of the infinite ladder $H$, and let $G$ be the graph obtained from $H$ by subdividing every edge in $B$. Then the set $D$ of edges that are incident with subdivision vertices has a finite and even intersection with every topological circuit $C$, finite or infinite, but it is not orthogonal to every element of $\mathcal{C}_{\text{top}}$, since it meets some of them infinitely.

For $\mathcal{F} = \mathcal{C}_{\text{alg}}$ we can re-use the example just given for $\mathcal{C}_{\text{top}}$, since for 1-ended graphs like the ladder the two spaces coincide.

However, if $G$ is 3-connected, an edge set is orthogonal to every element of $\mathcal{C}_{\text{top}}, \mathcal{C}_{\text{alg}}$, or $\mathcal{B}$ as soon as it is orthogonal to every minimal nonzero element:

**Theorem 3.** Let $G = (V, E)$ be a locally finite 3-connected graph, and $F, D \subseteq E$.

(i) $F \in C_{\text{top}}^+$ as soon as $F$ is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\text{top}}$, the topological circuits of $G$.

(ii) $F \in C_{\text{alg}}^+$ as soon as $F$ is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\text{alg}}$, the finite circuits and the edge sets of double rays in $G$. 

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(iii) $D \in \mathcal{B}^\perp$ as soon as $D$ is orthogonal to all the minimal nonzero elements of $\mathcal{B}$, the bonds of $G$.

Although Theorem 3 fails if we replace the assumption of 3-connectedness with 2-connectedness, it turns out that we need a little less than 3-connectedness. Recall that an end $\omega$ of $G$ has (combinatorial) vertex-degree $k$ if $k$ is the maximum number of vertex-disjoint rays in $\omega$. Halin [9] showed that every end in a $k$-connected locally finite graph has vertex-degree at least $k$. Let us call an end $\omega$ of $G$ $k$-padded if for every ray $R \in \omega$ there is a neighbourhood $U$ of $\omega$ such that for every vertex $u \in U$ there is a $k$-fan from $u$ to $R$ in $G$, a subdivided $k$-star with centre $u$ and leaves on $R$. If every end of $G$ is $k$-padded, we say that $G$ is $k$-padded at infinity. Note that $k$-connected graphs are $k$-padded at infinity. Our proof of Theorem 3(i) and (ii) will use only that every end has vertex-degree at least 3 and that $G$ is 2-connected. Similarly, and in a sense dually, our proof of Theorem 3(iii) uses only that every end has vertex-degree at least 2 and $G$ is 3-connected at infinity.

**Theorem 4.** Let $G = (V, E)$ be a locally finite 2-connected graph.

(i) If every end of $G$ has vertex-degree at least 3, then $F \in \mathcal{C}^\perp_{\text{top}}$ as soon as $F$ is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\text{top}}$, the topological circuits of $G$.

(ii) If every end of $G$ has vertex-degree at least 3, then $F \in \mathcal{C}^\perp_{\text{alg}}$ as soon as $F$ is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\text{alg}}$, the finite circuits and the edge sets of double rays in $G$.

(iii) If $G$ is 3-padded at infinity, then $D \in \mathcal{B}^\perp$ as soon as $D$ is orthogonal to all the minimal nonzero elements of $\mathcal{B}$, the bonds of $G$.

In general, our notation follows [4]. In particular, given an end $\omega$ in a graph $G$ and a finite set $S \subseteq V(G)$ of vertices, we write $C(S, \omega)$ for the unique component of $G - S$ that contains a ray $R \in \omega$. The vertex-degree of $\omega$ is the maximum number of vertex-disjoint rays in $\omega$. The mathematical background required for this paper is covered in [5, 6]. For earlier results on the cycle and cut space see Bruhn and Stein [1, 2].

## 2 Finding disjoint paths and fans

Menger’s theorem that the smallest cardinality of an $A$–$B$ separator in a finite graph is equal to the largest cardinality of a set of disjoint $A$–$B$ paths trivially extends to infinite graphs. Thus in a locally finite $k$-connected graph, there are $k$ internally disjoint paths between any two vertices. In Lemmas 5 and 6 we

\footnote{For example, if $G$ is the union of complete graphs $K_1, K_2, \ldots$ with $|K_i| = i$, each meeting the next in exactly one vertex (and these are all distinct), then the unique end of $G$ is $k$-padded for every $k \in \mathbb{N}$.}
show that, for two such vertices that are close to an end $\omega$, these connecting paths need not use vertices too far away from $\omega$.

In a graph $G$ with vertex sets $X, Y \subseteq V(G)$ and vertices $x, y \in V(G)$, a $k$-fan from $X$ (or $x$) to $Y$ is a subdivided $k$-star whose center lies in $X$ (or is $x$) and whose leaves lie in $Y$. A $k$-linkage from $x$ to $y$ is a union of $k$ internally disjoint $x$-$y$ paths. We may refer to a sequence $(v_i)_{i \in \mathbb{N}}$ simply by $(v_i)$, and use $\bigcup(v_i) := \bigcup_{i \in \mathbb{N}} \{v_i\}$ for brevity.

**Lemma 5.** Let $G$ be a locally finite graph with an end $\omega$, and let $(v_i)_{i \in \mathbb{N}}$ and $(w_i)_{i \in \mathbb{N}}$ be two sequences of vertices converging to $\omega$. Let $k$ be a positive integer.

(i) If for infinitely many $n \in \mathbb{N}$ there is a $k$-fan from $v_n$ to $\bigcup(w_i)$, then there are infinitely many disjoint such $k$-fans.

(ii) If for infinitely many $n \in \mathbb{N}$ there is a $k$-linkage from $v_n$ to $w_n$, then there are infinitely many disjoint such $k$-linkages.

**Proof.** For a contradiction, suppose $k \in \mathbb{N}$ is minimal such that there is a locally finite graph $G = (V, E)$ with sequences $(v_i)_{i \in \mathbb{N}}$ and $(w_i)_{i \in \mathbb{N}}$ in which either (i) or (ii) fails. Then $k > 1$, since for every finite set $S \subseteq V(G)$ the unique component $C(S, \omega)$ of $G - S$ that contains rays from $\omega$ is connected and contains all but finitely many vertices from $\bigcup(v_i)$ and $\bigcup(w_i)$.

For a proof of (i) it suffices to show that for every finite set $S \subseteq V(G)$ there is an integer $n \in \mathbb{N}$ and a $k$-fan from $v_n$ to $\bigcup(w_i)$ avoiding $S$. Suppose there is a finite set $S \subseteq V(G)$ that meets all $k$-fans from $\bigcup(v_i)$ to $\bigcup(w_i)$. By the minimality of $k$, there are infinitely many disjoint $(k - 1)$-fans from $\bigcup(v_i)$ to $\bigcup(w_i)$ in $C := C(S, \omega)$. Thus, there is a subsequence $(v'_i)_{i \in \mathbb{N}}$ of $(v_i)_{i \in \mathbb{N}}$ in $C$ and pairwise disjoint $(k - 1)$-fans $F_i \subseteq C$ from $v'_i$ to $\bigcup(w_i)$ for all $i \in \mathbb{N}$. For every $i \in \mathbb{N}$ there is by Menger’s theorem a $(k - 1)$-separator $S_i$ separating $v'_i$ from $\bigcup(w_i)$ in $C$, as by assumption there is no $k$-fan from $v'_i$ to $\bigcup(w_i)$ in $C$. Let $C_i$ be the component of $G - (S \cup S_i)$ containing $v'_i$.

Since $F_i$ is a subdivided $|S_i|$-star, $S_i \subseteq V(F_i)$. Hence for all $i \neq j$, our assumption of $F_i \cap F_j = \emptyset$ implies that $F_i \cap S_j = \emptyset$, and hence that $F_i \cap C_j = \emptyset$. But then also $C_i \cap C_j = \emptyset$, since any vertex in $C_i \cap C_j$ could be joined to $v'_i$ by a path $P$ in $C_j$ and to $v'_j$ by a path $Q$ in $C_i$, giving rise to a $v'_j - \bigcup(w_i)$ path in $P \cup Q \cup F_i$ avoiding $S_j$, a contradiction.

As $S \cup S_i$ separates $v'_i$ from $\bigcup(w_i)$ in $G$ and there is, by assumption, a $k$-fan from $v'_i$ to $\bigcup(w_i)$ in $G$, there are at least $k$ distinct neighbours of $C_i$ in $S \cup S_i$. Since $|S_i| = k - 1$, one of these lies in $S$. This holds for all $i \in \mathbb{N}$. As $C_i \cap C_j = \emptyset$ for distinct $i$ and $j$, this contradicts our assumption that $G$ is locally finite and $S$ is finite. This completes the proof of (i).

For (ii) it suffices to show that for every finite set $S \subseteq V(G)$ there is an integer $n \in \mathbb{N}$ such that there is a $k$-linkage from $v_n$ to $w_1$ avoiding $S$. Suppose there is a finite set $S \subseteq V(G)$ that meets all $k$-linkages from $v_i$ to $w_i$ for all $i \in \mathbb{N}$. By the minimality of $k$ there is an infinite family $(L_i)_{i \in I}$ of disjoint $(k - 1)$-linkages $L_i$ in $C := C(S, \omega)$ from $v_i$ to $w_i$. As earlier, there are pairwise disjoint $(k - 1)$-sets $S_i \subseteq V(L_i)$ separating $v_i$ from $w_i$ in $C$, for all $i \in I$. Let
$C_i, D_i$ be the components of $C - S_i$ containing $v_i$ and $w_i$, respectively. For no $i \in I$ can both $C_i$ and $D_i$ have $\omega$ in their closure, as they are separated by the finite set $S \cup S_i$. Thus for every $i \in I$ one of $C_i$ or $D_i$ contains at most finitely many vertices from $\bigcup_{i \in I} L_i$. By symmetry, and replacing $I$ with an infinite subset of itself if necessary, we may assume the following:

\begin{equation}
\begin{aligned}
\text{The components } C_i \text{ with } i \in I \text{ each contain only finitely many } \\
\text{vertices from } \bigcup_{i \in I} L_i.
\end{aligned}
\end{equation}

If infinitely many of the components $C_i$ are pairwise disjoint, then $S$ has infinitely many neighbours as earlier, a contradiction. By Ramsey’s theorem, we may thus assume that

\begin{equation}
C_i \cap C_j \neq \emptyset \text{ for all } i, j \in I.
\end{equation}

Note that if $C_i$ meets $L_j$ for some $j \neq i$, then $C_i \supseteq L_j$, since $L_j$ is disjoint from $L_i \supseteq S_i$. By (1), this happens for only finitely many $j > i$. We can therefore choose an infinite subset of $I$ such that $C_i \cap L_j = \emptyset$ for all $i < j$ in $I$. In particular, $(C_i \cup S_i) \cap S_j = \emptyset$ for $i < j$. By (2), this implies that

\begin{equation}
C_i \cup S_i \subseteq C_j \text{ for all } i < j.
\end{equation}

By assumption, there exists for each $i \in I$ some $v_i - w_i$ linkage of $k$ independent paths in $G$, one of which avoids $S_i$ and therefore meets $S$. Let $P_i$ denote its final segment from its last vertex in $S$ to $w_i$. As $w_i \in C \setminus (C_i \cup S_i)$ and $P_i$ avoids both $S_i$ and $S$ (after its starting vertex in $S$), we also have

\begin{equation}
P_i \cap C_i = \emptyset.
\end{equation}

On the other hand, $L_i$ contains $v_i \in C_i \subseteq C_{i+1}$ and avoids $S_{i+1}$, so $w_i \in L_i \subseteq C_{i+1}$. Hence $P_i$ meets $S_j$ for every $j \geq i + 1$ such that $P_i \not\subseteq S \cup C_j$. Since the $L_j \supseteq S_j$ are disjoint for different $j$, this happens for only finitely many $j > i$. Deleting those $j$ from $I$, and repeating that argument for increasing $i$ in turn, we may thus assume that $P_i \subseteq S \cup C_{i+1}$ for all $i \in I$. By (3) and (4) we deduce that $P_i \cap S$ are now disjoint for different values of $i \in I$. Hence $S$ contains a vertex of infinite degree, a contradiction.

Recall that $G$ is $k$-padded at an end $\omega$ if for every ray $R \in \omega$ there is a neighbourhood $U$ such that for all vertices $u \in U$ there is a $k$-fan from $u$ to $R$ in $G$. Our next lemma shows that, if we are willing to make $U$ smaller, we can find the fans locally around $\omega$:

**Lemma 6.** Let $G$ be a locally finite graph with a $k$-padded end $\omega$. For every ray $R \in \omega$ and every finite set $S \subseteq V(G)$ there is a neighbourhood $U \subseteq C(S, \omega)$ of $\omega$ such that from every vertex $u \in U$ there is a $k$-fan in $C(S, \omega)$ to $R$.

**Proof.** Suppose that, for some $R \in \omega$ and finite $S \subseteq V(G)$, every neighbourhood $U \subseteq C(S, \omega)$ of $\omega$ contains a vertex $u$ such that $C(S, \omega)$ contains no $k$-fan from $u$ to $R$. Then there is a sequence $u_1, u_2, \ldots$ of such vertices converging to $\omega$. As $\omega$ is $k$-padded there are $k$-fans from infinitely many $u_i$ to $R$ in $G$. By Lemma 5(i) we may assume that these fans are disjoint. By the choice of $u_1, u_2, \ldots$, all these disjoint fans meet the finite set $S$, a contradiction.

\end{enumerate}
3 The proof of Theorems 3 and 4

As pointed out in the introduction, Theorem 4 implies Theorem 3. It thus suffices to prove Theorem 4, of which we prove (i) first. Consider a set $F \neq \emptyset$ of edges that meets every circuit of $G$ evenly. We have to show that $F \in C^\perp_{\text{top}}$, i.e., that $F$ is a finite cut. (Recall that $C^\perp_{\text{top}}$ is known to equal $B_{\text{mn}}$, the finite-cut space [5].) As $F$ meets every finite cycle evenly it is a cut, with bipartition $(A, B)$ say. Suppose $F$ is infinite. Let $\mathcal{R}$ be a set of three disjoint rays that belong to an end $\omega$ in the closure of $F$. Every $R-R'$ path $P$ for two distinct $R, R' \in \mathcal{R}$ lies on the unique topological circle $C(R, R', P)$ that is contained in $R \cup R' \cup P \cup \{\omega\}$. As every circuit meets $F$ finitely, we deduce that no ray in $\mathcal{R}$ meets $F$ again and again. Replacing the rays in $\mathcal{R}$ with tails of themselves as necessary, we may thus assume that $F$ contains no edge from any of the rays in $\mathcal{R}$. Suppose $F$ separates $\mathcal{R}$, with the vertices of $R \in \mathcal{R}$ in $A$ and the vertices of $R', R'' \in \mathcal{R}$ in $B$ say. Then there are infinitely many disjoint $R-(R' \cup R'')$ paths each meeting $F$ at least once. Infinitely many of these disjoint paths avoid one of the rays in $B$, say $R''$. The union of these paths together with $R$ and $R'$ contains a ray $W \in \omega$ that meets $F$ infinitely often. For every $R''-W$ path $P$, the circle $C(W, R'', P)$ meets $F$ in infinitely many edges, a contradiction. Thus we may assume that $F$ does not separate $\mathcal{R}$, and that $G[A]$ contains $\bigcup \mathcal{R}$.

As $\omega$ lies in the closure of $F$, there is a sequence $(v_i)_{i \in \mathbb{N}}$ of vertices in $B$ converging to $\omega$. As $G$ is 2-connected there is a 2-fan from each $v_i$ to $\bigcup \mathcal{R}$ in $G$. By Lemma 5 there are infinitely many disjoint 2-fans from $\bigcup \{v_i\}$ to $\bigcup \mathcal{R}$. We may assume that every such fan has at most two vertices in $\bigcup \mathcal{R}$. Then infinitely many of these fans avoid some fixed ray in $\mathcal{R}$, say $R$. The two other rays plus the infinitely many 2-fans meeting only these together contain a ray $W \in \omega$ that meets $F$ infinitely often and is disjoint from $R$. Then for every $R-W$ path $P$ we get a contradiction, as $C(R, W, P)$ is a circle meeting $F$ in infinitely many edges.

For a proof of (ii), note first that the minimal elements of $C_{\text{alg}}$ are indeed the finite circuits and the edge sets of double rays in $G$. Indeed, these are clearly in $C_{\text{alg}}$ and minimal. Conversely, given any element of $C_{\text{alg}}$, a set $D$ of edges inducing even degrees at all the vertices, we can greedily find for any given edge $e \in D$ a finite circuit or double ray with all its edges in $D$ that contains $e$. We may thus decompose $D$ inductively into disjoint finite circuits and edge sets of double rays, since deleting finitely many such sets from $D$ clearly produces another element of $C_{\text{alg}}$, and including in each circuit or double ray chosen the smallest undeleted edge in some fixed enumeration of $D$ ensures that the entire set $D$ is decomposed. If $D$ is minimal in $C_{\text{alg}}$, it must therefore itself be a finite circuit or the edge set of a double ray.

Consider a set $F$ of edges that fails to meet some set $D \in C_{\text{alg}}$ evenly; we have to show that $F$ also fails to meet some finite circuit or double ray evenly. If $|F \cap D|$ is odd, then this follows from our decomposition of $D$ into disjoint finite circuits and edges sets of double rays. We thus assume that $F \cap D$ is infinite. Since $|G|$ is compact, we can find a sequence $e_1, e_2, \ldots$ of edges in $F \cap D$ that
converges to some end $\omega$. Let $R_1, R_2, R_3$ be disjoint rays in $\omega$, which exist by our assumption that $\omega$ has vertex-degree at least 3. Subdividing each edge $e_i$ by a new vertex $v_i$, and using that $G$ is 2-connected, we can find for every $i$ a 2-fan from $v_i$ to $W = V(R_1 \cup R_2 \cup R_3)$ that has only its last vertices and possibly $v_i$ in $W$. By Lemma 5, with $w_1, w_2, \ldots$ an enumeration of $W$, some infinitely many of these fans are disjoint. Renaming the rays $R_i$ and replacing $e_1, e_2, \ldots$ with a subsequence as necessary, we may assume that either all these fans have both endvertices on $R_1$, or that they all have one endvertex on $R_1$ and the other on $R_2$. In both cases all these fans avoid $R_3$, so we can find a ray $R$ in the union of $R_1, R_2$ and these fans (suppressing the subdividing vertices $v_i$ again) that contains infinitely many $e_i$ and avoids $R_3$. Linking $R$ to a tail of $R_3$ we thus obtain a double ray in $G$ that contains infinitely many $e_i$, as desired.

To prove (iii), let $D \subseteq E$ be a set of edges that meets every bond evenly. We have to show that $D \in \mathcal{B}^+$, i.e., that $D$ has an (only finite and) even number of edges also in every cut that is not a bond.

As $D$ meets every finite bond evenly, and hence every finite cut, it lies in $R_{\text{fin}} = C_{\text{top}}$. We claim that

$$D \text{ is a disjoint union of finite circuits.} \tag{\ast}$$

To prove $(\ast)$, let us show first that every edge $e \in D$ lies in some finite circuit $C \subseteq D$. If not, the endvertices $u, v$ of $e$ lie in different components of $(V, D \setminus \{e\})$, and we can partition $V$ into two sets $A, B$ so that $e$ is the only $A-B$ edge in $D$. The cut of $G$ of all its $A-B$ edges is a disjoint union of bonds [4], one of which meets $D$ in precisely $e$. This contradicts our assumption that $D$ meets every bond of $G$ evenly.

For our proof of $(\ast)$, we start by enumerating $D$, say as $D = \{e_1, e_2, \ldots\} =: D_0$. Let $C_0 \subseteq D_0$ be a finite circuit containing $e_0$, let $D_1 := D_0 \setminus C_0$, and notice that $D_1$, like $D_0$, meets every bond of $G$ evenly (because $C_0$ does). As before, $D_1$ contains a finite circuit $C_1$ containing the edge $e_i$ with $i = \min\{j \mid e_j \in D_1\}$. Continuing in this way we find the desired decomposition $D = C_1 \cup C_2 \cup \ldots$ of $D$ into finite circuits. This completes the proof of $(\ast)$.

As every finite circuit lies in $\mathcal{B}^+$, it suffices by $(\ast)$ to show that $D$ is finite. Suppose $D$ is infinite, and let $\omega$ be an end of $G$ in its closure. Let us say that two rays $R$ and $R'$ hug $D$ if every neighbourhood $U$ of $\omega$ contains a finite circuit $C \subseteq D$ that is neither separated from $R$ by $R'$ nor from $R'$ by $R$ in $U$. We shall construct two rays $R$ and $R'$ that hug $D$, inductively as follows.

Let $S_0 = \emptyset$, and let $R_0, R'_{0}$ be disjoint rays in $\omega$. (These exist as $G$ is 2-connected [9].) For step $j \geq 1$, assume that let $S_i, R_i$, and $R'_i$ have been defined for all $i < j$ so that $R_i$ and $R'_i$ each meet $S_i$ in precisely some initial segment (and otherwise lie in $C(S_i, \omega)$) and $S_i$ contains the $i$th vertex in some fixed enumeration of $V$. If the $j$th vertex in this enumeration lies in $C(S_{j-1}, \omega)$, add to $S_{j-1}$ this vertex and, if it lies on $R_{j-1}$ or $R'_{j-1}$, the initial segment of that ray up to it. Keep calling the enlarged set $S_{j-1}$. For the following choice of $S$ we apply Lemma 6 to $S_{j-1}$ and each of $R_{j-1}$ and $R'_{j-1}$. Let $S \supseteq S_{j-1}$ be a finite set such that from every vertex $v$ in $C(S, \omega)$ there are 3-fans in $C(S, \omega)$ both
to \( R_{j-1} \) and to \( R'_{j-1} \). By \((\ast)\) and the choice of \( \omega \), there is a finite circuit \( C_j \subseteq D \) in \( C(S, \omega) \). Then \( C_j \) can not be separated from \( R_{j-1} \) or \( R'_{j-1} \) in \( C(S_{j-1}, \omega) \) by fewer than three vertices, and thus there are three disjoint paths from \( C_j \) to \( R_{j-1} \cup R'_{j-1} \) in \( C(S_{j-1}, \omega) \).

There are now two possible cases. The first is that in \( C(S_{j-1}, \omega) \) the circuit \( C_j \) is neither separated from \( R_{j-1} \) by \( R'_{j-1} \) nor from \( R'_{j-1} \) by \( R_{j-1} \). This case is the preferable case. In the second case one ray separates \( C_j \) from the other. In this case we will reroute the two rays to obtain new rays as in the first case. We shall then ‘freeze’ a finite set containing initial parts of these rays, as well as paths from each ray to \( C_j \). This finite fixed set will not be changed in any later step of the construction of \( R \) and \( R' \). In detail, this process is as follows.

If \( C(S_{j-1}, \omega) \) contains both a \( C_j-R_{j-1} \) path \( P \) avoiding \( R'_j-1 \) and a \( C_j-R'_{j-1} \) path \( P' \) avoiding \( R_{j-1} \), let \( Q \) and \( Q' \) be the initial segments of \( R_{j-1} \) and \( R'_{j-1} \) up to \( P \) and \( P' \), respectively. Then let \( R_j = R_{j-1} \) and \( R'_j = R'_{j-1} \) and

\[
S_j = S_{j-1} \cup V(P) \cup V(P') \cup V(Q) \cup V(Q').
\]

This choice of \( S_j \) ensures that the rays \( R, R' \) constructed form the \( R_i \) and \( R'_i \) in the limit will not separate each other from \( C_j \), because they will satisfy \( R \cap S_j = R_j \cap S_j \) and \( R' \cap S_j = R'_j \cap S_j \).

If the ray \( R_{j-1} \) separates \( C_j \) from \( R'_j-1 \), let \( \mathcal{P}_j \) be a set of three disjoint \( C_j-R'_j-1 \) paths avoiding \( S_{j-1} \). All these paths meet \( R_{j-1} \). Let \( P_1 \in \mathcal{P}_j \) be the path which \( R_{j-1} \) meets first, and \( P_3 \in \mathcal{P}_j \) the one it meets last. Then \( R_{j-1} \cup C_j \cup P_1 \cup P_3 \) contains a ray \( R_j \) with initial segment \( R_{j-1} \cap S_{j-1} \) that meets \( C_j \) but is disjoint from the remaining path \( P_2 \in \mathcal{P} \) and from \( R'_j-1 \). Let \( R'_j = R'_j-1 \), and let \( S_j \) contain \( S_{j-1} \) and all vertices of \( \bigcup \mathcal{P}_j \), and the initial segments of \( R_{j-1} \) and \( R'_j-1 \) up to their last vertex in \( \bigcup \mathcal{P} \). Note that \( R_j \) meets \( C_j \), and that \( P_2 \) is a \( C_j-R'_j \) path avoiding \( R_j \).

If the ray \( R'_j-1 \) separates \( C_j \) from \( R_{j-1} \), reverse their roles in the previous part of the construction.

The edges that lie eventually in \( R_i \) or \( R'_i \) as \( i \to \infty \) form two rays \( R \) and \( R' \) that clearly hug \( D \).

Let us show that there are two disjoint combs, with spines \( R \) and \( R' \) respectively, and infinitely many disjoint finite circuits in \( D \) such that each of the combs has a tooth in each of these circuits. We build these combs inductively, starting with the rays \( R \) and \( R' \) and adding teeth one by one.

Let \( T_0 = R \) and \( T'_0 = R' \) and \( S_0 = \emptyset \). Given \( j \geq 1 \), assume that \( T_i, T'_i \) and \( S_i \) have been defined for all \( i < j \). By Lemma 6 there is a finite set \( S \supseteq S_{j-1} \) such that every vertex of \( C(S, \omega) \) sends a 3-fan to \( R \cup R' \) in \( C(S_{j-1}, \omega) \). As \( R \) and \( R' \) hug \( D \) there is a finite cycle \( C \) in \( C(S, \omega) \) with edges in \( D \), and which neither of the rays \( R \) or \( R' \) separates from the other. By the choice of \( S \), no one vertex of \( C(S_{j-1}, \omega) \) separates \( C \) from \( R \cup R' \) in \( C(S_{j-1}, \omega) \). Hence by Menger’s theorem there are disjoint \( (R \cup R')-C \) paths \( P \) and \( Q \) in \( C(S_{j-1}, \omega) \). If \( P \) starts on \( R \) and \( Q \) starts on \( R' \) (say), let \( P' := Q \). Assume now that \( P \) and \( Q \) start on the same ray \( R \) or \( R' \), say on \( R \). Let \( Q' \) be a path from \( R' \) to \( C \cup P \cup Q \) in \( C(S_{j-1}, \omega) \) that avoids \( R \). As \( Q' \) meets at most one of the paths \( P \) and \( Q \), we
may assume it does not meet \( P \). Then \( Q' \cup (Q \setminus R) \) contains an \( R' \cap C \) path \( P' \) disjoint from \( P \) and \( R \). In either case, let \( T_j = T_{j-1} \cup P \), let \( T'_j = T'_{j-1} \cup P' \), and let \( S_j \) consist of \( S_{j-1} \), the vertices in \( C \cup P \cup P' \), and the vertices on \( R \) and \( R' \) up to their last vertex in \( C \cup P \cup P' \).

The unions \( T = \bigcup_{i \in \mathbb{N}} T_i \) and \( T' = \bigcup_{i \in \mathbb{N}} T'_i \) are disjoint combs that have teeth in infinitely many common disjoint finite cycles whose edges lie in \( D \). Let \( A \) be the vertex set of the component of \( G - T \) containing \( T' \), and let \( B := V \setminus A \). Since \( T \) is connected, \( E(A, B) \) is a bond, and its intersection with \( D \) is infinite as every finite cycle that contains a tooth from both these combs meets \( E(A, B) \) at least twice. This contradiction implies that \( D \) is finite, as desired.

References


