A short proof of Halin's grid theorem

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We give a short proof of Halin's theorem that every thick end of a graph contains the infinite grid.

Introduction

An end of an infinite graph G is called *thick* if there are infinitely many disjoint *rays* (1-way infinite paths) in G all converging to that end. The infinite grid is an obvious example: it contains infinitely many disjoint rays, which all converge to its unique end. It is one of Halin's most striking theorems that this observation has a converse: whenever G has a thick end ω, it contains a subdivision of an infinite hexagonal grid whose rays all converge to ω.

The aim of this note is to give a short proof of Halin's theorem. We follow Halin [3] in defining an *end* of a graph as an equivalence class of rays, where two rays are *equivalent* if no finite set of vertices separates them.[1] In these terms, an end is *thick* if it contains infinitely many disjoint rays. The subrays of a ray are its *tails*, and an *A–B path* has no inner vertices in $A \cup B$. See [1] for details of this or any other undefined terms used below.

Let G be the $\mathbb{N} \times \mathbb{N}$ grid, ie., the graph on $\mathbb{N} \times \mathbb{N}$ in which two vertices are adjacent if and only if their Euclidean distance is 1. To define our standard copy H of the hexagonal grid, we delete from G the vertex $(0,0)$, the vertices (n,m) with $n > m$, and all edges $(n,m)(n+1,m)$ such that n and m have equal parity (Fig. 1).

Thus, H consists of the *vertical rays*

$$U_0 := G[\{(0,m) \mid 1 \leqslant m\}] \text{ and } U_n := G[\{(n,m) \mid n \leqslant m\}] \quad (n \geqslant 1)$$

joined by a set of *horizontal edges*,

$$E := \{(n,m)(n+1,m) \mid n \not\equiv m \pmod 2\}.$$

To enumerate these edges, as $e_1, e_2, \ldots$ say, we order them co-lexographically: the edge $(n,m)(n+1,m)$ precedes the edge $(n',m')(n'+1,m')$ if $m < m'$, and also if $m = m'$ and $n < n'$ (Fig. 1).

[1] For locally finite graphs, this definition agrees with the usual topological notion of an end; for graphs with infinite degrees it is more general [2].

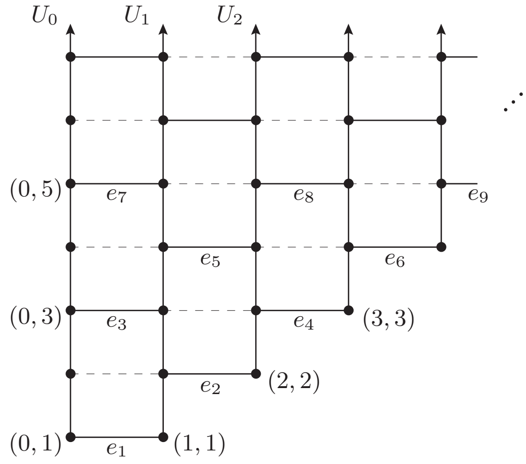

FIGURE 1. The hexagonal grid H

Theorem. (Halin [4], 1965)
Whenever a graph has a thick end, it has a subgraph isomorphic to a subdivision of the hexagonal grid H whose rays all belong to that end.

Proof. Given two infinite sets $\mathcal{P}, \mathcal{P}'$ of finite or infinite paths, let us write $\mathcal{P} \geqslant \mathcal{P}'$ if $\mathcal{P}'$ consists of final segments of paths in $\mathcal{P}$. (Thus, if $\mathcal{P}$ is a set of rays, then so is $\mathcal{P}'$.)

Let G be any graph with a thick end ω. Our task is to find disjoint rays in ω that can serve as 'vertical' (subdivided) rays U_n for our desired grid, and to link these up by suitable disjoint 'horizontal' paths. We begin by constructing a sequence $Q_0, Q_1, \ldots$ of rays (of which we shall later choose some tails Q'_n as 'vertical rays'), together with path systems $\mathcal{P}(Q_i)$ between the Q_i and suitable $Q_{p(i)}$ with $p(i) < i$ (from which we shall later choose the 'horizontal paths'). We shall aim to find the Q_n in 'supply sets' $\mathcal{R}_0 \geqslant \mathcal{R}_1 \geqslant \ldots$ of unused rays.

We start with any infinite set $\mathcal{R}_0$ of disjoint rays in ω; this exists by our assumption that ω is a thick end. At step $n \in \mathbb{N}$ of the construction, we shall choose the following:

(1) a ray $Q_n \in \omega$ disjoint from $Q_0 \cup \ldots \cup Q_{n-1}$;

(2) if $n \geqslant 1$, an integer $p(n) < n$;

(3) for every i with $1 \leqslant i \leqslant n$, an infinite set $\mathcal{P}_n(Q_i)$ of disjoint Q_i–$Q_{p(i)}$ paths, such that

  (i) $\bigcup \mathcal{P}_n(Q_i) \cap \bigcup \mathcal{P}_n(Q_j) = \emptyset$ for distinct $i, j \leqslant n$, and

  (ii) $\bigcup \mathcal{P}_n(Q_i) \cap Q_j = \emptyset$ for distinct $i, j \leqslant n$ with $j \neq p(i)$;

(4) an infinite set $\mathcal{R}_{n+1} \leqslant \mathcal{R}_n$ of rays disjoint from $Q_0 \cup \ldots \cup Q_n$ and from $\bigcup \mathcal{P}_n(Q_i)$ whenever $1 \leqslant i \leqslant n$.

Thus, while the rays Q_i and the predecessor map $i \mapsto p(i)$ remain unchanged once defined for some i, the path system $\mathcal{P}_n(Q_i)$ between Q_i and $Q_{p(i)}$

changes as n increases. More precisely, we shall have

(5) $\mathcal{P}_n(Q_i) \subseteq \mathcal{P}_{n-1}(Q_i)$ whenever $1 \leqslant i < n$.

Informally, we think of $\mathcal{P}_n(Q_i)$ as our best candidate at time n for a system of horizontal paths linking Q_i to $Q_{p(i)}$. But, as new rays Q_m with $m > n$ get selected, we may have to change our mind about $\mathcal{P}_n(Q_i)$ and, again and again, prune it to a smaller system $\mathcal{P}_m(Q_i)$. This may leave us with an empty system at the end of of the construction. Thus, when we later come to construct our grid, we shall have to choose its horizontal paths between Q_i and $Q_{p(i)}$ from these provisional sets $\mathcal{P}_n(Q_i)$, not from their (possibly empty) intersection over all n.

Let $n \in \mathbb{N}$ be given. If $n = 0$, choose any ray from $\mathcal{R}_0$ as Q_0, and put $\mathcal{R}_1 := \mathcal{R}_0 \setminus \{ Q_0 \}$. Then conditions (1)–(5) hold for $n = 0$.

Suppose now that $n \geqslant 1$, and consider a ray $R_n^0 \in \mathcal{R}_n$. By (4), R_n^0 is disjoint from

$$H_n := Q_0 \cup \ldots \cup Q_{n-1} \cup \bigcup_{i=1}^{n-1} \mathcal{P}_{n-1}(Q_i)\,.$$

By the choice of $\mathcal{R}_0$ and (4), we know that $R_n^0 \in \omega$. As also $Q_0 \in \omega$, there exists an infinite set $\mathcal{P}$ of disjoint R_n^0–H_n paths. If possible, we choose $\mathcal{P}$ so that $\bigcup \mathcal{P} \cap \bigcup \mathcal{P}_{n-1}(Q_i) = \emptyset$ for all $i \leqslant n-1$. We may then further choose $\mathcal{P}$ so that $\bigcup \mathcal{P} \cap Q_i \neq \emptyset$ for only one i, since by (1) the Q_i are disjoint for different i. We define $p(n)$ as this i, and put $\mathcal{P}_n(Q_j) := \mathcal{P}_{n-1}(Q_j)$ for all $j \leqslant n-1$.

If $\mathcal{P}$ cannot be chosen in this way, we may choose it so that all its vertices in H_n lie in $\bigcup \mathcal{P}_{n-1}(Q_i)$ for the same i, since by (3) the graphs $\bigcup \mathcal{P}_{n-1}(Q_i)$ are disjoint for different i. We can then find infinite disjoint subsets $\mathcal{P}_n(Q_i)$ of $\mathcal{P}_{n-1}(Q_i)$ and $\mathcal{P}'$ of $\mathcal{P}$. We continue infinitely many of the paths in $\mathcal{P}'$ along paths from $\mathcal{P}_{n-1}(Q_i) \setminus \mathcal{P}_n(Q_i)$ to Q_i or to $Q_{p(i)}$, to obtain an infinite set $\mathcal{P}''$ of disjoint R_n^0–Q_i or R_n^0–$Q_{p(i)}$ paths, and define $p(n)$ as i or as $p(i)$ accordingly. The paths in $\mathcal{P}''$ then avoid $\bigcup \mathcal{P}_n(Q_j)$ for all $j \leqslant n-1$ (with $\mathcal{P}_n(Q_j) := \mathcal{P}_{n-1}(Q_j)$ for $j \neq i$) and Q_j for all $j \neq p(n)$. We rename $\mathcal{P}''$ as $\mathcal{P}$, to simplify notation.

In either case, we have now defined $\mathcal{P}_n(Q_i)$ for all $i < n$ so as to satisfy (5) for n, chosen $p(n)$ as in (2), and found an infinite set $\mathcal{P}$ of disjoint R_n^0–$Q_{p(n)}$ paths that avoid all other Q_j and all the sets $\mathcal{P}_n(Q_i)$. All that can prevent us from choosing R_n^0 as Q_n and $\mathcal{P}$ as $\mathcal{P}_n(Q_n)$ and $\mathcal{R}_{n+1} \leqslant \mathcal{R}_n \setminus \{ R_n^0 \}$ is condition (4): if $\mathcal{P}$ meets all but finitely many rays in $\mathcal{R}_n$ infinitely, we cannot find an infinite set $\mathcal{R}_{n+1} \leqslant \mathcal{R}_n$ of rays avoiding $\mathcal{P}$.

However, we may now assume the following:

Whenever $R \in \mathcal{R}_n$ and $\mathcal{P}' \leqslant \mathcal{P}$ is an infinite set of R–$Q_{p(n)}$ paths, there is a ray $R' \neq R$ in $\mathcal{R}_n$ that meets $\mathcal{P}'$ infinitely. $(*)$

For if $(*)$ failed, we could choose R as Q_n and $\mathcal{P}'$ as $\mathcal{P}_n(Q_n)$, and select from every ray $R' \neq R$ in $\mathcal{R}_n$ a tail avoiding $\mathcal{P}'$ to form $\mathcal{R}_{n+1}$. This would satisfy conditions (1)–(5) for n.

Consider the paths in $\mathcal{P}$ as linearly ordered by the natural order of their starting vertices on R_n^0. This induces an ordering on every $\mathcal{P}' \leqslant \mathcal{P}$. If $\mathcal{P}'$ is a set of R–$Q_{p(n)}$ paths for some ray R, we shall call this ordering of $\mathcal{P}'$ *compatible* with R if the ordering it induces on the first vertices of its paths coincides with the natural ordering of those vertices on R.

Using assumption $(*)$, let us choose two sequences $R_n^0, R_n^1, \ldots$ and $\mathcal{P}^0 \geqslant \mathcal{P}^1 \geqslant \ldots$ such that every R_n^k is a tail of a ray in $\mathcal{R}_n$ and each $\mathcal{P}^k$ is an infinite set of R_n^k–$Q_{p(n)}$ paths whose ordering is compatible with R_n^k. The first path of $\mathcal{P}^k$ in this ordering will be denoted by P_k, its starting vertex on R_n^k by v_k, and the path in $\mathcal{P}^{k-1}$ containing P_k by P_k^- (Fig. 2). Clearly, $\mathcal{P}_0 := \mathcal{P}$ is as required for $k = 0$; put $P_0^- := P_0$. For $k \geqslant 1$, we may use $(*)$ with $R \supseteq R_n^{k-1}$ and $\mathcal{P}' = \mathcal{P}^{k-1}$ to find in $\mathcal{R}_n$ a ray $R' \not\supseteq R_n^{k-1}$ that meets $\mathcal{P}^{k-1}$ infinitely but has a tail R_n^k avoiding the finite subgraph $P_0^- \cup \ldots \cup P_{k-1}^-$. Let P_k^- be a path in $\mathcal{P}^{k-1}$ that meets R_n^k and let v be its 'highest' vertex on R_n^k, that is, the last vertex of R_n^k in $V(P_k^-)$. Replacing R_n^k with its tail vR_n^k, we can arrange that P_k^- has only the vertex v on R_n^k. Then $P_k := vP_k^-$ is an R_n^k–$Q_{p(n)}$ path starting at $v_k = v$. We may now select an infinite set $\mathcal{P}^k \leqslant \mathcal{P}^{k-1}$ of R_n^k–$Q_{p(n)}$ paths compatible with R_n^k and containing P_k is its first path.

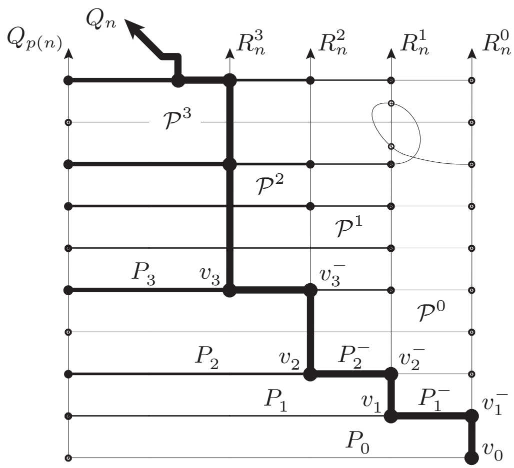

FIGURE 2. Constructing Q_n from condition $(*)$

Since P_k^- contains $v_k \in R_n^k$ but $R_n^k \cap P_{k-1} = \emptyset$, we have $P_k^- \neq P_{k-1}$, so the P_k are all disjoint. For each k, let v_{k+1}^- denote the starting vertex of P_{k+1}^- on R_n^k, and put $R_{n+1}^k := \mathring{v}_{k+1}^- R_n^k$. Then let

$$Q_n := v_0 R_n^0 v_1^- P_1^- v_1 R_n^1 v_2^- P_2^- v_2 R_n^2 \ldots$$

$$\mathcal{P}_n(Q_n) := \{ P_0, P_1, P_2, \ldots \}$$

$$\mathcal{R}_{n+1} := \{ R_{n+1}^k \mid k \in \mathbb{N} \}.$$

Let us check that these definitions satisfy (1)–(5) for n. We have already verified (2) and (5). For the disjointness requirements in (1) and (3), recall that Q_n and $\mathcal{P}_n(Q_n)$ consist of segments of paths in $\mathcal{R}_n$ and $\mathcal{P}$; these are disjoint from Q_i and $\mathcal{P}_n(Q_i)$ for all $i < n$ by definition of $\mathcal{P}$ and (4) for $n-1$ (together with (5) for n). For the disjointness requirement in (4) note that R^k_{n+1} does not meet Q_n or $\mathcal{P}_n(Q_n)$ inside any path P_j^- with $j > k+1$, since these P_j^- are proper final segments of R^k_n–$Q_{p(n)}$ paths in $\mathcal{P}^k$. Since R^k_{n+1} does not, by definition, meet Q_n or $\mathcal{P}_n(Q_n)$ inside any path P_j^- with $j \leqslant k+1$, condition (4) holds for n.

It remains to use our rays Q_n, path systems $\mathcal{P}_n(Q_i)$, and supply sets $\mathcal{R}_n$ of rays to construct the desired grid. By König's infinity lemma [1], there is a sequence $n_0 < n_1 < n_2 < \ldots$ such that either $p(n_i) = n_{i-1}$ for every $i \geqslant 1$ or $p(n_i) = n_0$ for every $i \geqslant 1$. We treat these two cases in turn.

In the first case, let us assume for notational simplicity that $n_i = i$ for all i, i.e. discard any Q_n with $n \notin \{ n_0, n_1, \ldots \}$. Then for every $i \geqslant 1$ and every $n \geqslant i$ we have an infinite set $\mathcal{P}_n(Q_i)$ of disjoint Q_i–Q_{i-1} paths. Our aim is to choose tails Q'_n of our rays Q_n that will correspond to the vertical rays $U_n \subseteq H$, and paths $S_1, S_2, \ldots$ between the Q'_n that will correspond to the horizontal edges $e_1, e_2, \ldots$ of H. We shall find the paths $S_1, S_2, \ldots$ inductively, choosing the Q'_n as needed as we go along (but also in the order of increasing n, starting with $Q'_0 := Q_0$). At every step of the construction, we shall have selected only finitely many S_k and only finitely many Q'_n.

Let k and n be minimal such that S_k and Q'_n are still undefined. We describe how to choose S_k, and Q'_n if the definition of S_k requires it. Let i be such that e_k joins U_{i-1} to U_i in H. If $i = n$, let Q'_n be a tail of Q_n that avoids the finitely many paths $S_1, \ldots, S_{k-1}$; otherwise, Q'_i has already been defined, and so has Q'_{i-1}. Now choose $S_k \in \mathcal{P}_n(Q_i)$ 'high enough' between Q'_{i-1} and Q'_i to mirror the position of e_k in H^∞, and to avoid $S_1 \cup \ldots \cup S_{k-1}$. By (3)(ii), S_k will also avoid every other Q'_j already defined. Since every Q'_n is chosen so as to avoid all previously defined S_k, and every S_k avoids all previously defined Q'_j (except Q'_{i-1} and Q'_i), the Q'_n and S_k are pairwise disjoint for all $n, k \in \mathbb{N}$, except for the required incidences. Our construction thus yields the desired subdivision of H.

It remains to treat the case that $p(n_i) = n_0$ for all $i \geqslant 1$. Let us rename Q_{n_0} as Q, and n_i as $i-1$ for $i \geqslant 1$. Then our sets $\mathcal{P}_n(Q_i)$ consist of disjoint Q_i–Q paths. We choose rays $Q'_n \subseteq Q_n$ and paths S_k inductively as before, except that S_k now consists of three parts: an initial segment from $\mathcal{P}_n(Q_{i-1})$, followed by a middle segment from Q, and a final segment from $\mathcal{P}_n(Q_i)$. Such S_k can again be found, since at every stage of the construction only a finite part of Q has been used. □

References

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Version 1.12.2004