

# A simple existence criterion for normal spanning trees in infinite graphs

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Halin proved in 1978 that there exists a normal spanning tree in every connected graph  $G$  that satisfies the following two conditions: (i)  $G$  contains no subdivision of a ‘fat’  $K_{\aleph_0}$ , one in which every edge has been replaced by uncountably many parallel edges; and (ii)  $G$  has no  $K_{\aleph_0}$  subgraph. We show that the second condition is unnecessary.

## Introduction

A spanning tree of an infinite graph is *normal* if the endvertices of any chord are comparable in the tree order defined by some arbitrarily chosen root. (In finite graphs, these are their ‘depth-first search’ trees; see [3] for precise definitions.) Normal spanning trees are perhaps the most important single structural tool for analysing an infinite graph – see [4] for a typical example, and the exercises in [3, Chapter 8] for many more – but they do not always exist. The question of which graphs have normal spanning trees thus is an important question.

All countable connected graphs have normal spanning trees [3]. But not all connected graphs do. For example, if  $T$  is a normal spanning tree of  $G$  and  $G$  is complete, then  $T$  defines a chain on its vertex set. Hence  $T$  must be a single path or ray, and  $G$  is countable.

For connected graphs of arbitrary order, there are three characterizations of the graphs that admit a normal spanning tree:

**Theorem 1.** *The following statements are equivalent for connected graphs  $G$ .*

- (i)  $G$  has a normal spanning tree;
- (ii)  $V(G)$  is a countable union of dispersed sets (Jung [7, 2]);
- (iii)  $|G|$  is metrizable [1];
- (iv)  $G$  contains neither an  $(\aleph_0, \aleph_1)$ -graph nor an Aronszajn-tree graph as a minor [5].

Here, a set of vertices in  $G$  is *dispersed* if every ray can be separated from it by some finite set of vertices. (The levels of a normal spanning tree are dispersed; see [3].) The dispersed vertex sets in a graph  $G$  are precisely those that are closed in the topological space  $|G|$  of (iii), which consists of  $G$  and its ends [1]. The space  $|G|$  will not concern us in this note, so we refer to [1] for the definition of the topology on  $|G|$ . But we shall use the equivalence of (i) and (iv) in our proof, and the forbidden minors mentioned in (iv) will be defined in Section 2.

Despite the variety in Theorem 1, it can still be hard in practice to decide whether a given graph has a normal spanning tree.<sup>1</sup> In most applications, none of these characterizations is used, but a simpler sufficient condition due to Halin. This condition, however, is much stronger, and hence does not always hold even if a normal spanning tree exists. It is the purpose of this note to show that this condition can be considerably weakened.

## 1. The result

Halin's [6] most-used sufficient condition for the existence of a normal spanning tree in a connected graph is that it does not contain a  $TK_{\aleph_0}$ . This is usually easier to check than the conditions in Theorem 1, but it is also quite a strong assumption. However, Halin [6] also proved that this assumption can be replaced by the conjunction of two independent much weaker assumptions:

- $G$  contains no *fat*  $TK_{\aleph_0}$ : a subdivision of the multigraph obtained from a  $K_{\aleph_0}$  by replacing every edge with  $\aleph_1$  parallel edges;
- $G$  contains no  $K_{\aleph_0}$  (as a subgraph).

We shall prove that the second condition is unnecessary:

**Theorem 2.** *Every connected graph not containing a fat  $TK_{\aleph_0}$  has a normal spanning tree.*

We remark that all the graphs we consider are simple, including our fat  $TK_{\aleph_0}$ s. When we say, without specifying any graph relation, that a graph  $G$  *contains* another graph  $H$ , we mean that  $H$  is isomorphic to a subgraph of  $G$ . Any other undefined terms can be found in [3].

## 2. The proof

Our proof of Theorem 2 will be based on the equivalence (i) $\leftrightarrow$ (iv) in Theorem 1, so let us recall from [5] the terms involved here.

An *Aronszajn tree* is a poset  $(T, \leq)$  with the following properties:<sup>2</sup>

- $T$  that has a least element, its *root*;
- the down-closure of every point in  $T$  is well-ordered;
- $T$  is uncountable, but all chains and all levels in  $T$  are countable.

Here, the *down-closure*  $[t]$  of a point  $t \in T$  is the set  $\{x \mid x \leq t\}$ ; its *up-closure* is the set  $\lceil t \rceil := \{y \mid t \leq y\}$ . More generally, if  $x < y$  we say that  $x$  lies *below*

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<sup>1</sup> In particular, the two types of graph mentioned in (iv) are not completely understood; see [5] for the – quite intriguing – problem of how to properly understand (or meaningfully classify) the  $(\aleph_0, \aleph_1)$ -graphs.

<sup>2</sup> Unlike the perhaps better known Suslin trees – Aronszajn trees in which even every anti-chain must be countable – Aronszajn trees can be shown to exist without any set-theoretic assumptions in addition to ZFC.

$y$  and  $y$  above  $x$ . The *height* of a point  $t \in T$  is the order type of the chain  $[t] \setminus \{t\}$ , and the *levels* of  $T$  are its maximal subsets of points of equal height.

An *Aronszajn-tree graph* or *AT-graph*, is a graph  $G$  on whose vertex set there exists an Aronszajn tree  $T$  such that

- the endvertices of every edge of  $G$  are comparable in  $T$ ;
- for all  $x < y$ , the vertex  $y$  has a neighbour  $x'$  such that  $x \leq x' < y$ .

The second condition says that each vertex is joined cofinally to the vertices below it. The idea behind this is that if we were to construct any order tree  $T$  on  $V(G)$  satisfying the first condition, a tree satisfying also the second condition would be one that minimizes the level of each vertex.

Note that *intervals* in  $T$ , sets of the form  $\{t \mid x \leq t < y\}$  for some given points  $x < y$ , span connected subgraphs in  $G$ . This is because every  $t > x$  has a neighbour  $t'$  with  $x \leq t' < t$ , by the second condition, and hence the interval contains for each of its elements  $t$  the vertices of a  $t$ - $x$  path in  $G$ . Similarly,  $G$  itself is connected, because every vertex can be linked to the unique root of  $T$ .

An  $(\aleph_0, \aleph_1)$ -*graph with bipartition*  $(A, B)$  is a bipartite graph with vertex classes  $A$  of size  $\aleph_0$  and  $B$  of size  $\aleph_1$  such that every vertex in  $B$  has infinite degree.

Replacing the vertices  $x$  of a graph  $X$  with disjoint connected graphs  $H_x$ , and the edges  $xy$  of  $X$  with non-empty sets of  $H_x$ - $H_y$  edges, yields a graph that we shall call an *IX* (for ‘inflated  $X$ ’). More formally, a graph  $H$  is an *IX* if its vertex set admits a partition  $\{V_x \mid x \in V(X)\}$  into connected subsets  $V_x$  such that distinct vertices  $x, y \in X$  are adjacent in  $X$  if and only if  $H$  contains a  $V_x$ - $V_y$  edge. The sets  $V_x$  are the *branch sets of the IX*. Thus,  $X$  arises from  $H$  by contracting the subgraphs  $H_x$ , without deleting any vertices or edges (other than loops or parallel edges arising in the contraction). A graph  $X$  is a *minor* of a graph  $G$  if  $G$  contains an *IX* as a subgraph. See [3] for more details.

For our proof of Theorem 2 from Theorem 1 (i) $\leftrightarrow$ (iv) it suffices to show the following:

*Every IX, where  $X$  is either an  $(\aleph_0, \aleph_1)$ -graph or an AT-graph, contains a fat  $TK_{\aleph_0}$  (as a subgraph).* (\*)

The rest of this section is devoted to the proof of (\*).

**Lemma 3.** *Let  $X$  be an  $(\aleph_0, \aleph_1)$ -graph, with bipartition  $(A, B)$  say.*

- (i)  *$X$  has an  $(\aleph_0, \aleph_1)$ -subgraph  $X'$  with bipartition into  $A' \subseteq A$  and  $B' \subseteq B$  such that every vertex in  $A'$  has uncountable degree in  $X'$ .*
- (ii) *For every finite set  $F \subseteq A$  and every uncountable set  $U \subseteq B$ , there exists a vertex  $a \in A \setminus F$  that has uncountably many neighbours in  $U$ .*

**Proof.** (i) Delete from  $X$  all the vertices in  $A$  that have only countable degree, together with their neighbours in  $B$ . Since this removes only countably many vertices from  $B$ , the remaining set  $B' \subseteq B$  is still uncountable. Every  $b' \in B'$

has all its  $X$ -neighbours in the set  $A'$  of the vertices in  $A$  that we did not delete, as otherwise  $b'$  would have been deleted too. Thus,  $b'$  still has infinite degree in the subgraph  $X'$  of  $X$  induced by  $A'$  and  $B'$ . In particular,  $A'$  is still infinite, and  $X'$  is the desired  $(\aleph_0, \aleph_1)$ -subgraph of  $X$ .

(ii) If there is no vertex  $a \in A \setminus F$  as claimed, then each vertex  $a \in A \setminus F$  has only countably many neighbours in  $U$ . As  $A \setminus F$  is countable, this means that  $U \setminus N(A \setminus F) \neq \emptyset$ . But every vertex in this set has all its neighbours in  $F$ , and thus has finite degree. This contradicts our assumption that  $X$  is an  $(\aleph_0, \aleph_1)$ -graph.  $\square$

**Lemma 4.** *Let  $X$  be an  $(\aleph_0, \aleph_1)$ -graph with bipartition  $(A, B)$ . Let  $A' \subseteq A$  be infinite and such that for every two vertices  $a, a'$  in  $A'$  there is some uncountable set  $B(a, a')$  of common neighbours of  $a$  and  $a'$  in  $B$ . Then  $A'$  is the set of branch vertices of a fat  $TK_{\aleph_0}$  in  $X$  whose subdivided edges all have the form  $aba'$  with  $b \in B(a, a')$ .*

**Proof.** We have to find a total of  $\aleph_0^2 \cdot \aleph_1 = \aleph_1$  independent paths in  $X$  between vertices in  $A'$ . Let us enumerate these desired paths as  $(P_\alpha)_{\alpha < \omega_1}$ ; it is then easy to find them recursively on  $\alpha$ , keeping them independent.  $\square$

**Lemma 5.** *Every  $IX$ , where  $X$  is an  $(\aleph_0, \aleph_1)$ -graph, contains a fat  $TK_{\aleph_0}$ .*

**Proof.** Let  $H$  be an  $IX$  for an  $(\aleph_0, \aleph_1)$ -graph  $X$  with bipartition  $(A, B)$ , with branch sets  $V_x$  for vertices  $x \in X$ . Replacing  $X$  with an appropriate  $(\aleph_0, \aleph_1)$ -subgraph  $Y$  (and  $H$  with the corresponding  $IY \subseteq H$ ) if necessary, we may assume by Lemma 3 (i) that every vertex in  $A$  has uncountable degree in  $X$ . We shall find our desired fat  $TK_{\aleph_0}$  in  $H$  as follows.

We construct, inductively, an infinite set  $A' = \{a_0, a_1, \dots\} \subseteq A$  such that, for each  $a_i \in A'$ , there is an uncountable subdivided star  $S(a_i) \subseteq H[V_{a_i}]$  whose leaves send edges of  $H$  to the branch sets of distinct vertices  $b \in B$ . The sets  $B_i$  of these  $b$  will be nested as  $B_0 \supseteq B_1 \supseteq \dots$ . We shall then apply Lemma 4 to find a fat  $TK_{\aleph_0}$  in  $X$ , and translate this to the desired fat  $TK_{\aleph_0}$  in  $H$ .

Pick  $a_0 \in A$  arbitrarily. For each of the uncountably many neighbours  $b$  of  $a_0$  in  $B$  we can find a vertex  $v_b \in V_b$  that sends an edge of  $H$  to  $V_{a_0}$ . For every  $b$ , pick one neighbour  $u_b$  of  $v_b$  in  $V_{a_0}$ . Consider a minimal connected subgraph  $H_0$  of  $H[V_{a_0}]$  containing all these vertices  $u_b$ , and add to it all the edges  $u_b v_b$  to obtain the graph  $T = T(a_0)$ . By the minimality of  $H_0$ ,

*$T$  is a tree in which every edge lies on a path between two vertices of the form  $v_b$ .* (1)

Since there are uncountably many  $b$  and their  $v_b$  are distinct,  $T$  is uncountable and hence has a vertex  $s_0$  of uncountable degree. For every edge  $e$  of  $T$  at  $s_0$  pick a path in  $T$  from  $s_0$  through  $e$  to some  $v_b$ ; this is possible by (1). Let  $S(a_0)$  be the union of all these paths. Then  $S(a_0)$  is an uncountable subdivided star with centre  $s_0$  all whose non-leaves lie in  $V_{a_0}$  and whose leaves lie in the

branch sets  $V_b$  of distinct vertices  $b \in B$ . Let  $B_0 \subseteq B$  be the (uncountable) set of these  $b$ , and rename the vertices  $v_b$  with  $b \in B_0$  as  $v_b^0$ .

Assume now that, for some  $n \geq 1$ , we have picked distinct vertices  $a_0, \dots, a_{n-1}$  from  $A$  and defined uncountable subsets  $B_0 \supseteq \dots \supseteq B_{n-1}$  of  $B$  so that each  $a_i$  is adjacent in  $X$  to every vertex in  $B_i$ . By Lemma 3 (ii) there exists an  $a_n \in A \setminus \{a_0, \dots, a_{n-1}\}$  which, in  $X$ , has uncountably many neighbours in  $B_{n-1}$ . As before, we can find an uncountable subdivided star  $S(a_n)$  in  $H$  whose centre  $s_n$  and any other non-leaves lie in  $V_{a_n}$  and whose leaves  $v_b^n$  lie in the branch sets  $V_b$  of (uncountably many) distinct vertices  $b \in B_{n-1}$ . We let  $B_n$  be the set of those  $b$ . Then  $B_n$  is an uncountable subset of  $B_{n-1}$ , and  $a_n$  is adjacent in  $X$  to all the vertices in  $B_n$ , as required for  $n$  by our recursion.

By construction, every two vertices  $a_i, a_j$  in  $A' := \{a_0, a_1, \dots\}$  have uncountably many common neighbours in  $B$ : those in  $B_j$  if  $i < j$ . By Lemma 4 applied with  $B(a_i, a_j) := B_j$  for  $i < j$ , we deduce that  $A'$  is the set of branch vertices of a fat  $TK_{\aleph_0}$  in  $X$  whose subdivided edges  $a_i \dots a_j$  with  $i < j$  have the form  $a_i b a_j$  with  $b \in B_j$ . Replacing each of these paths  $a_i b a_j$  with the concatenation of paths  $s_i \dots v_b^i \subseteq S(a_i)$  and  $v_b^i \dots v_b^j \subseteq H[V_b]$  and  $v_b^j \dots s_j \subseteq S(a_j)$ , we obtain a fat  $TK_{\aleph_0}$  in  $H$  with  $s_0, s_1, \dots$  as branch vertices. (It is important here that  $b$  is not just any common neighbour of  $a_i$  and  $a_j$  but one in  $B_j$ : only then do we know that  $S(a_i)$  and  $S(a_j)$  both have a leaf in  $V_b$ .)  $\square$

Let us now turn to the case of (\*) where  $X$  is an AT-graph. As before, we shall first prove that  $X$  itself contains a fat  $TK_{\aleph_0}$ , and later refine this to a fat  $TK_{\aleph_0}$  in any  $IX$ . In this second step we shall be referring to the details of the proof of the lemma below, not just to the lemma itself.

**Lemma 6.** *Every AT-graph contains a fat  $TK_{\aleph_0}$ .*

**Proof.** Let  $X$  be an AT-graph, with Aronszajn tree  $T$ , say. Let us pick the branch vertices  $a_0, a_1, \dots$  of our desired  $TK_{\aleph_0}$  inductively, as follows.

Let  $t_0$  be the root of  $T_0 := T$ , and  $X_0 := X$ . Since  $X_0$  is connected, it has a vertex  $a_0$  of uncountable degree. Uncountably many of its neighbours lie above it in  $T_0$ , because its down-closure is a chain and hence countable, and all its neighbours are comparable with it (by definition of an AT-graph). As levels in  $T_0$  are countable,  $a_0$  has a successor  $t_1$  in  $T_0$  such that uncountably many  $X_0$ -neighbours of  $a_0$  lie above  $t_1$ ; let  $B_0$  be some uncountable set of neighbours of  $a_0$  in  $[t_1]_{T_0}$ . (We shall specify  $B_0$  more precisely later.)

Let  $T_1$  be the down-closure of  $B_0$  in  $[t_1]_{T_0}$ . Since  $T_1$  is an uncountable subposet of  $T_0$  with least element  $t_1$ , it is again an Aronszajn tree, and the subgraph  $X_1$  it induces in  $X_0$  is an AT-graph with respect to  $T_1$ .

Starting with  $t_0, T_0$  and  $X_0$  as above, we may in this way select for  $n = 0, 1, \dots$  an infinite sequence  $T_0 \supseteq T_1 \supseteq \dots$  of Aronszajn subtrees of  $T$  with roots  $t_0 < t_1 < \dots$  satisfying the following:

- $X_n := X[T_n]$  is an AT-graph with respect to  $T_n$ ;

- the predecessor  $a_n$  of  $t_{n+1}$  in  $T_n$  has an uncountable set  $B_n$  of  $X_n$ -neighbours above  $t_{n+1}$  in  $T_n$ ;
- $T_{n+1} = \lfloor t_{n+1} \rfloor_{T_n} \cap \lceil B_n \rceil_{T_n}$ .

By the last item above, there exists for every  $b \in T_{n+1}$  a vertex  $b' \in B_n \cap \lfloor b \rfloor$  (possibly  $b' = b$ ). Applied to vertices  $b$  in  $B_{n+1} \subseteq T_{n+1}$  this means that, inductively,

*Whenever  $i < j$ , every vertex in  $B_j$  has some vertex of  $B_i$  in its up-closure.* (2)

Let us now make  $a_0, a_1, \dots$  into the branch vertices of a fat  $TK_{\aleph_0}$  in  $X$ . As earlier, we enumerate the desired subdivided edges as one  $\omega_1$ -sequence, and find independent paths  $P_\alpha \subseteq X$  to serve as these subdivided edges recursively for all  $\alpha < \omega_1$ . When we come to construct the path  $P_\alpha$ , between  $a_i$  and  $a_j$  with  $i < j$  say, we have previously constructed only the countably many paths  $P_\beta$  with  $\beta < \alpha$ . The down-closure  $D_\alpha$  in  $T$  of all their vertices and all the  $a_n$  is a countable set, since the down-closure of each vertex is a chain in  $T$  and hence countable. We can thus find a vertex  $b \in B_j$  outside  $D_\alpha$ , and a vertex  $b' \geq b$  in  $B_i$  by (2). The interval of  $T$  between  $b$  and  $b'$  thus avoids  $D_\alpha$ , and since it is connected in  $X$  it contains the vertices of a  $b'-b$  path  $Q_\alpha$  in  $X - D_\alpha$ . We choose  $P_\alpha := a_i b' Q_\alpha b a_j$  as the  $\alpha$ th subdivided edge for our fat  $TK_{\aleph_0}$  in  $X$ .  $\square$

**Lemma 7.** *Every  $IX$ , where  $X$  is an AT-graph, contains a fat  $TK_{\aleph_0}$ .*

**Proof.** Let  $H$  be an  $IX$  with branch sets  $V_x$  for vertices  $x \in X$ , where  $X$  is an AT-graph with respect to an Aronszajn tree  $T$ . Rather than applying Lemma 6 to  $X$  formally, let us re-do its proof for  $X$ . We shall choose the sets  $B_n$  more carefully this time, so that we can turn the fat  $TK_{\aleph_0}$  found in  $X$  into one in  $H$ .

Given  $n$ , the set  $B_n$  chosen in the proof of Lemma 6 was an arbitrary uncountable set of upper neighbours of  $a_n$  in  $T_n$  above some fixed successor  $t_n$  of  $a_n$ . We shall replace  $B_n$  with a subset of itself, chosen as follows. For every  $b \in B_n$ , pick a vertex  $v_b^n \in V_b$  that sends an edge of  $H$  to a vertex  $u_b^n \in V_{a_n}$ . As in the proof of Lemma 5, there is a subdivided uncountable star  $S_n$  in  $H$  whose leaves are among these  $v_b^n$  and all whose non-leaves, including its centre  $s_n$ , lie in  $V_{a_n}$ . Let us replace  $B_n$  with its (uncountable) subset consisting of only those  $b$  whose  $v_b^n$  is a leaf of  $S_n$ .

Let  $K \subseteq X$  be the fat  $TK_{\aleph_0}$  found by the proof of Lemma 6 for these revised sets  $B_n$ . In order to turn  $K$  into the desired  $TK_{\aleph_0}$  in  $H$ , we replace its branch vertices  $a_n$  by the centres  $s_n$  of the stars  $S_n$ , and its subdivided edges  $P_\alpha = a_i b' Q_\alpha b a_j$  between branch vertices  $a_i, a_j$  by the concatenation of paths  $s_i \dots v_b^i \subseteq S_i$  and  $Q'_\alpha = v_b^i \dots v_b^j$  and  $v_b^j \dots s_j \subseteq S_j$ , where  $Q'_\alpha$  is a path in  $H$  expanded from  $Q_\alpha$ , i.e. whose vertices lie in the branch sets of the vertices of  $Q_\alpha$ . These paths  $P'_\alpha$  are internally disjoint for distinct  $\alpha$ , because the  $P_\alpha$  were internally disjoint.  $\square$

**Proof of Theorem 2.** Let  $G$  be a connected graph without a normal spanning tree; we show that  $G$  contains a fat  $TK_{\aleph_0}$ . By Theorem 1,  $G$  has an  $X$ -minor such that  $X$  is either an  $(\aleph_0, \aleph_1)$ -graph or an Aronszajn-tree graph. Equivalently,  $G$  has a subgraph  $H$  that is an  $IX$ , with  $X$  as above. By Lemmas 5 and 7, this subgraph  $H$ , and hence  $G$ , contains a fat  $TK_{\aleph_0}$ .  $\square$

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