Orthogonality and minimality in the homology of locally finite graphs

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Abstract

Given a finite set E, a subset $D \subseteq E$ (viewed as a function $E \to \mathbb{F}_2$) is orthogonal to a given subspace $\mathcal{F}$ of the $\mathbb{F}_2$-vector space of functions $E \to \mathbb{F}_2$ as soon as D is orthogonal to every $\subseteq$-minimal element of $\mathcal{F}$. This fails in general when E is infinite.

However, we prove the above statement for the six subspaces $\mathcal{F}$ of the edge space of any 3-connected locally finite graph that are relevant to its homology: the topological, algebraic, and finite cycle and cut spaces. This solves a problem of [5].

1 Introduction

Let G be a 2-connected locally finite graph, and let $\mathcal{E} = \mathcal{E}(G)$ be its edge space over $\mathbb{F}_2$. We think of the elements of $\mathcal{E}$ as sets of edges, possibly infinite. Two sets of edges are *orthogonal* if their intersection has (finite and) even cardinality. A set $D \in \mathcal{E}$ is *orthogonal* to a subspace $\mathcal{F} \subseteq \mathcal{E}$ if it is orthogonal to every $F \in \mathcal{F}$. See [4, 5] for any definitions not given below.

The topological *cycle space* $\mathcal{C}_{\text{top}}(G)$ of G is the subspace of $\mathcal{E}(G)$ generated (via thin sums, possibly infinite) by the *circuits* of G, the edge sets of the topological circles in the Freudenthal compactification $|G|$ of G. This space $\mathcal{C}_{\text{top}}(G)$ contains precisely the elements of $\mathcal{E}$ that are orthogonal to $\mathcal{B}_{\text{fin}}(G)$, the finite-cut space of G [4]. The *algebraic cycle space* $\mathcal{C}_{\text{alg}}(G)$ of G is the subspace of $\mathcal{E}$ consisting of the edge sets inducing even degrees at all the vertices. It contains precisely the elements of $\mathcal{E}$ that are orthogonal to the *skew cut space* $\mathcal{B}_{\text{skew}}(G)$ [3], the subspace of $\mathcal{E}$ consisting of all the cuts of G with one side finite. The *finite-cycle space* $\mathcal{C}_{\text{fin}}(G)$ is the subspace of $\mathcal{E}$ generated (via finite sums) by the finite circuits of G. This space $\mathcal{C}_{\text{fin}}(G)$ contains precisely the elements of $\mathcal{E}$ that are orthogonal to $\mathcal{B}(G)$, the cut space of G [4, 5]. Thus,

$$\mathcal{C}_{\text{top}} = \mathcal{B}_{\text{fin}}^{\perp}, \quad \mathcal{C}_{\text{alg}} = \mathcal{B}_{\text{skew}}^{\perp}, \quad \mathcal{C}_{\text{fin}} = \mathcal{B}^{\perp}.$$

Conversely,

$$\mathcal{C}_{\text{top}}^{\perp} = \mathcal{B}_{\text{fin}}, \quad \mathcal{C}_{\text{alg}}^{\perp} = \mathcal{B}_{\text{skew}}, \quad \mathcal{C}_{\text{fin}}^{\perp} = \mathcal{B}.$$

Thus, for any of the six spaces $\mathcal{F}$ just mentioned, we have $\mathcal{F}^{\perp\perp} = \mathcal{F}$.

Proofs of most of the above six identities were first published by Casteels and Richter [3], in a more general setting. Any remaining proofs can be found in [5], except for the inclusion $\mathcal{C}_{\rm alg}^{\perp} \supseteq \mathcal{B}_{\rm skew}$, which is easy.

The six subspaces of $\mathcal{E}$ mentioned above are the the ones most relevant to the homology of locally finite infinite graphs. See [5], Diestel and Sprüssel [6], and Georgakopoulos [7, 8]. Our aim in this note is to facilitate orthogonality proofs for these spaces by showing that, whenever $\mathcal{F}$ is one of them, a set D of edges is orthogonal to $\mathcal{F}$ as soon as it is orthogonal to the minimal nonzero elements of $\mathcal{F}$.

This is easy when $\mathcal{F}$ is $\mathcal{C}_{\rm fin}$ or $\mathcal{B}_{\rm fin}$ or $\mathcal{B}_{\rm skew}$:

Proposition 1. *Let $\mathcal{F}$ be a subspace of $\mathcal{E}$ all whose elements are finite sets of edges. Then $\mathcal{F}$ is generated (via finite sums) by its $\subseteq$-minimal nonzero elements.*

Proof. For a contradiction suppose that some $F \in \mathcal{F}$ is not a finite sum of finitely many minimal nonzero elements of $\mathcal{F}$. Choose F with $|F|$ minimal. As F is not minimal itself, by assumption, it properly contains a minimal nonzero element F' of $\mathcal{F}$. As F is finite, $F + F' = F \setminus F' \in \mathcal{F}$ has fewer elements than F, so there is a finite family $(M_i)_{i \leq n}$ of minimal nonzero elements of $\mathcal{F}$ with $\sum_{i \leq n} M_i = F + F'$. This contradicts our assumption, as $F' + \sum_{i \leq n} M_i = F$. □

Corollary 2. *If $\mathcal{F} \in \{\mathcal{C}_{\rm fin}, \mathcal{B}_{\rm fin}, \mathcal{B}_{\rm skew}\}$, a set D of edges is orthogonal to $\mathcal{F}$ as soon as D is orthogonal to all the minimal nonzero elements of $\mathcal{F}$.* □

When $\mathcal{F} \in \{\mathcal{C}_{\rm top}, \mathcal{C}_{\rm alg}, \mathcal{B}\}$, the statement of Corollary 2 is generally false for graphs that are not 3-connected. Here are some examples.

For $\mathcal{F} = \mathcal{B}$, let G be the graph obtained from the $\mathbb{N} \times \mathbb{Z}$ grid by doubling every edge between two vertices of degree 3 and subdividing all the new edges. The set D of the edges that lie in a K^3 of G is orthogonal to every bond F of G: their intersection $D \cap F$ is finite and even. But D is not orthogonal to every element of $\mathcal{F} = \mathcal{B}$, since it meets some cuts that are not bonds infinitely.

For $\mathcal{F} = \mathcal{C}_{\rm top}$, let B be an infinite bond of the infinite ladder H, and let G be the graph obtained from H by subdividing every edge in B. Then the set D of edges that are incident with subdivision vertices has a finite and even intersection with every topological circuit C, finite or infinite, but it is not orthogonal to every element of $\mathcal{C}_{\rm top}$, since it meets some of them infinitely.

For $\mathcal{F} = \mathcal{C}_{\rm alg}$ we can re-use the example just given for $\mathcal{C}_{\rm top}$, since for 1-ended graphs like the ladder the two spaces coincide.

However, if G is 3-connected, an edge set is orthogonal to every element of $\mathcal{C}_{\rm top}, \mathcal{C}_{\rm alg}$ or $\mathcal{B}$ as soon as it is orthogonal to every minimal nonzero element:

Theorem 3. *Let $G = (V, E)$ be a locally finite 3-connected graph, and $F, D \subseteq E$.*

(i) *$F \in \mathcal{C}_{\rm top}^{\perp}$ as soon as F is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\rm top}$, the topological circuits of G.*

(ii) *$F \in \mathcal{C}_{\rm alg}^{\perp}$ as soon as F is orthogonal to all the minimal nonzero elements of $\mathcal{C}_{\rm alg}$, the finite circuits and the edge sets of double rays in G.*

(iii) $D \in \mathcal{B}^{\perp}$ *as soon as* D *is orthogonal to all the minimal nonzero elements of* $\mathcal{B}$*, the bonds of* G*.*

Although Theorem 3 fails if we replace the assumption of 3-connectedness with 2-connectedness, it turns out that we need a little less than 3-connectedness. Recall that an end ω of G has (combinatorial) *vertex-degree* k if k is the maximum number of vertex-disjoint rays in ω. Halin [9] showed that every end in a k-connected locally finite graph has vertex-degree at least k. Let us call an end ω of G *k-padded* if for every ray $R \in \omega$ there is a neighbourhood U of ω such that for every vertex $u \in U$ there is a *k-fan* from u to R in G, a subdivided k-star with centre u and leaves on R.[1] If every end of G is k-padded, we say that G is *k-padded at infinity.* Note that k-connected graphs are k-padded at infinity. Our proof of Theorem 3(i) and (ii) will use only that every end has vertex-degree at least 3 and that G is 2-connected. Similarly, and in a sense dually, our proof of Theorem 3(iii) uses only that every end has vertex-degree at least 2 and G is 3-connected at infinity.

Theorem 4. *Let* $G = (V, E)$ *be a locally finite* 2*-connected graph.*

(i) *If every end of* G *has vertex-degree at least* 3*, then* $F \in \mathcal{C}_{\mathrm{top}}^{\perp}$ *as soon as* F *is orthogonal to all the minimal nonzero elements of* $\mathcal{C}_{\mathrm{top}}$*, the topological circuits of* G*.*

(ii) *If every end of* G *has vertex-degree at least* 3*, then* $F \in \mathcal{C}_{\mathrm{alg}}^{\perp}$ *as soon as* F *is orthogonal to all the minimal nonzero elements of* $\mathcal{C}_{\mathrm{alg}}$*, the finite circuits and the edge sets of double rays in* G*.*

(iii) *If* G *is* 3*-padded at infinity, then* $D \in \mathcal{B}^{\perp}$ *as soon as* D *is orthogonal to all the minimal nonzero elements of* $\mathcal{B}$*, the bonds of* G*.*

In general, our notation follows [4]. In particular, given an end ω in a graph G and a finite set $S \subseteq V(G)$ of vertices, we write $C(S, \omega)$ for the unique component of $G - S$ that contains a ray $R \in \omega$. The *vertex-degree* of ω is the maximum number of vertex-disjoint rays in ω. The mathematical background required for this paper is covered in [5, 6]. For earlier results on the cycle and cut space see Bruhn and Stein [1, 2].

2 Finding disjoint paths and fans

Menger's theorem that the smallest cardinality of an A–B separator in a finite graph is equal to the largest cardinality of a set of disjoint A–B paths trivially extends to infinite graphs. Thus in a locally finite k-connected graph, there are k internally disjoint paths between any two vertices. In Lemmas 5 and 6 we

[1] For example, if G is the union of complete graphs $K_1, K_2, \ldots$ with $|K_i| = i$, each meeting the next in exactly one vertex (and these are all distinct), then the unique end of G is k-padded for every $k \in \mathbb{N}$.

show that, for two such vertices that are close to an end ω, these connecting paths need not use vertices too far away from ω.

In a graph G with vertex sets $X, Y \subseteq V(G)$ and vertices $x, y \in V(G)$, a *k-fan* from X (or x) to Y is a subdivided k-star whose center lies in X (or is x) and whose leaves lie in Y. A *k-linkage* from x to y is a union of k internally disjoint x–y paths. We may refer to a sequence $(v_i)_{i\in\mathbb{N}}$ simply by (v_i), and use $\bigcup(v_i) := \bigcup_{i\in\mathbb{N}}\{v_i\}$ for brevity.

Lemma 5. *Let G be a locally finite graph with an end ω, and let $(v_i)_{i\in\mathbb{N}}$ and $(w_i)_{i\in\mathbb{N}}$ be two sequences of vertices converging to ω. Let k be a positive integer.*

(i) *If for infinitely many $n \in \mathbb{N}$ there is a k-fan from v_n to $\bigcup(w_i)$, then there are infinitely many disjoint such k-fans.*

(ii) *If for infinitely many $n \in \mathbb{N}$ there is a k-linkage from v_n to w_n, then there are infinitely many disjoint such k-linkages.*

Proof. For a contradiction, suppose $k \in \mathbb{N}$ is minimal such that there is a locally finite graph $G = (V, E)$ with sequences $(v_i)_{i\in\mathbb{N}}$ and $(w_i)_{i\in\mathbb{N}}$ in which either (i) or (ii) fails. Then $k > 1$, since for every finite set $S \subseteq V(G)$ the unique component $C(S, \omega)$ of $G - S$ that contains rays from ω is connected and contains all but finitely many vertices from $\bigcup(v_i)$ and $\bigcup(w_i)$.

For a proof of (i) it suffices to show that for every finite set $S \subseteq V(G)$ there is an integer $n \in \mathbb{N}$ and a k-fan from v_n to $\bigcup(w_i)$ avoiding S. Suppose there is a finite set $S \subseteq V(G)$ that meets all k-fans from $\bigcup(v_i)$ to $\bigcup(w_i)$. By the minimality of k, there are infinitely many disjoint $(k-1)$-fans from $\bigcup(v_i)$ to $\bigcup(w_i)$ in $C := C(S, \omega)$. Thus, there is a subsequence $(v_i')_{i\in\mathbb{N}}$ of $(v_i)_{i\in\mathbb{N}}$ in C and pairwise disjoint $(k-1)$-fans $F_i \subseteq C$ from v_i' to $\bigcup(w_i)$ for all $i \in \mathbb{N}$. For every $i \in \mathbb{N}$ there is by Menger's theorem a $(k-1)$-separator S_i separating v_i' from $\bigcup(w_i)$ in C, as by assumption there is no k-fan from v_i' to $\bigcup(w_i)$ in C. Let C_i be the component of $G - (S \cup S_i)$ containing v_i'.

Since F_i is a subdivided $|S_i|$-star, $S_i \subseteq V(F_i)$. Hence for all $i \neq j$, our assumption of $F_i \cap F_j = \emptyset$ implies that $F_i \cap S_j = \emptyset$, and hence that $F_i \cap C_j = \emptyset$. But then also $C_i \cap C_j = \emptyset$, since any vertex in $C_i \cap C_j$ coud be joined to v_j' by a path P in C_j and to v_i' by a path Q in C_i, giving rise to a v_j'–$\bigcup(w_i)$ path in $P \cup Q \cup F_i$ avoiding S_j, a contradiction.

As $S \cup S_i$ separates v_i' from $\bigcup(w_i)$ in G and there is, by assumption, a k-fan from v_i' to $\bigcup(w_i)$ in G, there are at least k distinct neighbours of C_i in $S \cup S_i$. Since $|S_i| = k-1$, one of these lies in S. This holds for all $i \in \mathbb{N}$. As $C_i \cap C_j = \emptyset$ for distinct i and j, this contradicts our assumption that G is locally finite and S is finite. This completes the proof of (i).

For (ii) it suffices to show that for every finite set $S \subseteq V(G)$ there is an integer $n \in \mathbb{N}$ such that there is a k-linkage form v_n to w_n avoiding S. Suppose there is a finite set $S \subseteq V(G)$ that meets all k-linkages from v_i to w_i for all $i \in \mathbb{N}$. By the minimality of k there is an infinite family $(L_i)_{i\in I}$ of disjoint $(k-1)$-linkages L_i in $C := C(S, \omega)$ from v_i to w_i. As earlier, there are pairwise disjoint $(k-1)$-sets $S_i \subseteq V(L_i)$ separating v_i from w_i in C, for all $i \in I$. Let

C_i, D_i be the components of $C - S_i$ containing v_i and w_i, respectively. For no $i \in I$ can both C_i and D_i have ω in their closure, as they are separated by the finite set $S \cup S_i$. Thus for every $i \in I$ one of C_i or D_i contains at most finitely many vertices from $\bigcup_{i \in I} L_i$. By symmetry, and replacing I with an infinite subset of itself if necessary, we may assume the following:

$$\text{The components } C_i \text{ with } i \in I \text{ each contain only finitely many vertices from } \bigcup_{i\in I} L_i. \tag{1}$$

If infinitely many of the components C_i are pairwise disjoint, then S has infinitely many neighbours as earlier, a contradiction. By Ramsey's theorem, we may thus assume that

$$C_i \cap C_j \neq \emptyset \text{ for all } i, j \in I. \tag{2}$$

Note that if C_i meets L_j for some $j \neq i$, then $C_i \supseteq L_j$, since L_j is disjoint from $L_i \supseteq S_i$. By (1), this happens for only finitely many $j > i$. We can therefore choose an infinite subset of I such that $C_i \cap L_j = \emptyset$ for all $i < j$ in I. In particular, $(C_i \cup S_i) \cap S_j = \emptyset$ for $i < j$. By (2), this implies that

$$C_i \cup S_i \subseteq C_j \text{ for all } i < j. \tag{3}$$

By assumption, there exists for each $i \in I$ some v_i–w_i linkage of k independent paths in G, one of which avoids S_i and therefore meets S. Let P_i denote its final segment from its last vertex in S to w_i. As $w_i \in C \setminus (C_i \cup S_i)$ and P_i avoids both S_i and S (after its starting vertex in S), we also have

$$P_i \cap C_i = \emptyset. \tag{4}$$

On the other hand, L_i contains $v_i \in C_i \subseteq C_{i+1}$ and avoids S_{i+1}, so $w_i \in L_i \subseteq C_{i+1}$. Hence P_i meets S_j for every $j \geq i+1$ such that $P_i \not\subseteq S \cup C_j$. Since the $L_j \supseteq S_j$ are disjoint for different j, this happens for only finitely many $j > i$. Deleting those j from I, and repeating that argument for increasing i in turn, we may thus assume that $P_i \subseteq S \cup C_{i+1}$ for all $i \in I$. By (3) and (4) we deduce that $P_i \setminus S$ are now disjoint for different values of $i \in I$. Hence S contains a vertex of infinite degree, a contradiction. □

Recall that G is *k-padded* at an end ω if for every ray $R \in \omega$ there is a neighbourhood U such that for all vertices $u \in U$ there is a k-fan from u to R in G. Our next lemma shows that, if we are willing to make U smaller, we can find the fans locally around ω:

Lemma 6. *Let G be a locally finite graph with a k-padded end ω. For every ray $R \in \omega$ and every finite set $S \subseteq V(G)$ there is a neighbourhood $U \subseteq C(S, \omega)$ of ω such that from every vertex $u \in U$ there is a k-fan in $C(S, \omega)$ to R.*

Proof. Suppose that, for some $R \in \omega$ and finite $S \subseteq V(G)$, every neighbourhood $U \subseteq C(S, \omega)$ of ω contains a vertex u such that $C(S, \omega)$ contains no k-fan from u to R. Then there is a sequence $u_1, u_2, \ldots$ of such vertices converging to ω. As ω is k-padded there are k-fans from infinitely many u_i to R in G. By Lemma 5(i) we may assume that these fans are disjoint. By the choice of $u_1, u_2, \ldots$, all these disjoint fans meet the finite set S, a contradiction. □

3 The proof of Theorems 3 and 4

As pointed out in the introduction, Theorem 4 implies Theorem 3. It thus suffices to prove Theorem 4, of which we prove (i) first. Consider a set $F \neq \emptyset$ of edges that meets every circuit of G evenly. We have to show that $F \in \mathcal{C}_{\text{top}}^{\perp}$, i.e., that F is a finite cut. (Recall that $\mathcal{C}_{\text{top}}^{\perp}$ is known to equal $\mathcal{B}_{\text{fin}}$, the finite-cut space [5].) As F meets every finite cycle evenly it is a cut, with bipartition (A, B) say. Suppose F is infinite. Let $\mathcal{R}$ be a set of three disjoint rays that belong to an end ω in the closure of F. Every R–R' path P for two distinct $R, R' \in \mathcal{R}$ lies on the unique topological circle $C(R, R', P)$ that is contained in $R \cup R' \cup P \cup \{\omega\}$. As every circuit meets F finitely, we deduce that no ray in $\mathcal{R}$ meets F again and again. Replacing the rays in $\mathcal{R}$ with tails of themselves as necessary, we may thus assume that F contains no edge from any of the rays in $\mathcal{R}$. Suppose F separates $\mathcal{R}$, with the vertices of $R \in \mathcal{R}$ in A and the vertices of $R', R'' \in \mathcal{R}$ in B say. Then there are infinitely many disjoint R–$(R' \cup R'')$ paths each meeting F at least once. Infinitely many of these disjoint paths avoid one of the rays in B, say R''. The union of these paths together with R and R' contains a ray $W \in \omega$ that meets F infinitely often. For every R''–W path P, the circle $C(W, R'', P)$ meets F in infinitely many edges, a contradiction. Thus we may assume that F does not separate $\mathcal{R}$, and that $G[A]$ contains $\bigcup \mathcal{R}$.

As ω lies in the closure of F, there is a sequence $(v_i)_{i \in \mathbb{N}}$ of vertices in B converging to ω. As G is 2-connected there is a 2-fan from each v_i to $\bigcup \mathcal{R}$ in G. By Lemma 5 there are infinitely many disjoint 2-fans from $\bigcup(v_i)$ to $\bigcup \mathcal{R}$. We may assume that every such fan has at most two vertices in $\bigcup \mathcal{R}$. Then infinitely many of these fans avoid some fixed ray in $\mathcal{R}$, say R. The two other rays plus the infinitely many 2-fans meeting only these together contain a ray $W \in \omega$ that meets F infinitely often and is disjoint from R. Then for every R–W path P we get a contradiction, as $C(R, W, P)$ is a circle meeting F in infinitely many edges.

For a proof of (ii), note first that the minimal elements of $\mathcal{C}_{\text{alg}}$ are indeed the finite circuits and the edge sets of double rays in G. Indeed, these are clearly in $\mathcal{C}_{\text{alg}}$ and minimal. Conversely, given any element of $\mathcal{C}_{\text{alg}}$, a set D of edges inducing even degrees at all the vertices, we can greedily find for any given edge $e \in D$ a finite circuit or double ray with all its edges in D that contains e. We may thus decompose D inductively into disjoint finite circuits and edge sets of double rays, since deleting finitely many such sets from D clearly produces another element of $\mathcal{C}_{\text{alg}}$, and including in each circuit or double ray chosen the smallest undeleted edge in some fixed enumeration of D ensures that the entire set D is decomposed. If D is minimal in $\mathcal{C}_{\text{alg}}$, it must therefore itself be a finite circuit or the edge set of a double ray.

Consider a set F of edges that fails to meet some set $D \in \mathcal{C}_{\text{alg}}$ evenly; we have to show that F also fails to meet some finite circuit or double ray evenly. If $|F \cap D|$ is odd, then this follows from our decomposition of D into disjoint finite circuits and edges sets of double rays. We thus assume that $F \cap D$ is infinite. Since $|G|$ is compact, we can find a sequence $e_1, e_2, \ldots$ of edges in $F \cap D$ that converges to some end ω. Let R_1, R_2, R_3 be disjoint rays in ω, which exist by our assumption that ω has vertex-degree at least 3. Subdividing each edge e_i by a new vertex v_i, and using that G is 2-connected, we can find for every i a 2-fan from v_i to $W = V(R_1 \cup R_2 \cup R_3)$ that has only its last vertices and possibly v_i in W. By Lemma 5, with $w_1, w_2, \dots$ an enumeration of W, some infinitely many of these fans are disjoint. Renaming the rays R_i and replacing $e_1, e_2, \dots$ with a subsequence as necessary, we may assume that either all these fans have both endvertices on R_1, or that they all have one endvertex on R_1 and the other on R_2. In both cases all these fans avoid R_3, so we can find a ray R in the union of R_1, R_2 and these fans (suppressing the subdividing vertices v_i again) that contains infinitely many e_i and avoids R_3. Linking R to a tail of R_3 we thus obtain a double ray in G that contains infinitely many e_i, as desired.

To prove (iii), let $D \subseteq E$ be a set of edges that meets every bond evenly. We have to show that $D \in \mathcal{B}^\perp$, i.e., that D has an (only finite and) even number of edges also in every cut that is not a bond.

As D meets every finite bond evenly, and hence every finite cut, it lies in $\mathcal{B}_{\text{fin}}^\perp = \mathcal{C}_{\text{top}}$. We claim that

$$D \textit{ is a disjoint union of finite circuits.} \tag{$\star$}$$

To prove ($\star$), let us show first that every edge $e \in D$ lies in some finite circuit $C \subseteq D$. If not, the endvertices u, v of e lie in different components of $(V, D \smallsetminus \{e\})$, and we can partition V into two sets A, B so that e is the only A–B edge in D. The cut of G of all its A–B edges is a disjoint union of bonds [4], one of which meets D in precisely e. This contradicts our assumption that D meets every bond of G evenly.

For our proof of ($\star$), we start by enumerating D, say as $D =: \{e_1, e_2, \dots\} =: D_0$. Let $C_0 \subseteq D_0$ be a finite circuit containing e_0, let $D_1 := D_0 \smallsetminus C_0$, and notice that D_1, like D_0, meets every bond of G evenly (because C_0 does). As before, D_1 contains a finite circuit C_1 containing the edge e_i with $i = \min\{j \mid e_j \in D_1\}$. Continuing in this way we find the desired decomposition $D = C_1 \cup C_2 \cup \dots$ of D into finite circuits. This completes the proof of ($\star$).

As every finite circuit lies in $\mathcal{B}^\perp$, it suffices by ($\star$) to show that D is finite. Suppose D is infinite, and let ω be an end of G in its closure. Let us say that two rays R and R' *hug* D if every neighbourhood U of ω contains a finite circuit $C \subseteq D$ that is neither separated from R by R' nor from R' by R in U. We shall construct two rays R and R' that hug D, inductively as follows.

Let $S_0 = \emptyset$, and let R_0, R_0' be disjoint rays in ω. (These exist as G is 2-connected [9].) For step $j \geq 1$, assume that let S_i, R_i, and R_i' have been defined for all $i < j$ so that R_i and R_i' each meet S_i in precisely some initial segement (and otherwise lie in $C(S_i, \omega)$) and S_i contains the ith vertex in some fixed enumeration of V. If the jth vertex in this enumeration lies in $C(S_{j-1}, \omega)$, add to S_{j-1} this vertex and, if it lies on R_{j-1} or R_{j-1}', the initial segement of that ray up to it. Keep calling the enlarged set S_{j-1}. For the following choice of S we apply Lemma 6 to S_{j-1} and each of R_{j-1} and R_{j-1}'. Let $S \supseteq S_{j-1}$ be a finite set such that from every vertex v in $C(S, \omega)$ there are 3-fans in $C(S_{j-1}, \omega)$ both to R_{j-1} and to R'_{j-1}. By $(\star)$ and the choice of ω, there is a finite circuit $C_j \subseteq D$ in $C(S, \omega)$. Then C_j can not be separated from R_{j-1} or R'_{j-1} in $C(S_{j-1}, \omega)$ by fewer than three vertices, and thus there are three disjoint paths from C_j to $R_{j-1} \cup R'_{j-1}$ in $C(S_{j-1}, \omega)$.

There are now two possible cases. The first is that in $C(S_{j-1}, \omega)$ the circuit C_j is neither separated from R_{j-1} by R'_{j-1} nor from R'_{j-1} by R_{j-1}. This case is the preferable case. In the second case one ray separates C_j from the other. In this case we will reroute the two rays to obtain new rays as in the first case. We shall then 'freeze' a finite set containing initial parts of these rays, as well as paths from each ray to C_j. This finite fixed set will not be changed in any later step of the construction of R and R'. In detail, this process is as follows.

If $C(S_{j-1}, \omega)$ contains both a C_j–R_{j-1} path P avoiding R'_{j-1} and a C_j–R'_{j-1} path P' avoiding R_{j-1}, let Q and Q' be the initial segments of R_{j-1} and R'_{j-1} up to P and P', respectively. Then let $R_j = R_{j-1}$ and $R'_j = R'_{j-1}$ and

$$S_j = S_{j-1} \cup V(P) \cup V(P') \cup V(Q) \cup V(Q').$$

This choice of S_j ensures that the rays R, R' constructed form the R_i and R'_i in the limit will not separate each other from C_j, because they will satisfy $R \cap S_j = R_j \cap S_j$ and $R' \cap S_j = R'_j \cap S_j$.

If the ray R_{j-1} separates C_j from R'_{j-1}, let $\mathcal{P}_j$ be a set of three disjoint C_j–R'_{j-1} paths avoiding S_{j-1}. All these paths meet R_{j-1}. Let $P_1 \in \mathcal{P}_j$ be the path which R_{j-1} meets first, and $P_3 \in \mathcal{P}_j$ the one it meets last. Then $R_{j-1} \cup C_j \cup P_1 \cup P_3$ contains a ray R_j with initial segment $R_{j-1} \cap S_{j-1}$ that meets C_j but is disjoint from the remaining path $P_2 \in \mathcal{P}$ and from R'_{j-1}. Let $R'_j = R'_{j-1}$, and let S_j contain S_{j-1} and all vertices of $\bigcup \mathcal{P}_j$, and the initial segments of R_{j-1} and R'_{j-1} up to their last vertex in $\bigcup \mathcal{P}$. Note that R_j meets C_j, and that P_2 is a C_j–R'_j path avoiding R_j.

If the ray R'_{j-1} separates C_j from R_{j-1}, reverse their roles in the previous part of the construction.

The edges that lie eventually in R_i or R'_i as $i \to \infty$ form two rays R and R' that clearly hug D.

Let us show that there are two disjoint combs, with spines R and R' respectively, and infinitely many disjoint finite circuits in D such that each of the combs has a tooth in each of these circuits. We build these combs inductively, starting with the rays R and R' and adding teeth one by one.

Let $T_0 = R$ and $T'_0 = R'$ and $S_0 = \emptyset$. Given $j \geq 1$, assume that T_i, T'_i and S_i have been defined for all $i < j$. By Lemma 6 there is a finite set $S \supseteq S_{j-1}$ such that every vertex of $C(S, \omega)$ sends a 3-fan to $R \cup R'$ in $C(S_{j-1}, \omega)$. As R and R' hug D there is a finite cycle C in $C(S, \omega)$ with edges in D, and which neither of the rays R or R' separates from the other. By the choice of S, no one vertex of $C(S_{j-1}, \omega)$ separates C from $R \cup R'$ in $C(S_{j-1}, \omega)$. Hence by Menger's theorem there are disjoint $(R \cup R')$–C paths P and Q in $C(S_{j-1}, \omega)$. If P starts on R and Q starts on R' (say), let $P' := Q$. Assume now that P and Q start on the same ray R or R', say on R. Let Q' be a path from R' to $C \cup P \cup Q$ in $C(S_{j-1}, \omega)$ that avoids R. As Q' meets at most one of the paths P and Q, we may assume it does not meet P. Then $Q' \cup (Q \setminus R)$ contains an R'–C path P' disjoint from P and R. In either case, let $T_j = T_{j-1} \cup P$, let $T'_j = T'_{j-1} \cup P'$, and let S_j consist of S_{j-1}, the vertices in $C \cup P \cup P'$, and the vertices on R and R' up to their last vertex in $C \cup P \cup P'$.

The unions $T = \bigcup_{i \in \mathbb{N}} T_i$ and $T' = \bigcup_{i \in \mathbb{N}} T'_i$ are disjoint combs that have teeth in infinitely many common disjoint finite cycles whose edges lie in D. Let A be the vertex set of the component of $G - T$ containing T', and let $B := V \setminus A$. Since T is connected, $E(A, B)$ is a bond, and its intersection with D is infinite as every finite cycle that contains a tooth from both these combs meets $E(A, B)$ at least twice. This contradiction implies that D is finite, as desired. □

References

[1] H. Bruhn and M. Stein. On end degrees and infinite circuits in locally finite graphs. *Combinatorica*, 27:269–291, 2007.

[2] Henning Bruhn and Maya Stein. Duality of ends. *Comb., Probab. Comput.*, 19:47–60, 2010.

[3] K. Casteels and B. Richter. The Bond and Cycle Spaces of an Infinite Graph. *J. Graph Theory*, 59(2):162–176, 2008.

[4] R. Diestel. *Graph Theory.* Springer, 4th edition, 2010.

[5] R. Diestel. Locally finite graphs with ends: a topological approach. *Discrete Math.*, 310–312: 2750–2765 (310); 1423–1447 (311); 21–29 (312), 2010–11. arXiv:0912.4213.

[6] R. Diestel and P. Sprüssel. The homology of locally finite graphs with ends. *Combinatorica*, 30:681–714, 2010.

[7] A. Georgakopoulos. Graph topologies induced by edge lengths. In Diestel, Hahn, and Mohar, editors, *Infinite graphs: introductions, connections, surveys*, volume Discrete Mathematics 311, pages 1523–1542, 2011.

[8] A. Georgakopoulos. Cycle decompositions: from graphs to continua. *Advances in Mathematics*, 229:935–967, 2012.

[9] R. Halin. A note on Menger's theorem for infinite locally finite graphs. *Abh. Math. Sem. Univ. Hamburg*, 40:111–114, 1974.