Dense minors in graphs of large girth

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We show that a graph of girth greater than $6\log k+3$ and minimum degree at least 3 has a minor of minimum degree greater than k. This is best possible up to a factor of at most $9/4$. As a corollary, every graph of girth at least $6\log r+3\log\log r+c$ and minimum degree at least 3 has a K_r minor.

1. Introduction

Thomassen [9] proved that, in graphs of minimum degree at least 3, sufficiently high girth forces a minor of any given minimum degree:

Theorem. (Thomassen 1983)
For any integer k, every graph G of girth $g(G)\geqslant 4k-3$ and $\delta(G)\geqslant 3$ has a minor H with $\delta(H)\geqslant k$.

Our aim in this note is to reduce the upper bound for the required girth to the correct order of magnitude:

Theorem 1. *For any integer k, every graph G of girth $g(G)>6\log k+3$ and $\delta(G)\geqslant 3$ has a minor H with $\delta(H)>k$.*

The best lower bound implied by known examples is $\frac{8}{3}\log k-c$, but we note that existing conjectures about cubic graphs of large girth would raise this to about $4\log k$.

Since an average degree of at least $cr\sqrt{\log r}$ forces a K_r minor [5, 10], Theorem 1 has the following consequence:

Corollary 2. *There exists a constant $c\in\mathbb{R}$ such that every graph G of girth $g(G)\geqslant 6\log r+3\log\log r+c$ and $\delta(G)\geqslant 3$ has a K_r minor.* □

Asymptotically, Thomason [11] showed that a K_r minor is forced by an average degree of $(d+o(1))\,r\sqrt{\log r}$, where $d=0.53131\ldots$ is an explicit constant that is best possible. This means that, for large enough r, Corollary 2 holds with $c=-2.4742$.

We adopt the notation of [4]. All our logarithms are binary, all graphs considered finite, and $0\in\mathbb{N}$.

2. A lower bound

Minimum-order cubic graphs of girth at least some given integer g are called *g-cages* and have been studied in some detail (see [1] for an overview). Their exact order is known for $g \leqslant 12$. The best more general upper bound for the order of g-cages is due to Biggs & Hoare [2] and Weiss [12]:

Lemma 2.1. *There is a constant $c^* > 0$ such that for infinitely many integers g there exists a cubic graph of girth at least g and order at most $c^* 2^{3g/4}$.*

Now suppose that a graph G as in Lemma 2.1 has a minor of minimum degree k (say). Then G has at least $k+1$ branch sets, each of which sends out at least k edges and hence contains at least $k-2$ vertices (since G is cubic). Therefore

$$(k+1)(k-2) \leqslant |G| \leqslant c^* 2^{3g/4},$$

giving $g \geqslant \frac{8}{3} \log k - c$ for a suitable constant c. Choosing $k = k(g) \in \mathbb{N}$ maximal with this last inequality, we can thus deduce from Lemma 2.1 the following counterpart to Theorem 1:

Proposition 2.2. *There is a constant $c \in \mathbb{R}$ such that for infinitely many $k \in \mathbb{N}$ there exist cubic graphs of girth at least $\frac{8}{3} \log k - c$ that have no minor H with $\delta(H) > k$.* □

Any improvement on the bound in Lemma 2.1 will result in a corresponding improvement to Proposition 2.2. It has been conjectured (see [3] or [8]) that g-cages exist on as few as about $2^{g/2}$ vertices. This would increase our lower girth bound to $4 \log k - c$.

3. The upper bound

In this section we prove Theorem 1. Following Mader [7], we start from the observation that in a graph G of girth $g(G) > 2d+1$ and $\delta(G) \geqslant 3$ the d-ball around a vertex x is a tree T_x sending at least $|T_x| - 2$ edges to the rest of G. Our main effort will go into proving that, depending on our lower bound for $g(G)$, not too many of these edges can go to the same tree T_y. Then partitioning $V(G)$ into such trees and contracting these will give us a minor of large minimum degree.

Given a tree T with root r and vertices $t, t' \in T$, we say that t' lies *above* t in T (and t *below* t') if $t \leqslant t'$ in the tree-order on $V(T)$ associated with r, ie. if t separates t' from r in T. Any neighbour of t above it is a *successor* of t in T, its unique neighbour below is its *predecessor*. For $i \in \mathbb{N}$ we write L^i_T for the set of *leaves* (maximal elements) of T at distance i from r.

Given a graph G, a vertex $x \in G$, and $d \in \mathbb{N}$, let us write $V^d_{G,x}$ for the set of vertices of G at distance exactly d from x. We need the following easy lemma:

Lemma 3.1. *Let T be a tree with root r in which no vertex has exactly one successor, and let $d \in \mathbb{N}$. Then $\sum_{i \geqslant d} 2^{d-i}|L_T^i| \geqslant |V_{T,r}^d|$.* $\square$

We are now ready to prove our main result, which we restate:

Theorem 1. *For any integer k, every graph G of girth $g(G) > 6\log k + 3$ and $\delta(G) \geqslant 3$ has a minor H with $\delta(H) > k$.*

Proof. Put $\lfloor \log k \rfloor =: d$. Let X be a maximal set of vertices such that $d(x,y) > 2d$ for all distinct $x, y \in X$. Beginning with $T_x^0 := \{x\}$, let us define trees T_x^i rooted at x, for all $x \in X$ and $i = 0, \dots, 2d$. Assume that for some i the T_x^i have been defined and partition the set of vertices of G at distance at most i from X. We then add each vertex v at distance $i+1$ from X to one T_x^i to which it is adjacent, thereby obtaining a similar set of disjoint trees T_x^{i+1}. By the choice of X, the trees $T_x := T_x^{2d}$ partition the entire vertex set of G, and

$$T_x \text{ contains all the vertices of } G \text{ at distance at most } d \text{ from } x. \tag{1}$$

As $g(G) > 4d+1$, the T_x are induced subgraphs in G. Finally, we have

$$d(w,y) \leqslant d(v,x) + 1 \text{ whenever } vw \in E(G) \text{ with } v \in T_x \text{ and } w \in T_y\,, \tag{2}$$

as otherwise w would have been added to T_x after v rather than to T_y.

Let us use Lemma 3.1 to estimate the number of edges leaving a tree T_x. For all $i \in \mathbb{N}$ let

$$E_x^i := \{ vw \in E(G) \mid v \in T_x,\ w \in G - T_x,\ d(v,x) = i \}.$$

Let T'_x denote the subgraph of G induced by T_x and all its neighbours in G. As $g(G) > 4d+3$, T'_x is again a tree. Every vertex $v \in T_x$ has degree $d_G(v) \geqslant 3$ in T'_x, while all the vertices of $T'_x - T_x$ are leaves in T'_x. As $|E_x^i| = |L_{T'_x}^{i+1}|$ for all i, and $|L_{T'_x}^d| = 0$ by (1), Lemma 3.1 yields

$$\sum_{i \geqslant d} 2^{d-i-1}|E_x^i| = \sum_{i \geqslant d} 2^{d-i-1}|L_{T'_x}^{i+1}| = \sum_{i \geqslant d} 2^{d-i}|L_{T'_x}^i| \geqslant |V_{T'_x,x}^d| = |V_{G,x}^d|.$$

Multiplying by 2^{d+1} and setting $V_x^d := V_{G,x}^d$ we obtain

$$\sum_{i \geqslant d} 2^{2d-i}|E_x^i| \geqslant 2^{d+1}|V_x^d|.$$

Every edge in E_x^i joins T_x to a tree T_y distinct from T_x. This defines a partition of E_x^i into sets $A_{x,y}^i$ ($y \in X \setminus \{x\}$). Then the above inequality can be rewritten as

$$2^{d+1}|V_x^d| \leqslant \sum_y \sum_{i \geqslant d} 2^{2d-i}|A_{x,y}^i|\,, \tag{3}$$

where the first sum is taken over all $y \in X \smallsetminus \{x\}$ such that G contains a T_x–T_y edge. We shall prove that, for each of these y,

$$\sum_{i \geqslant d} 2^{2d-i} |A^i_{x,y}| \leqslant |V^d_x|, \tag{4}$$

so that (3) can be satisfied only if there are at least 2^{d+1} distinct y, ie. if T_x sends edges to at least 2^{d+1} other trees T_y. Contracting all the trees T_x with $x \in X$ we then obtain a minor of G of minimum degree at least $2^{d+1} > k$, as desired.

For the proof of (4) let now x and y be fixed distinct vertices in X. Consider a T_x–T_y edge $e = vw$ of G, with $v \in T_x$ and $w \in T_y$ say. Then $i := d(v, x) \geqslant d$, by (1) and $w \notin T_x$. Let z_e be the vertex below v in T_x at distance d from v, ie. in V^{i-d}_x, and let B_e be the set of vertices in V^d_x that lie above z_e in T_x. These vertices have distance $2d - i$ from z_e, so

$$|B_e| \geqslant 2^{2d-i}. \tag{5}$$

Let us show that

$$B_e \cap B_{e'} = \emptyset \text{ for all distinct } T_x\text{–}T_y \text{ edges } e, e'. \tag{6}$$

Suppose not, ie. suppose that z_e and $z_{e'}$ are comparable in T_x, say $z_e \leqslant z_{e'}$. Write $e =: vw$ and $e' =: v'w'$ with $v, v' \in T_x$ and $w, w' \in T_y$, and put $i := d(v, x)$. We show that the unique cycle C in $T_x \cup T_y + e + e'$ has length less than $g(G)$ (Fig. 1).

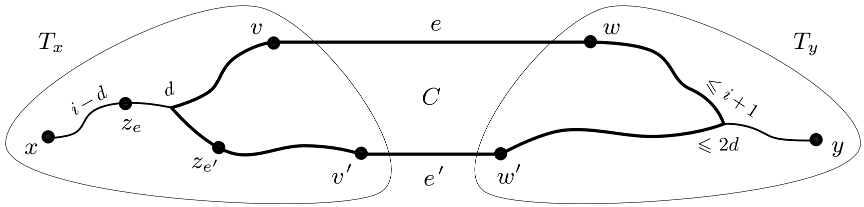

FIGURE 1. The cycle C between T_x and T_y.

The portion of C in T_x is a subpath of the walk $v \ldots z_e \ldots v'$ in T_x, which has length at most $d + (2d - (i - d)) = 4d - i$. Its portion in T_y is a subpath of the walk $w \ldots y \ldots w'$ in T_y, which has length at most $(i + 1) + 2d$ by (2). Thus $|C| \leqslant 6d + 3 < g(G)$, as desired. This completes the proof of (6).

Now (5), (6) and the definition of the B_e imply (4):

$$\sum_{i \geqslant d} 2^{2d-i} |A^i_{x,y}| = \sum_{i \geqslant d} \sum_{e \in A^i_{x,y}} 2^{2d-i} \underset{(5)}{\leqslant} \sum_{i \geqslant d} \sum_{e \in A^i_{x,y}} |B_e| \underset{(6)}{\leqslant} |V^d_x|.$$

□

In order to improve the bound in Theorem 1 further, we have considered the question of whether the set X might be chosen more effectively. For the proof of (1) we need its points to be more than $2d$ apart. But if they were placed in G so that every other vertex v had distance $d(v, X) \leqslant \alpha d$ from X for some $\alpha < 2$ (rather than just $d(v, X) \leqslant 2d$, which we get simply by choosing X maximal), we would instantly shorten the cycle C in the proof of (6) to at most $(2+2\alpha)d+3$, improving the girth bound in the theorem to $(2+2\alpha)\log k+3$. Note that the theoretical optimum of $\alpha = 1$ would give us exactly (up to the additive constant) the conjectured lower bound from Section 2.

The problem of whether such a set X exists for given values of d and α has been shown to be NP-hard [8], and so we did not pursue this approach further. However, Kühn and Osthus [6] have recently shown that a random choice of X can indeed reduce the leading factor of 6 in Theorem 1 to the conjectured optimum of 4.

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