Proof. By Lemma 12.3.4, we have \( tw(K^4) \geq 3 \). By Proposition 12.4.2, therefore, a graph of tree-width \(< 3\) cannot contain \( K^4 \) as a minor.

Conversely, let \( G \) be a graph without a \( K^4 \) minor; we assume that \( |G| \geq 3 \). Add edges to \( G \) until the graph \( G' \) obtained is edge-maximal without a \( K^4 \) minor. By Proposition 8.3.1, \( G' \) can be constructed recursively from triangles by pasting along \( K^2 \)'s. By induction on the number of recursion steps and Lemma 12.3.4, every graph constructible in this way has a tree-decomposition into triangles (as in the proof of Proposition 12.3.8). Such a tree-decomposition of \( G' \) has width 2, and by Lemma 12.3.1 it is also a tree-decomposition of \( G \). \( \square \)

A question converse to the above is to ask for which \( X \) (other than \( K^3 \) and \( K^4 \)) the tree-width of the graphs in \( \text{Forb}_\prec(X) \) is bounded. Interestingly, it is not difficult to show that any such \( X \) must be planar. Indeed, consider the graph on \( \{1, \ldots, n\}^2 \) with the edge set

\[
\{ (i, j)(i', j') : |i - i'| + |j - j'| = 1 \};
\]

this graph is called the \( n \times n \) grid. Clearly, the \( n \times n \) grid is planar (for every \( n \)), and hence lies in every class \( \text{Forb}_\prec(X) \) with non-planar \( X \). On the other hand, it is not difficult to show that the tree-width of the \( n \times n \) grid tends to infinity with \( n \) (Exercise 19). Therefore, the tree-width of the graphs in \( \text{Forb}_\prec(X) \) cannot be bounded unless \( X \) is planar.

The following deep and surprising theorem says that, conversely, the tree-width of the graphs in \( \text{Forb}_\prec(X) \) is bounded for every planar \( X \):

**Theorem 12.4.4.** (Robertson & Seymour 1986) The tree-width of the graphs in \( \text{Forb}_\prec(X) \) is bounded if and only if \( X \) is planar.

The proof of Theorem 12.4.4 is too involved to be presented here. However, there is a similar result on the related notion of ‘path-width’, which we shall prove instead: its proof is much simpler, but it gives an indication of some of the techniques used for the proof of Theorem 12.4.4.

A tree-decomposition whose tree is a path is called a path-decomposition. We usually denote a path-decomposition \( (P, \mathcal{V}) \) simply by listing the sets \( V_1, \ldots, V_s \in \mathcal{V} \) in the order defined by \( P \). The least width of a path-decomposition of \( G \) is the path-width \( \text{pw}(G) \) of \( G \).

The analogue of Theorem 12.4.4 for path-width is obtained simply by replacing planarity with acyclicity:

**Theorem 12.4.5.** (Robertson & Seymour 1983) The path-width of the graphs in \( \text{Forb}_\prec(X) \) is bounded if and only if \( X \) is a forest.

The forward implication of Theorem 12.4.5 is again easy. All we have to show is that trees can have arbitrarily large path-width: since
Forbₜ(X) contains all trees if X has a cycle, this will imply that forbidding X cannot bound the path-width unless X is a forest.

How can one show that a graph—in our case, a tree—has large path-width? Let \((V_1, \ldots, V_s)\) be a path-decomposition of some connected graph \(G\), of width \(\text{pw}(G)\) and such that \(V_1, V_s \neq \emptyset\). Pick vertices \(v_1 \in V_1\) and \(v_s \in V_s\), and let \(Q\) be a \(v_1-v_s\) path in \(G\). By Lemma 12.3.2, \(Q\) meets every \(V_r\), \(r = 1, \ldots, s\). Hence, the path-decomposition \((V_1 \setminus V(Q), \ldots, V_s \setminus V(Q))\) of \(G-Q\) has width at most \(\text{pw}(G)-1\), so

\[
\text{pw}(G-Q) < \text{pw}(G) .
\]

Thus every connected graph \(G\) contains a path whose deletion reduces the path-width of \(G\). If we may further assume (e.g. by some suitable induction hypothesis) that \(G-Q\) has large path-width for every path \(Q \subseteq G\), then \(G\) has even larger path-width.

We now use this idea to show that trees can have arbitrarily large path-width. Let \(T^k_3\) denote the tree in which one specified vertex \(r\) has degree 3, all other vertices (except the leaves) have degree 4, and all leaves have distance \(k\) from \(r\). If \(T = T^k_3+1\) and \(Q\) is any path in \(T\), then \(Q\) contains at most two of the three edges at \(r\); hence, \(T-Q\) contains a component of \(T-r\), which is a copy of \(T^k_3\). Induction on \(k\) thus shows that \(\text{pw}(T^k_3) \geq k\) for all \(k\).

For the proof of the backward implication of Theorem 12.4.5 we need some definitions and two lemmas. Let \(G = (V,E)\) be a graph. For \(X \subseteq V\), we denote by \(\partial X\) the set of all vertices in \(X\) with a neighbour \(\partial X\) in \(G-X\). For every integer \(n \geq 0\) we define a set \(B_n = B_n(G)\) of subsets of \(V\) by the following recursion:

\[
\begin{align*}
(i) & \emptyset \in B_n; \\
(ii) & \text{if } X \in B_n, X \subseteq Y \subseteq V \text{ and } |\partial X| + |Y \setminus X| \leq n, \text{ then } Y \in B_n \\
& \text{ (Fig. 12.4.1).}
\end{align*}
\]

Fig. 12.4.1. If \(X\) lies in \(B_n\), then so does \(Y\)

Thus, a set \(X \subseteq V\) lies in \(B_n\) if and only if there is a sequence

\[
\emptyset = X_0 \subseteq \ldots \subseteq X_s = X
\]

such that \(|\partial X_r| + |X_{r+1} \setminus X_r| \leq n\) for all \(r < s\). For example, if \((V_1, \ldots, V_s)\) is a path-decomposition of \(G\) of width \(< n\), then all its
12.4 Tree-width and forbidden minors

‘initial segments’ \( V_1 \cup \ldots \cup V_r \) (\( r \leq s \)) lie in \( B_n \), including \( V \) for \( r = s \) (exercise). Conversely, we have the following:

**Lemma 12.4.6.** If \( V \in B_n \), then \( \text{pw}(G) < n \).

**Proof.** If \( V \in B_n \), then there is a sequence \( \emptyset = X_0 \subseteq \ldots \subseteq X_s = V \) such that \( |\partial X_r| + |X_{r+1} \setminus X_r| \leq n \) for all \( r < s \). We set

\[
V_{r+1} := \partial X_r \cup (X_{r+1} \setminus X_r)
\]

and show that \( (V_1, \ldots, V_s) \) is a path-decomposition of \( G \) (Fig. 12.4.2).

**Fig. 12.4.2.** Constructing a path-decomposition from \( B_n \)

Induction on \( r \) shows that \( X_r = V_1 \cup \ldots \cup V_r \) for all \( r \leq s \); in particular, \( V = X_s = V_1 \cup \ldots \cup V_s \). Hence (T1) holds. For the proof of (T2), let \( xy \in E \) be given. Let \( r(x) \) be minimum with \( x \in X_{r(x)} \), and \( r(y) \) minimum with \( y \in X_{r(y)} \). We assume that \( r(x) \leq r(y) =: r \), and show that \( x \), like \( y \), lies in \( V_r \). This is clear if \( r(x) = r \). Yet if \( r(x) < r \), then \( x \) lies in \( X_{r-1} \), and hence in \( \partial X_{r-1} \subseteq V_r \) since \( xy \in E \). For the proof of (T3), finally, let \( p < q < r \) and \( x \in V_p \cap V_r \) be given. Then \( x \in V_p \subseteq V_1 \cup \ldots \cup V_{q-1} = X_{q-1} \subseteq X_{r-1} \), so \( x \in \partial X_{r-1} \cap V_r \). By definition of \( V_r \) this implies \( x \in \partial X_{r-1} \), so \( x \in \partial X_{r-1} \cap X_{q-1} \subseteq \partial X_{q-1} \subseteq V_q \). \( \square \)

**Lemma 12.4.7.** Let \( Y \in B_n \) and \( Z \subseteq Y \). If there is a family \( (P_z)_{z \in \partial Z} \) of disjoint \( Z-\partial Y \) paths in \( G \) with \( z \in P_z \) for all \( z \in \partial Z \), then \( Z \in B_n \) (Fig. 12.4.3).

**Fig. 12.4.3.** Five paths \( P_z \); three of them trivial
Proof. By definition of $B_n$, there are sets $\emptyset = Y_0 \subseteq \ldots \subseteq Y_s = Y$ such that

$$|\partial Y_r| + |Y_{r+1} \setminus Y_r| \leq n$$

for all $r < s$. We shall deduce from this that, setting $Z_r := Y_r \cap Z$, we also have

$$|\partial Z_r| + |Z_{r+1} \setminus Z_r| \leq n$$

for all $r < s$; then $Z = Z_s \in B_n$.

Fix $r$. Since $Z_{r+1} \setminus Z_r = Z_{r+1} \setminus Y_r \subseteq Y_{r+1} \setminus Y_r$, it suffices to show that $|\partial Z_r| \leq |\partial Y_r|$. We prove this by constructing an injective map $z \mapsto y$ from $\partial Z_r \setminus \partial Y_r$ to $\partial Y_r \setminus \partial Z_r$ (Fig. 12.4.4).

![Fig. 12.4.4. An injective path linkage between $\partial Z_r \setminus \partial Y_r$ and $\partial Y_r \setminus \partial Z_r$](image)

Consider a vertex $z \in \partial Z_r \setminus \partial Y_r$. Then $z$ has a neighbour in $Y_r \setminus Z_r = Y_r \setminus Z$, so $z \in \partial Z$. Now $P_z$ is a path from $(Z_r \subseteq) Y_r$ to $\partial Y$, so $P_z$ has a vertex $y$ in $\partial Y$; note that $y \neq z$ by the choice of $z$. As $z$ is the only vertex of $P_z$ in $Z$, we have $y \in \partial Y_r \setminus \partial Z_r$. Since the paths $P_z$ are disjoint, these vertices $y$ are distinct for different $z$, so $|\partial Z_r| \leq |\partial Y_r|$ as claimed. \(\square\)

**Proof of Theorem 12.4.5.** The forward implication of the theorem was proved earlier. For the converse, we prove the following:

If $\text{pw}(G) \geq n \in \mathbb{N}$, then $G$ contains every forest $F$ with $|F| - 1 = n$ as a minor. \((\ast)\)

Clearly, by \((\ast)\), if $X$ is any forest then every graph in $\text{Forb}_< (X)$ has path-width less than $|X| - 1$.

So let $\text{pw}(G) \geq n$, and assume without loss of generality that $F$ is a tree. Let $(v_1, \ldots, v_{n+1})$ be an enumeration of $V(F)$ as in Corollary 1.5.2, i.e. so that $v_{i+1}$ has exactly one neighbour in $\{v_1, \ldots, v_i\}$, for all $i \leq n$. 

---

\[(\text{1.5.2})\]  
\[(\text{3.3.1})\]
For every $i = 0, \ldots, n$, we shall define a family $X^i = (X^i_0, \ldots, X^i_i)$ of disjoint subsets of $V$, such that $X^i_j \subseteq X^i_\ell$ whenever $j \leq k \leq \ell$ and all $X^i_j$ with $j > 0$ are connected in $G$. We then write

$$X^i := X^i_0 \cup \ldots \cup X^i_i.$$ 

For each $i$, the following three statements will hold:

(i) $G$ contains an $X^i_j$--$X^i_k$ edge whenever $1 \leq j < k \leq i$ and $v_j v_k \in E(F)$ (so $F[v_1, \ldots, v_i]$ is a minor of $G[X^i_1 \cup \ldots \cup X^i_i]$);

(ii) $|X^i_j \cap \partial X^i| = 1$ for all $1 \leq j \leq i$;

(iii) $X^i$ is maximal in $B_n$ with $|\partial X^i| \leq i$.

Note that (ii) and (iii) together imply $|\partial X^i| = i$.

![Figure 12.4.5. Constructing an $F$ minor in $G$](image)

Let $X^0_0 \in B_n$ be maximal with $|\partial X^0_0| = 0$ (possibly $X^0_0 = \emptyset$). Then (i)–(iii) hold for $i = 0$. Assume now that $X^i$ has been defined so that (i)–(iii) hold, for given $i \leq n$. If $i = 0$, let $x$ be any vertex of $G - X^0$; note that $G - X^0 \neq \emptyset$, since $X^0 \in B_n$ but $V \notin B_n$ by Lemma 12.4.6. If $i > 0$, consider the unique $j \leq i$ with $v_j v_{i+1} \in E(F)$, and let $x \in G - X^i$ be a neighbour of the unique vertex in $X^i_j \cap \partial X^i$; cf. (ii). Set

$$X := X^i \cup \{x\}.$$ 

If $i = n$, we have $F \preceq G[X]$ by (i) and the choice of $x$, so we are done. Assume then that $i < n$. Then $X \in B_n$ and $|\partial X| > i$, by (iii) and the definition of $B_n$. Since $\partial X \cap X^i \subseteq \partial X^i$, this means that

$$|\partial X| = i + 1$$

and

$$\partial X = \partial X^i \cup \{x\}.$$
Let $Y \in \mathcal{B}_n$ be maximal with $X \subseteq Y$ and

$$|\partial Y| = i + 1;$$

this set $Y$ will later become $X^{i+1}$.

By Menger’s theorem (3.3.1), there exist a set $P$ of disjoint $X-\partial Y$ paths in $G[Y]$ and a set $S \subseteq Y$ which separates $X$ from $\partial Y$ in $G[Y]$ and contains exactly one vertex from each path in $P$ (but no other vertices).

Let $Z$ denote the union of $S$ with the vertex sets of the components of $G-S$ that meet $X$. Clearly,

$$\partial Z \subseteq S$$

and $X^i \subseteq X \subseteq Z$; let us show that even

$$X \subseteq Z \subseteq Y.$$  

Let $z \in Z$ be given. If $z \in S$, then $z \in Y$ by the choice of $S$. If $z \notin S$, then $z$ can be reached from $X$ by a path avoiding $S$. If $z \notin Y$, then by $X \subseteq Y$ this path contains an $X-\partial Y$ path in $G[Y]$, contradicting the definition of $S$.

Thus $Z \subseteq Y \in \mathcal{B}_n$, so $Z \in \mathcal{B}_n$ by Lemma 12.4.7 applied to the $Z-\partial Y$ paths contained in the paths from $P$. By (iii), $i < |\partial Z| \leq |S| = |P|$. As every path in $P$ meets $\partial X$, this gives $i < |P| \leq |\partial X| = i + 1$ and hence

$$|P| = i + 1,$$

so $P$ links $\partial X$ to $\partial Y$ bijectively.

We now define $X^{i+1}$. For $1 \leq k \leq i$ let $X^{i+1}_k := X^i_k \cup V(P_k)$, where $P_k$ is the path in $P$ containing the unique vertex of $\partial X^i$ in $X^i_k$; cf. (ii). Similarly, let $X^{i+1}_0$ be the vertex set of the path in $P$ that contains $x$. Finally, put $X^{i+1}_0 := Y \setminus (X^{i+1}_1 \cup \ldots \cup X^{i+1}_i)$. Clearly,

$$X^{i+1} = Y.$$

Condition (i) for $i + 1$ holds by choice of $x$; (ii) holds by $X^{i+1} = Y$ and definition of $P$; (iii) holds by $X^{i+1} = Y$ and the choice of $Y$, combined with $X^i \subseteq Y$ and (iii) for $i$.

As remarked earlier, $F \preceq G$ follows from the definition of $X$ when $i = n$. \qed