Ends and tangles

To the memory of Rudolf Halin

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We show that an arbitrary infinite graph can be compactified by its $\aleph_0$-tangles in much the same way as the ends of a locally finite graph compactify it in its Freudenthal compactification. In general, the ends then appear as a subset of its $\aleph_0$-tangles.

The $\aleph_0$-tangles of a graph are shown to form an inverse limit of the ultrafilters on the sets of components obtained by deleting a finite set of vertices. The $\aleph_0$-tangles that are ends are precisely the limits of principal ultrafilters.

The $\aleph_0$-tangles that correspond to a highly connected part, or $\aleph_0$-block, of the graph are shown to be precisely those that are closed in the topological space of its finite-order separations.

Introduction

Much of Halin's legacy in graph theory stems from the fact that, in his seminal paper [10] of 1964, he initiated the study of *ends* for infinite graphs. Our aim in this paper is to unify this notion with that of a *tangle* introduced by Robertson and Seymour [15] in 1991. It turns out that Halin's ends can be viewed as a special case of tangles of infinite order. These can, in turn, be used to compactify an arbitrary infinite graph in the same way as ends compactify locally finite graphs in their well-known Freudenthal compactification.

Inspired by Carathéodory's notion of *Primenden* of regions in the complex plane [1], but unaware of Freudenthal's [8] generalization of these to more general locally compact Hausdorff spaces, Halin [10] defined an *end* of a graph G as an equivalence class of *rays*, or 1-way infinite paths, in G. Here, two rays are *equivalent* if no finite set of vertices separates them in G.

Halin does not require these graphs to be locally finite. If they are, his notion of an end is equivalent to Freudenthal's. If a graph is locally finite and connected, there is a natural topology that makes it and its ends into a compact space, its *Freudenthal compactification* [4, 5]. The rays of an end, in Halin's definition, then converge to their ends. If the graph is not locally finite, Halin's notion of an end no longer agrees with Freudenthal's [9] but is more general [6]. There is no longer an obvious topology on the graph and its ends, in either definition, that makes rays converge to 'their' ends – let alone one that makes the graph and its ends compact.

An end ω of a graph G orients its separations $\{A, B\}$ of finite order in that either every ray from ω has a tail in A, or every ray from ω has a tail in B. These orientations of finite-order separations are *consistent* in a number of ways; for example, if $\{C, D\}$ is another such separation with $C \subseteq A$ and $D \supseteq B$, then ω must orient $\{C, D\}$ towards D if it orients $\{A, B\}$ towards B.

Robertson and Seymour [15], independently, introduced another notion of consistently orienting all the low-order separations of G: the notion of a *tangle*. It is easy to see that every end ω defines an $\aleph_0$-tangle, one that orients all the separations of finite order: if we orient them all towards the side where the rays in ω have their tails, we obtain an $\aleph_0$-tangle. Conversely, every $\aleph_0$-tangle of a locally finite and connected graph G is defined by an end in this way. Thus, if G is locally finite, its $\aleph_0$-tangles are just another way of identifying the points at infinity in its Freudenthal compactification $|G|$.

When G is not locally finite, however, things get interesting. Now adding its ends no longer compactifies G. But there can also be $\aleph_0$-tangles that are not defined by an end. And it turns out that adding these, in addition to the ends, does compactify G. More precisely, we shall define a topology on the union of G, viewed as a 1-complex, and its set Θ of $\aleph_0$-tangles so that the resulting space $|G| = G \cup \Theta$ satisfies the following:

Theorem 1. *Let G be any graph.*

(i) *$|G|$ is a compact space in which G is dense and $|G| \setminus G$ is totally disconnected.*

(ii) *If G is locally finite and connected, then all its $\aleph_0$-tangles are ends, and $|G|$ coincides with the Freudenthal compactification of G.*

This compactification of G differs from its Stone-Čech compactification and from other compactifications that have been suggested for graphs that are not locally finite, e.g. by Cartwright, Soardi and Woess [3] or by Polat [13].

In order to prove Theorem 1, we shall need to understand the tangles that are not ends. When X is a finite set of vertices of G, then every bipartition of the set $\mathcal{C}_X$ of the components of $G - X$ defines a finite-order separation of G: if $\mathcal{C}_X = \mathcal{C}_1 \cup \mathcal{C}_2$, say, then $\{\bigcup \mathcal{C}_1 \cup X, \bigcup \mathcal{C}_2 \cup X\}$ is such a separation. Every ultrafilter U on $\mathcal{C}_X$ contains exactly one $\mathcal{C}_i$ from such a bipartition. If we think of U as orienting the corresponding separation of G towards this $\mathcal{C}_i$, and there are no other finite-order separations in G, then U defines an $\aleph_0$-tangle of G. Conversely, every $\aleph_0$-tangle of G will orient all the separations of the above form in such a way that the $\mathcal{C}_i$ it points to form an ultrafilter on $\mathcal{C}_X$. We shall call all the ultrafilters on such sets $\mathcal{C}_X$ with X finite the *ultrafilters of cofinite components* of G.

Of course, there will normally be other finite-order separations $\{A, B\}$ of G. But each of these also has the above form, with $X := A \cap B$. And it turns out that the ultrafilters U_X which a tangle defines for different choices of X are all compatible in a simple and natural way: they form the limits $(\, U_X \mid X \in \mathcal{X}\,)$ of a natural inverse system $(\, \mathcal{U}_X \mid X \in \mathcal{X}\,)$ of the sets of ultrafilters of cofinite components (where $\mathcal{X}$ is the set of finite sets of vertices of G). And conversely, every such limit of ultrafilters comes from a tangle:

Theorem 2. *Let G be any graph.*

(i) *The $\aleph_0$-tangles of G are precisely the limits of the inverse system of its sets of ultrafilters of cofinite components.*

(ii) *The ends of G are precisely those of its $\aleph_0$-tangles whose ultrafilters of cofinite components are all principal.*

When a tangle is defined by an end, every ray in that end can be used as an 'oracle' to determine which of the two orientations of a given separation lies in the tangle: it is the orientation that points to a tail of that ray. For tangles that are not defined by an end we can use an ultrafilter in a similar way:

Theorem 3. *Every $\aleph_0$-tangle τ in G satisfies exactly one of the following:*

- *There is a ray R in G such that every separation in τ points to a tail of R.*
- *There is a non-principal ultrafilter U of cofinite components in G such that every separation in τ points to an element of U.*

In a finite graph, the tangles of order some fixed $k \in \mathbb{N}$, those that orient every separation of order $< k$, can be thought of as pointing towards some 'highly connected substructure' of that graph. This is clearly not the case for all $\aleph_0$-tangles: the ends of a tree, for example, are hardly highly connected substructures. However, we can identify those $\aleph_0$-tangles of an infinite graph G that do point to a highly connected substructure in a meaningful sense: they are the $\aleph_0$-tangles that are closed in the set $\vec{S}$ of all oriented separations of finite order of the graph, with respect to a natural topology on this set.

End tangles of trees are not closed in this topology, but $\aleph_0$-tangles pointing to some fixed infinite complete subgraph, for example, are. More generally, tangles pointing to a fixed $\aleph_0$-block are closed: a *κ-block* in a graph G is a maximal set of at least κ vertices no two of which can be separated in G by fewer than κ vertices. As it turns out, the ends defined by an $\aleph_0$-block are precisely the $\aleph_0$-tangles that are closed in $\vec{S}$:

Theorem 4. *Let G be any graph.*

(i) *The $\aleph_0$-tangles of G that are not ends are never closed in $\vec{S}$.*

(ii) *An end tangle of G is closed in $\vec{S}$ if and only if it is defined by an $\aleph_0$-block.*

The paper is organized as follows. We begin in Section 1 with a review of tangles, especially those of infinite order, and their relationship to ends. In Section 2 we introduce the inverse system of the sets $\mathcal{U}_X$ of ultrafilters of cofinite components, and prove Theorem 2. In Section 3 we take a closer look at how tangles not defined by an end arise in this inverse system: we show that each of them is already determined by a single non-principal ultrafilter among those of which it is a limit. In Section 4 we topologize the set of $\aleph_0$-tangles by viewing it as $\varprojlim \mathcal{U}_X$, with the $\mathcal{U}_X$ carrying the Stone-Čech topology. We then extend this space to include G itself, and prove Theorem 1. In Section 5, finally, we introduce a topology on the set $\vec{S}$ of finite-order separations of a graph, and prove Theorem 4. We close with a short section on the potential for applications of Theorem 1 and possible further questions, one of which appears to be quite far-reaching.

Any graph-theoretic notation not explained here can be found in [5]. Inverse systems and inverse limits, including their topology, are explained in [14].

1. Tangles and ends

A *separation* of a graph $G = (V, E)$ is a set $\{A, B\}$ such that $A \cup B = V$ and G has no edge between $A \smallsetminus B$ and $B \smallsetminus A$. The *order* of a separation $\{A, B\}$, and of its orientations (see below), is the cardinal number $|A \cap B|$.

The ordered pairs (A, B) and (B, A) are the *orientations* of a separation $\{A, B\}$ and are also called *(oriented) separations*. Given a set S of separations, we write $\vec{S}$ for the set of their orientations. An *orientation of* S is a subset O of $\vec{S}$ that contains for every $\{A, B\} \in S$ exactly one of (A, B) and (B, A).

Mapping every $(A, B) \in \vec{S}$ to its *inverse* (B, A) is an involution on $\vec{S}$ that reverses the partial ordering

$$(A, B) \leqslant (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D,$$

since the above is equivalent to $(D, C) \leqslant (B, A)$. Informally, we think of (A, B) as *pointing towards* B and *away from* A. Similarly, if $(A, B) \leqslant (C, D)$, then (A, B) *points towards* $\{C, D\}$ and its orientations, while (C, D) *points away from* $\{A, B\}$ and its orientations.

A set σ of oriented separations is a *star* if they all point towards each other: if $(A, B) \leqslant (B', A')$ for all distinct $(A, B), (A', B') \in \sigma$. Note that if σ is a star then $\bigcap \{\, B \mid (A, B) \in \sigma \,\}$ contains $A' \cap B'$ for every $(A', B') \in \sigma$.

A set of oriented separations is *consistent* if no two of them point away from each other: if it contains no distinct separations (B, A) and (C, D) with $(A, B) < (C, D)$. For an orientation O of S, consistency is tantamount to being closed down in $\vec{S}$: that $(A, B) < (C, D) \in O$ with $\{A, B\} \in S$ implies $(A, B) \in O$.

For the rest of this paper, let $G = (V, E)$ be a fixed infinite graph. Let $S = S_{\aleph_0}$ be the set of all its separations of finite order, and let $\mathcal{S}$ denote the set of stars in $\vec{S}$. Let $\mathcal{X}$ be the set of finite subsets of V. As usual, we write $\Omega = \Omega(G)$ for the set of ends of G, defined as in the Introduction or in [5].

Lemma 1.1. *Every consistent orientation O of S contains every separation (A, V) of G with A finite.*

Proof. Pick $v \in V \smallsetminus A$, and let $A' := A \cup \{v\}$. Since O contains one of (A', V) and (V, A'), and $(A, V) < (A', V)$ as well as $(A, V) < (V, A')$, the consistency of O requires that $(V, A) \notin O$. Hence $(A, V) \in O$, as claimed. $\square$

Given a set $\mathcal{F} \subseteq \mathcal{S}$ of stars of separations, we call a consistent orientation of S an *$\mathcal{F}$-tangle* if it has no subset in $\mathcal{F}$. Let us consider some particular choices for $\mathcal{F}$. For integers $n \geqslant 1$, let

$$\mathcal{T}_n := \left\{ \{(A_1, B_1), \ldots, (A_n, B_n)\} \in \mathcal{S} : B_1 \cap \ldots \cap B_n \text{ is finite} \right\}.$$

These (A_i, B_i) need not be distinct, so $\mathcal{T}_n$ is a set of stars in $\vec{S}$ of up to n separations each. In particular, $\mathcal{T}_m \subseteq \mathcal{T}_n$ for all $m \leqslant n$, so every $\mathcal{T}_n$-tangle of S is also a $\mathcal{T}_m$-tangle.

Let us note the following observation for later use:

Lemma 1.2. *Any $\mathcal{T}_3$-tangle O of S containing two separations $(A, B), (A', B')$ also contains $(A \cup A', B \cap B')$.*

Proof. $\{A \cup A', B \cap B'\}$ is a clearly a separation of G. It lies in S, because its separator is a subset of $(A \cap B) \cup (A' \cap B')$ and hence has finite order. Suppose $(B \cap B', A \cup A') \in O$. Since also $(A \cap B', B \cup A') \leqslant (A, B) \in O$ lies in O, because O is consistent, and (A', B') does by assumption, we have found three separations in O forming a star in $\mathcal{T}_3$, a contradiction. □

Robertson and Seymour [15] defined tangles slightly differently, as follows.[0] Given a cardinal κ, a *tangle of order κ*, or *κ-tangle*, of G is an orientation of the set S_κ of its separations of order $< \kappa$ that has no subset of the form

$$\{ (A_1, B_1), (A_2, B_2), (A_3, B_3) : G[A_1] \cup G[A_2] \cup G[A_3] = G \}, \tag{T}$$

where $G[A_i]$ denotes the subgraph of G induced by A_i. As before, the (A_i, B_i) need not be distinct, so in particular every κ-tangle is consistent.

Our $\mathcal{T}_3$-tangles of $S = S_{\aleph_0}$, however, coincide with its $\aleph_0$-tangles as defined by Robertson and Seymour:

Lemma 1.3. *The $\mathcal{T}_3$-tangles of S are precisely the $\aleph_0$-tangles of G.*

Proof. As remarked, both $\mathcal{T}_3$-tangles of S and $\aleph_0$-tangles of G are consistent orientations of S. It was shown in [7, Lemma 5.2] that consistent orientations of S with no star subset as in (T) have no subset as in (T) at all, star or not. Since any set as in (T) satisfies

$$\textstyle\bigcap_{i=1}^3 B_i = V \cap \bigcap_{i=1}^3 B_i = \bigcup_{i=1}^3 A_i \cap \bigcap_{i=1}^3 B_i \subseteq \bigcup_{i=1}^3 (A_i \cap B_i),$$

which is finite, all $\mathcal{T}_3$-tangles of S are $\aleph_0$-tangles of G.

Conversely, suppose some consistent orientation O of S is not a $\mathcal{T}_3$-tangle but contains a star $\sigma = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\} \in \mathcal{T}_3$. We show that O

[0] Formally, their definition of a k-tangle (for finite k and G) differs slightly from our definition below, but is easily seen to be equivalent.

also has a subset as in (T), and hence is not an $\aleph_0$-tangle. As $X := B_1 \cap B_2 \cap B_3$ is finite, we have $(A', B_1) := (A_1 \cup X, B_1) \in \vec{S}$. Using that σ is a star, one easily checks that $G[A'] \cup G[A_2] \cup G[A_3] = G$. Hence if $(A', B_1) \in O$, then

$$\{(A', B_1), (A_2, B_2), (A_3, B_3)\}$$

is our desired subset of O as in (T). But if not, then $(B_1, A') \in O$. But then $\{(A_1, B_1), (B_1, A')\}$ is a subset of O as in (T), since $G[A_1] \cup G[B_1] = G$. □

We noted earlier that, trivially, every $\mathcal{T}_n$-tangle of S is also a $\mathcal{T}_m$-tangle for all $m \leqslant n$. By the particular nature of our $S = S_{\aleph_0}$, we also have a converse:

Lemma 1.4. *Every $\mathcal{T}_n$-tangle of S is also a $\mathcal{T}_{n+1}$-tangle, for all $n \geqslant 3$.*

Proof. Let $O \subseteq \vec{S}$ be a $\mathcal{T}_n$-tangle of S, and let $\sigma \in \mathcal{T}_{n+1}$ be given; we have to show that $\sigma \nsubseteq O$. Pick distinct $(A_1, B_1), (A_2, B_2) \in \sigma$. We may assume that $(A_1, B_1), (A_2, B_2) \in O$, as otherwise $\sigma \nsubseteq O$ as desired.

Let $A := A_1 \cup A_2$ and $B := B_1 \cap B_2$, and let σ' be obtained from σ by replacing (A_1, B_1) and (A_2, B_2) with (A, B). It is easy to check that, since σ is a star in $\mathcal{S}$, so is σ'. Hence $\sigma' \in \mathcal{T}_n$, giving $\sigma' \nsubseteq O$ by the choice of O.

Hence if $\sigma \subseteq O$ then $(A, B) \notin O$, and thus $(B, A) \in O$. (Here we use that $\{A, B\} \in S$, which can fail for arbitrary S but clearly holds for our $S = S_{\aleph_0}$.) But now O contains the set $\{(B, A), (A_1, B_2), (A_2, B_2)\} \in \mathcal{T}_3 \subseteq \mathcal{T}_n$, contrary to its definition. Thus, $\sigma \nsubseteq O$ as desired. □

For $\mathcal{T}_{<\aleph_0} := \bigcup_{n=3}^{\infty} \mathcal{T}_n$, Lemma 1.4 implies

Corollary 1.5. *The $\mathcal{T}_{<\aleph_0}$-tangles of S are exactly its $\mathcal{T}_3$-tangles.* □

As we have seen, all choices of $\mathcal{T}_n$ or of $\mathcal{T}_{<\aleph_0}$ as $\mathcal{F}$ yield the same $\mathcal{F}$-tangles: if a consistent orientation of S avoids all 3-stars whose target sides have a finite intersection, it avoids all finite such stars. In view of Lemma 1.3, we shall from now on refer to all these $\mathcal{F}$-tangles of S as the $\aleph_0$-*tangles* in G, and write $\Theta = \Theta(G)$ for the set of all these.

But how about excluding infinite stars of separations as well? Let

$$\mathcal{T} := \Big\{ \{(A_i, B_i) \mid i \in I\} \in \mathcal{S} : \bigcap_{i \in I} B_i \text{ is finite} \Big\},$$

where the I are arbitrary index sets. Thus $\mathcal{T}_{<\aleph_0} \subseteq \mathcal{T}$, and hence all $\mathcal{T}$-tangles are $\aleph_0$-tangles. Unlike in Lemma 1.4, the converse does not hold. But before we give an example, let us show something more surprising: the $\mathcal{T}$-tangles of S correspond precisely to the ends of G!

One way of this correspondence is easy. Given an end ω of G, exactly one of the two orientations (A, B) of each separation in S has the property that B contains a tail of every ray in ω (for which we say that ω *lives in* B): this is immediate from the notion of ray-equivalence in the definition of an end. Hence ω defines an orientation of S, which is easily seen to be consistent. This orientation τ has no subset $\sigma = \{(A_i, B_i) \mid i \in I\}$ in $\mathcal{T}$. Indeed, as $X := \bigcap_i B_i$ is finite, our end ω lives in some component C of $G - X$. If $C \cap A_i \neq \emptyset$, say,[1] then $C \subseteq A_i \setminus B_i$, because $A_i \cap B_i \subseteq X$ since σ is a star. Therefore $(B_i, A_i) \in \tau$, and hence $\sigma \nsubseteq \tau$ as claimed.

We have thus defined a map

$$\omega \mapsto \tau_\omega \tag{1}$$

from the ends of G to its $\aleph_0$-tangles, whose images are in fact $\mathcal{T}$-tangles. We shall say that τ_ω is *defined by* ω.

The map in (1) is clearly injective. Indeed, distinct ends are, by definition, distinguished by some $X \in \mathcal{X}$ in the sense that they live in different components of $G - X$. Then any separation $\{A, B\} \in S$ with $A \cap B = X$ for which these components lie on different sides will get oriented differently by the corresponding two $\aleph_0$-tangles.

Conversely, every $\mathcal{T}$-tangle is defined by an end in this way. To prove this we need a result from [6] (see also [16]), which requires another definition.

A *direction* in G is a function f that assigns to every finite set $X \subseteq V$ one of the components of $G - X$ so that $f(X') \subseteq f(X)$ whenever $X \subseteq X'$. Clearly, every end ω of G defines a direction f by taking as $f(X)$ the unique component of $G - X$ in which ω lives. It was shown in [6] that this map

$$\omega \mapsto f_\omega \tag{2}$$

is a bijection: not only do different ends define different directions (which is immediate), but every direction is defined by an end in the way indicated. Hence all we have to show is that the $\mathcal{T}$-tangles in a graph correspond to its directions:

Lemma 1.6. *For every $\mathcal{T}$-tangle τ of S there is a unique direction $f = f_\tau$ in G such that, for every $X \in \mathcal{X}$ and every component C of $G - X$, we have $(V \setminus C, X \cup C) \in \tau$ if and only if $C = f(X)$. This map*

$$\tau \mapsto f_\tau \tag{3}$$

is a bijection from the $\mathcal{T}$-tangles of S to the directions in G, which commutes with the maps from (1) *and* (2).

Proof. To define f given τ, let $X \in \mathcal{X}$ be given. There is a unique component C of $G - X$ such that $(V \setminus C, X \cup C) \in \tau$: existence follows from $\tau \in \mathcal{T}$, while uniqueness follows from the consistency of τ. Let $f(X) := C$.

[1] To simplify notation, we do not always distinguish between graphs and their vertex sets.

To show that $f\colon X \mapsto C$ is a direction of G, consider $X \subseteq X' \in \mathcal{X}$. As $C' := f(X')$ is connected, it lies inside a component of $G - X$. If this component was not C, the separations $(V \smallsetminus C, X \cup C), (V \smallsetminus C', X' \cup C') \in \tau$ would make τ inconsistent, contradicting our assumptions.

$$\begin{tikzcd} \omega \arrow[r, mapsto, "(1)"'] \arrow[rr, bend right, "(2)"'] & \tau \arrow[r, mapsto, "(3)"'] & f \end{tikzcd}$$

FIGURE 1. Ends, tangles, and directions

Clearly, our map $\tau \mapsto f$ composes with the map $\omega \mapsto \tau$ from (1) as shown in Figure 1, to yield the bijection $\omega \mapsto f$ from (2). In particular, it is surjective. In order to show that it is injective, consider distinct $\mathcal{T}$-tangles of S, say τ and τ'. Let $\{A, B\} \in S$ be a separation which they orient differently, with $(A, B) \in \tau$ and $(B, A) \in \tau'$ say. Since τ is consistent, f_τ maps $X = A \cap B$ to a component of $G - X$ contained in $B \smallsetminus A$. Similarly, $f_{\tau'}$ maps X to a component of $G - X$ contained in $A \smallsetminus B$. Thus $f_\tau \neq f_{\tau'}$, as desired. $\square$

Corollary 1.7. *The map $\omega \to \tau_\omega$ defined in* (1) *is a bijection from the ends of G to its $\mathcal{T}$-tangles.* $\square$

Corollary 1.7 says that those $\mathcal{T}_{<\aleph_0}$-tangles of S that are even $\mathcal{T}$-tangles are precisely the $\aleph_0$-tangles that are defined by an end via (1). We shall call these $\aleph_0$-tangles the *end tangles* of G. Let us look at an example.

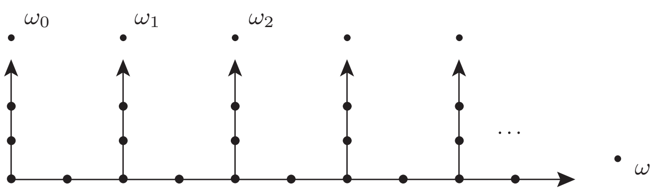

FIGURE 2. All $\aleph_0$-tangles in this graph are end tangles

Example 1.8. If G is the *comb of rays* shown in Figure 2, then every $\aleph_0$-tangle τ in G is an end tangle. Indeed, if there exists an $i \in \mathbb{N}$ such that ω_i lives in B for every $(A, B) \in \tau$, then $\omega_i \mapsto \tau$ in the map of (1), so τ is an end tangle. Suppose, then, that there is no such i.

We claim that ω lives in every B with $(A, B) \in \tau$, so that $\omega \mapsto \tau$ in (1), again making τ an end tangle. If not then, for some $(A, B) \in \tau$, the end ω lives in A. Then only finitely many ω_i live in B. By assumption, each of these ω_i lives in A_i for some $(A_i, B_i) \in \tau$. By the consistency of τ, these (A_i, B_i) can be chosen so as to form a (finite) star $\sigma \subseteq \tau$ together with (A, B). But the intersection of B with these B_i is finite. Hence $\tau \supseteq \sigma \in \mathcal{T}_{<\aleph_0}$, so τ is not an $\aleph_0$-tangle, contrary to our assumption. $\square$

Next, let us see an example of an $\aleph_0$-tangle that is not an end tangle: a consistent orientation of S that has infinite but no finite stars in $\mathcal{T}$.

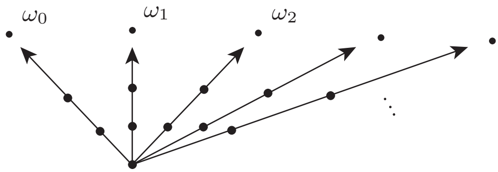

FIGURE 3. This graph has both end and ultrafilter tangles

Example 1.9. If G is the graph shown in Figure 3, it has $\aleph_0$-tangles that are not end tangles. Indeed, let U be any ultrafilter on $\mathbb{N}$. Every separation $\{A, B\} \in S$ induces a bipartition of $\mathbb{N}$ into

$$\bar{A} = \{\, i \in \mathbb{N} \mid \omega_i \text{ lives in } A \,\} \quad \text{and} \quad \bar{B} = \{\, i \in \mathbb{N} \mid \omega_i \text{ lives in } B \,\}.$$

As U is an ultrafilter, exactly one of these sets is an element of U. Hence

$$\tau = \{\, (A, B) \in \vec{S} \mid \bar{B} \in U \,\}$$

is an orientation of S. Since the intersection of two sets in U lies in U and hence is non-empty, τ is consistent. Similarly, let $\sigma \subseteq \tau$ be a finite star. The set of all $i \in \mathbb{N}$ whose ω_i lives in every B with $(A, B) \in \sigma$ is a finite intersection of sets in U, and hence is non-empty. Consider any i in this set, and a ray in ω_i. This ray has a tail outside every A with $(A, B) \in \sigma$, and hence has a tail in $\bigcap \{B \mid (A, B) \in \sigma\}$. In particular, this intersection is infinite, and hence $\sigma \notin \mathcal{T}_{<\aleph_0}$. Thus, τ is indeed an $\aleph_0$-tangle.

If U is a principal ultrafilter generated by $\{n\}$, say, then τ is an end tangle defined by ω_n. If U is a non-principal ultrafilter, then τ is not defined by any ω_i and hence is not an end tangle. □

We shall see in Section 3 that every $\aleph_0$-tangle that is not an end tangle is defined by a non-principal ultrafilter in a way similar to Example 1.9.

We conclude this section with a couple of simple lemmas about $\aleph_0$-tangles. The first is that if we change a separation in an $\aleph_0$-tangle τ only finitely, the resulting separation will again lie in τ. For sets $A, A' \subseteq V$ let us write $A \sim A'$ if their symmetric difference is finite.

Lemma 1.10. *Let τ be an $\aleph_0$-tangle of S and $(A, B) \in \tau$. Let (A', B') be a separation such that $A' \sim A$ and $B' \sim B$. Then $(A', B') \in \tau$.*

Proof. It suffices to show that $(A, B) \in \tau$ implies $(A \cup A', B \cup B') \in \tau$: then also $(A', B') \in \tau$, since otherwise $(B', A') \in \tau$ and therefore $(B \cup B', A \cup A') \in \tau$.

As $(A, B \cup B') \leqslant (A, B)$, we have $(A, B \cup B') \in \tau$ by the consistency of τ. But $\{(A, B \cup B'), (B \cup B', A \cup A')\} \in \mathcal{T}_2$. Therefore $(B \cup B', A \cup A') \notin \tau$, and hence $(A \cup A', B \cup B') \in \tau$ as desired. □

Lemma 1.11. *For every $\aleph_0$-tangle τ and $(A,B) \in \tau$, the set B is infinite.*

Proof. If B is finite, then $A \sim V$. Since $(B,V) \in \tau$ by Lemma 1.1, this implies $(B,A) \in \tau$ by Lemma 1.10. $\square$

2. Tangles and ultrafilters

We are considering a fixed infinite graph $G = (V,E)$, with $\mathcal{X}$ denoting the set of finite subsets of V. For each $X \in \mathcal{X}$, write $\mathcal{C}_X$ for the set of components of $G - X$, and let $\mathcal{U}_X$ denote the set of all ultrafilters on $\mathcal{C}_X$.

Our aim in this section is to study those $\aleph_0$-tangles of G that are not end tangles. We shall see that they define ultrafilters on the sets $\mathcal{C}_X$, and conversely that every way of choosing such ultrafilters consistently defines an $\aleph_0$-tangle. More precisely, we shall see that the $\aleph_0$-tangles of G correspond to the points of a natural inverse limit of the sets $\mathcal{U}_X$, with the end tangles among them corresponding to the limits of principal ultrafilters.

Recall that every end tangle $\tau = \tau_\omega$ of G defines a direction f in G: a way of choosing for every $X \in \mathcal{X}$ one component of $G - X$, the component C in which ω lives. As we saw in Example 1.9, an arbitrary $\aleph_0$-tangle τ may not select one $C \in \mathcal{C}_X$ in this way, but it still cannot orient the separations $\{A,B\}$ with $A \cap B = X$ arbitrarily. Indeed, as τ is consistent and has no subset in $\mathcal{T}_{<\aleph_0}$, it defines an ultrafilter on $\mathcal{C}_X$, the collection of all $\mathcal{C} \subseteq \mathcal{C}_X$ that such that $\bigcup \mathcal{C} = B \setminus A$ for some $(A,B) \in \tau$ with $A \cap B = X$:

$$U(\tau, X) := \left\{ \mathcal{C} \subseteq \mathcal{C}_X \mid \left(\bigcup (\mathcal{C}_X \setminus \mathcal{C}) \cup X, \bigcup \mathcal{C} \cup X \right) \in \tau \right\} \in \mathcal{U}_X.$$

It is easy to check that this is indeed an ultrafilter. Every $(A,B) \in \tau$ with $A \cap B = X$ partitions $\mathcal{C}_X$ into two sets, the components in A versus those in B, and τ puts the latter set in the ultrafilter. The intersection of two such subsets of $\mathcal{C}_X$ are also chosen by τ: if $(A,B),(A',B') \in \tau$ then also $(A \cup A', B \cap B') \in \tau$ by Lemma 1.2. By Lemma 1.1, these filter sets are non-empty, and they are closed under taking supersets in $\mathcal{C}_X$ because τ is consistent.

For every $X \in \mathcal{X}$, we thus have a map

$$\tau \mapsto U(\tau, X) \tag{4}$$

from Θ to $\mathcal{U}_X$. These maps are not in general injective: if some $C \in \mathcal{C}_X$ is home to more than one end, for example, then $\{C\}$ will generate $U(\tau, X)$ for all the corresponding end tangles τ. However, we shall see in Lemma 3.3 that distinct $\tau, \tau' \in \Theta$ can never map to the same non-principal ultrafilter as in (4).

We shall also see later in Lemma 3.7 that the maps in (4) are nearly surjective: only principal ultrafilters in $\mathcal{U}_X$ generated by $\{C\}$ for a *finite* component C of $G - X$ are not of the form $U(\tau, X)$.

The ultrafilters $U(\tau, X)$ for a given τ but variable X are compatible for $X \subseteq X'$ just as the choices of $f(X) \in \mathcal{C}_X$ and $f(X') \in \mathcal{C}_{X'}$ are compatible when f is a direction in G. Indeed, consider the maps

$$f_{X',X} : \mathcal{U}_{X'} \to \mathcal{U}_X$$

defined for all $X \subseteq X' \in \mathcal{X}$ by mapping an ultrafilter $U' \in \mathcal{U}_{X'}$ to $U' \restriction X \subseteq 2^{\mathcal{C}_X}$, where $U' \restriction X$ is the set of all supersets in $\mathcal{C}_X$ of sets of the form $\mathcal{C}' \restriction X$ with $\mathcal{C}' \in U'$ and $\mathcal{C}' \restriction X := \{ C \in \mathcal{C}_X \mid \exists C' \in \mathcal{C}' : C' \subseteq C \}$. Less formally, from an ultrafilter $U' \in \mathcal{U}_{X'}$ we obtain an ultrafilter $U = f_{X',X}(U') \in \mathcal{U}_X$ by putting the following sets $\mathcal{C} \subseteq \mathcal{C}_X$ in U: pick some $\mathcal{C}' \in U'$, and let $\mathcal{C}$ consist of at least all those components of $G - X$ that contain some $C' \in \mathcal{C}'$ as a subset (Fig. 4).

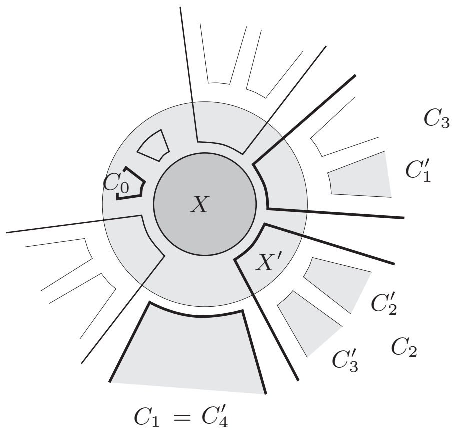

FIGURE 4. $\mathcal{C} = \{C_0, C_1, C_2, C_3\} \in U$ if $\mathcal{C}' = \{C_1', \dots, C_4'\} \in U'$

Lemma 2.1. *The maps $f_{X',X}$ make $(\mathcal{U}_X \mid X \in \mathcal{X})$ into an inverse system, with $\mathcal{X}$ partially ordered by inclusion.*

Proof. It is immediate that for $X \subseteq X' \subseteq X''$ we have $f_{X',X} \circ f_{X'',X'} = f_{X'',X}$, as required for an inverse system, as long as these compositions are defined. But to establish this we first have to show that, given $X \subseteq X'$ and $U' \in \mathcal{U}_{X'}$, the set $U := f_{X',X}(U') \subseteq 2^{\mathcal{C}_X}$ is indeed an ultrafilter.

To this end, notice that every component C' of $G - X'$ lies inside some component C of $G - X$, because C' is connected and does not meet $X \subseteq X'$. Let $f : C' \mapsto C$ be this map from $\mathcal{C}_{X'}$ to $\mathcal{C}_X$. Now consider a partition $\mathcal{C}_X = \mathcal{C}_1 \cup \mathcal{C}_2$. For $i = 1, 2$ define $\mathcal{C}_i' := \{ C' \in \mathcal{C}_{X'} \mid f(C') \in \mathcal{C}_i \}$. Since f has domain all of $\mathcal{C}_{X'}$, the $\mathcal{C}_i'$ partition $\mathcal{C}_{X'}$. As U' is an ultrafilter on $\mathcal{C}_{X'}$, it contains exactly one of the $\mathcal{C}_i'$ as an element. By definition of $f_{X',X}$, the corresponding $\mathcal{C}_i$ lies in U: it is a superset[2] of $\mathcal{C}_i' \restriction X = \{ f(C') \mid C' \in \mathcal{C}_i' \}$.

[2] The superset can be strict if $G - X$ has (finite) components contained in X'.

The other two properties required of an ultrafilter, that it is closed under taking finite intersections and does not contain $\emptyset$, follow for U from the corresponding properties of U' in a similar way. $\square$

For want of a better expression, let us call the ultrafilters in the sets $\mathcal{U}_X$ the *ultrafilters of cofinite components* in G, and put

$$\mathcal{U} := \varprojlim \left(\mathcal{U}_X \mid X \in \mathcal{X} \right).$$

Conveniently, our maps $\tau \mapsto U(\tau, X)$ commute with the maps $f_{X',X}$:

Lemma 2.2. *For all $X \subseteq X' \in \mathcal{X}$ and all $\aleph_0$-tangles τ we have*

$$f_{X',X}\big(U(\tau, X')\big) = U(\tau, X).$$

Every $\aleph_0$-tangle τ therefore defines a limit $\upsilon_\tau = \left(U_X \mid X \in \mathcal{X} \right) \in \mathcal{U}$ in which $U_X = U(\tau, X)$ for all X.

Proof. Let $U := f_{X',X}\big(U(\tau, X')\big)$, consider any $\mathcal{C} \in U$, and let $(A, B) \in \vec{S}$ be such that $A \cap B = X$ and $B \smallsetminus A = \bigcup \mathcal{C}$. Our aim is to show that $(A, B) \in \tau$: then $\mathcal{C} \in U(\tau, X)$, giving $U \subseteq U(\tau, X)$, with equality since both are ultrafilters.

By definition of $f_{X',X}$, there exists $\mathcal{C}' \in U(\tau, X')$ such that $\mathcal{C} \supseteq \mathcal{C}' \restriction X$. The separation $(A', B') \in \vec{S}$ with $A' \cap B' = X'$ and $B' \smallsetminus A' = \bigcup \mathcal{C}'$ thus lies in τ. Since $(A, B \cup X') \leqslant (A', B')$, this implies $(A, B \cup X') \in \tau$ by the consistency of τ. Now $(A, B) \in \tau$ follows by Lemma 1.10. $\square$

Conversely, every $\upsilon \in \mathcal{U}$ comes from a tangle in this way:

Lemma 2.3. *For every limit $\left(U_X \mid X \in \mathcal{X} \right) \in \mathcal{U}$ there exists a unique $\aleph_0$-tangle τ in G such that $U_X = U(\tau, X)$ for all $X \in \mathcal{X}$. The map*

$$\tau \mapsto \upsilon_\tau \tag{5}$$

defined in Lemma 2.2, therefore, is a bijection from Θ to $\mathcal{U}$.

Proof. Let $\tau := \{ (A, B) \in \vec{S} \mid \mathcal{C}_{A \cap B} \cap 2^B \in U_{A \cap B} \}$. Clearly, $U(\tau, U) = U_X$ for all $X \in \mathcal{X}$ if τ is indeed an $\aleph_0$-tangle, so let us check this; uniqueness will be clear, since distinct $\aleph_0$-tangles τ, τ' differ on some separation $\{A, B\}$, so that $U(\tau, X) \neq U(\tau', X)$ for $X = A \cap B$.

For every separation $\{A, B\} \in S$, the sets $\mathcal{C}_X \cap 2^A$ and $\mathcal{C}_X \cap 2^B$ partition $\mathcal{C}_X$, so U_X contains exactly one of them. Hence, τ is an orientation of S.

To show that τ is consistent, consider $(A, B) < (A', B') \in \tau$. Let $X := A \cap B$ and $X' := A' \cap B'$, and put $X'' := X \cup X'$. Let $\mathcal{C} := \mathcal{C}_X \cap 2^A$ and $\mathcal{C}' := \mathcal{C}_{X'} \cap 2^{B'}$. Note that $\mathcal{C}, \mathcal{C}' \subseteq \mathcal{C}_{X''}$, since $(A, B) \leqslant (A', B')$. Now if $(B, A) \in \tau$, then

$$f_{X'',X} : U_{X''} \mapsto U_X \ni \mathcal{C} \supseteq \mathcal{C}'' \restriction X$$

for some $\mathcal{C}'' \in U_{X''}$. Since $\mathcal{C} \subseteq \mathcal{C}_{X''}$, this means that in fact $\mathcal{C} \supseteq \mathcal{C}''$, and hence that $\mathcal{C} \in U_{X''}$. Similarly, $\mathcal{C}' \in U_{X''}$. But $\mathcal{C} \cap \mathcal{C}' = \emptyset$, a contradiction.

It remains to show that τ has no subset $\sigma = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$ in $\mathcal{T}_3$. Suppose it does. For $i = 1, 2, 3$ let $X_i := A_i \cap B_i$, and put $X := \bigcap_i B_i$. This is finite by definition of $\mathcal{T}_3$, and includes every X_i since σ is a star. Then

$$f_{X,X_i} : U_X \mapsto U_{X_i} \ni \mathcal{C}_{X_i} \cap 2^{B_i} \supseteq \mathcal{C}_i \upharpoonright X_i$$

for some $\mathcal{C}_i \in U_X$, for each i. Every $C \in \mathcal{C}_i$ is a subset of some $C_i \in \mathcal{C}_{X_i} \cap 2^{B_i}$, and hence of B_i. Hence any $C \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$ is both a subset of $\bigcap_i B_i = X$ and an element of $\mathcal{C}_X$, which is impossible. As $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 \in U_X$ is non-empty, this is a contradiction. $\square$

We have seen that every end ω of G defines a tangle τ_ω by (1), which in turn defines a limit $\upsilon \in \mathcal{U}$ by (5). The composition

$$\omega \mapsto \tau_\omega \mapsto \upsilon_{\tau_\omega} \tag{6}$$

maps ω to a limit $\upsilon = (\, U_X \mid X \in \mathcal{X}\,)$ in which every U_X is a principal ultrafilter in $\mathcal{U}_X$, generated by $\{C\}$ say, where C is the component of $G - X$ in which ω lives. The converse of this is also true: if every U_X in $(\, U_X \mid X \in \mathcal{X}\,) = \upsilon_\tau$ is principal, then τ is an end tangle:

Lemma 2.4. *The following assertions are equivalent for all $\aleph_0$-tangles τ:*

- *τ is an end tangle;*
- *$U(\tau, X)$ is a principal ultrafilter, for all $X \in \mathcal{X}$.*

Proof. We only have to show the backward implication. For each $X \in \mathcal{X}$, let C_X be the unique component of $G - X$ such that $\{C_X\}$ generates $U(\tau, X)$. By Lemma 2.2, the map $f\colon X \mapsto C_X$ is a direction in G. Since the map $\omega \mapsto f$ is surjective [6], there is an end ω of G such that $f = f_\omega$. This ω lives in every C_X, giving $\tau = \tau_\omega$ as desired. $\square$

Let us call $\tau \in \Theta$ an *ultrafilter tangle* if at least one of the ultrafilters U_X in $\upsilon_\tau = (\, U_X \mid X \in \mathcal{X}\,)$ is non-principal. Of every such X we say that it *witnesses* that τ is an ultrafilter tangle.

Lemma 2.4 tells us that the $\aleph_0$-tangles in G divide into its end tangles and its ultrafilter tangles. We have thus proved Theorem 2:

Theorem 2. *Let G be any graph.*

(i) *The $\aleph_0$-tangles of G are precisely the limits of the inverse system of its sets of ultrafilters of cofinite components.*

(ii) *The ends of G are precisely those of its $\aleph_0$-tangles whose ultrafilters of cofinite components are all principal.*

3. A closer look at ultrafilter tangles

Recall that our aim was to understand better those $\aleph_0$-tangles that are not defined by an end. We have seen that these are the ultrafilter tangles, those $\tau \in \Theta$ such that at least one of the $U_X = U(\tau, X)$ in their $\upsilon_\tau = (\, U_X \mid X \in \mathcal{X}\,)$ is non-principal. In this section we show that each of these U_X already determines τ. Thus, every ultrafilter tangle is determined by a single ultrafilter of cofinite components, not just by a limit of such ultrafilters.

In particular, if $U = U(\tau, X)$ is non-principal and $X \subsetneq X'$, then both υ_τ and $U' = U(\tau, X')$ are determined by U, since U determines τ and τ determines υ_τ and U'. However we shall prove this directly, without involving τ: we show that for every element U of the set

$$\mathcal{U}^*_X := \{\, U \in \mathcal{U}_X \mid U \text{ is non-principal}\,\}$$

there is a unique $U' \in \mathcal{U}_{X'}$ such that $f_{X',X}(U') = U$. (This U' will also lie in $\mathcal{U}^*_{X'}$.) Thus, the maps $f_{X',X}$ have inverses on the sets $\mathcal{U}^*_X$ of non-principal ultrafilters. Hence there is also a unique $\upsilon \in \mathcal{U} = \varprojlim\,(\, U_X \mid X \in \mathcal{X}\,)$ with $U_X = U$.

We finally show that the set $\mathcal{X}_\tau$ of all $X \in \mathcal{X}$ witnessing that a given τ is an ultrafilter tangle has a least element X. From its corresponding $U = U_X$ we can thus directly construct all the ultrafilters of cofinite components induced by τ, the filters $U(\tau, X')$ with $X' \in \mathcal{X}_\tau$, by applying the inverses of the maps $f_{X',X}$.

Lemma 3.1. *Let $X \subsetneq X' \in \mathcal{X}$, and let $U \in \mathcal{U}^*_X$ be a non-principal ultrafilter on $\mathcal{C}_X$. Then there is a unique ultrafilter U' on $\mathcal{C}_{X'}$ such that $f_{X',X}(U') = U$. This ultrafilter $U' \in \mathcal{U}_{X'}$ is also non-principal, and it satisfies*

$$U' = \{\, \mathcal{C} \subseteq \mathcal{C}_{X'} \mid \exists\, \mathcal{D} \in U \colon\ \mathcal{D} \subseteq \mathcal{C}\,\}.$$

Proof. As $f_{X',X}$ maps principal ultrafilters to principal ultrafilters, it is clear that any $U' \in \mathcal{U}_{X'}$ satisfying $f_{X',X}(U') = U$ lies in $\mathcal{U}^*_{X'}$. Let us show that there is a unique such $U' \in \mathcal{U}_{X'}$.

Consider any bipartition $\mathcal{C}_{X'} = \mathcal{C} \cup \mathcal{C}'$. Since every component of $G - X$ that does not meet X' is also a component of $G - X'$, our partition of $\mathcal{C}_{X'}$ induces a partition $\mathcal{D}^- \cup \mathcal{D} \cup \mathcal{D}'$ of $\mathcal{C}_X$, where $\mathcal{D}^-$ is the set of components of $G - X$ meeting X', and $\mathcal{D} \subseteq \mathcal{C}$ and $\mathcal{D}' \subseteq \mathcal{C}'$.

As U is non-principal, it does not contain the finite set $\mathcal{D}^-$. Hence exactly one of $\mathcal{D}$ and $\mathcal{D}'$ lies in U, say $\mathcal{D}$. Then any $U' \in \mathcal{U}_{X'}$ satisfying $f_{X',X}(U') = U$ contains $\mathcal{C}$: otherwise it would contain $\mathcal{C}'$, and then U_X would contain $\mathcal{D}^- \cup \mathcal{D}' \supseteq \mathcal{C}' \restriction X$ which it does not.

Let U' be the set of all $\mathcal{C}$ obtained in this way: the set of all $\mathcal{C} \subseteq \mathcal{C}_{X'}$ such that $\mathcal{C} \cap \mathcal{C}_X \in U$. If U' is a filter, it will be an ultrafilter, because it contains a set from every bipartition of $\mathcal{C}_{X'}$. It thus remains to show that U' is indeed a filter, and that it satisfies $f_{X',X}(U') = U$; we have already seen that U' will then be unique in $\mathcal{U}_{X'}$ with this property.

Since every $\mathcal{C} \in U'$ has a subset $\mathcal{D} \in U$, clearly $\emptyset \notin U'$. And for $\mathcal{C}_1, \mathcal{C}_2 \in U'$, with $\mathcal{D}_i \subseteq \mathcal{C}_i$ in U say, we have $U \ni \mathcal{D}_1 \cap \mathcal{D}_2 \subseteq \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_X$, giving $\mathcal{C}_1 \cap \mathcal{C}_2 \in U'$. Thus, $U' \in \mathcal{U}_{X'}$.

It is straightforward from the definitions that $f_{X',X}(U') \subseteq U$. Since both these are ultrafilters on the same set (cf. Lemma 2.1), this implies the desired equality. $\square$

As a consequence of Lemma 3.1, we have inverses of the maps $f_{X',X}$ on the non-principal ultrafilters:

Lemma 3.2. *For all $X \subseteq X' \in \mathcal{X}$ there exists a map $g_{X,X'}\colon \mathcal{U}^*_X \to \mathcal{U}_{X'}$ such that $f_{X',X} \circ g_{X,X'}$ is the identity on $\mathcal{U}^*_X$.* $\square$

Our aim was to show that if $\tau \in \Theta$ is an ultrafilter tangle and X witnesses this, then $U(\tau, X)$ alone determines τ. This is an easy consequence of Lemma 3.1:

Lemma 3.3. *Let $\tau \in \Theta$ and $X \in \mathcal{X}$ be such that $U(\tau, X)$ is non-principal. Then $U(\tau', X) \neq U(\tau, X)$ for every $\tau' \in \Theta \setminus \{\tau\}$.*

Proof. As $\tau \neq \tau'$, there exists $(A, B) \in \tau$ such that $(B, A) \in \tau'$. For $X' := A \cap B$ this gives $U(\tau, X') \neq U(\tau', X')$. Then also $U(\tau, X \cup X') \neq U(\tau', X \cup X')$: if these were the same filter $U \in \mathcal{U}_{X \cup X'}$ we would have $U(\tau, X') = f_{X \cup X', X'}(U) = U(\tau', X')$ by Lemma 2.2. By Lemma 3.1, the fact that $U(\tau, X \cup X') \neq U(\tau', X \cup X')$ implies $U(\tau', X) \neq U(\tau, X)$. $\square$

Every $U \in \mathcal{U}^*_X$ can be used to define the unique τ with $U(\tau, X) = U$ directly, by telling us which orientation of a given separation $\{A, B\} \in S$ lies in τ:

Lemma 3.4. *Let $\tau \in \Theta$ be an ultrafilter tangle, witnessed by $X \in \mathcal{X}$. Then*

$$\tau = \left\{ (A', B') \in \vec{S} \mid \exists \mathcal{C} \in U(\tau, X) : \bigcup \mathcal{C} \subseteq B' \right\}.$$

Proof. For a proof of '$\subseteq$' let $(A', B') \in \tau$ be given, and put $X' := A' \cap B'$. Pick $(A, B) \in \tau$ with $A \cap B = X$. By Lemma 1.2, also $(A'', B'') \in \tau$ for $A'' = A \cup A'$ and $B'' = B \cap B'$. For $X'' = X \cup X'$ then also $(A'', X'' \cup B'') \leqslant (A'', B'')$ lies in τ, by the consistency of τ. The set $\mathcal{C}''$ of components of $G - X''$ contained in B'' then lies in $U(\tau, X'')$. As $f_{X'',X}(U(\tau, X'')) = U(\tau, X)$ by Lemma 2.2, the set $\mathcal{C}''$ has a subset $\mathcal{C} \in U(\tau, X)$ by Lemma 3.1. Then $\bigcup \mathcal{C} \subseteq \bigcup \mathcal{C}'' \subseteq B'' \subseteq B'$ as desired.

Conversely, any $(A', B') \in \vec{S}$ with $\bigcup \mathcal{C} \subseteq B'$ for some $\mathcal{C} \in U(\tau, X)$ must lie in τ: otherwise $(B', A') \in \tau$ with $\bigcup \mathcal{C}' \subseteq A'$ for some $\mathcal{C}' \in U(\tau, X)$, as shown above, but $\mathcal{C} \cap \mathcal{C}'$ is finite and hence not in $U(\tau, X)$, a contradiction. $\square$

Note that Lemma 3.4 also implies Lemma 3.3. Also, we could replace the requirement of $\bigcup \mathcal{C} \subseteq B$ with $\bigcup \mathcal{C} \subseteq B \smallsetminus A$: our proof yield this directly, but it also follows retrospectively since U is non-principal and $A \cap B$ can meet only finitely many elements of $\mathcal{C}$.

For an end tangle $\tau = \tau_\omega$, a single ray $R \in \omega$ can be used as an 'oracle' to determine which of the two orientations of a given separation $\{A, B\} \in S$ is in τ: it is the unique orientation that points to a tail of R. Lemma 3.4 says that for ultrafilter tangles we have similar oracles, given by a single ultrafilter in

$$\mathcal{U}_{\mathcal{X}} := \bigcup_{X \in \mathcal{X}} \mathcal{U}_X.$$

We have thus proved the following more precise version of Theorem 3:

Theorem 3.5. *Every $\aleph_0$-tangle τ in G satisfies exactly one of the following:*

- *$\exists$ ray $R \subseteq G$ such that $\tau = \{ (A, B) \in \vec{S} : G[B]$ contains a tail of $R \}$;*
- *$\exists$ ultrafilter $U \in \mathcal{U}_{\mathcal{X}}$ such that $\tau = \{ (A, B) \in \vec{S} \mid \exists \mathcal{C} \in U : \bigcup \mathcal{C} \subseteq B \smallsetminus A \}$.*

Proof. We have shown everything claimed, except that no tangle can satisfy both statements. But this is clear: an ultrafilter U as in the second statement cannot be principal since, given its generating set $\{C\}$, we can choose $\{A, B\} \in S$ with $A \cap B \cap C \neq \emptyset$, in which case $\bigcup \mathcal{C} \supseteq C$ will not be contained in either $B \smallsetminus A$ or $A \smallsetminus B$, so $\tau \cap \{(A, B), (B, A)\} = \emptyset$, a contradiction. □

Let us now prove that, for every ultrafilter tangle $\tau \in \Theta$, the set

$$\mathcal{X}_\tau := \{ X \in \mathcal{X} \mid U(\tau, X) \in \mathcal{U}^*_X \}$$

of all X witnessing that τ is an ultrafilter tangle is the up-closure in $\mathcal{X}$ of a single element:

Theorem 3.6. *For every ultrafilter tangle τ the set $\mathcal{X}_\tau$ has a least element X_τ. Then $\mathcal{X}_\tau = \{ X \in \mathcal{X} \mid X_\tau \subseteq X \}$.*

Proof. We already noted in Lemma 3.1 that $\mathcal{X}_\tau$ is closed upwards in $\mathcal{X}$. It remains to show that it has a least element.

Suppose not. Let X', X'' be incomparable minimal elements of $\mathcal{X}_\tau$. Pick $x' \in X' \smallsetminus X''$, and let $X := X' \smallsetminus \{x'\}$. By the minimality of X', the ultrafilter $U(\tau, X)$ is principal, and hence generated by $\{C\}$ for some $C \in \mathcal{C}_X$. As $U(\tau, X')$ is non-principal and $f_{X',X}(U(\tau, X')) = U(\tau, X)$, the set $\mathcal{C}$ of components of $C - x'$ lies in $U(\tau, X')$. As X'' meets only finitely many elements of $\mathcal{C}$, the others form a set $\mathcal{C}' \in U(\tau, X')$. Similarly, pick $x'' \in X'' \smallsetminus X'$ and find a set $\mathcal{C}'' \in U(\tau, X'')$ of components of $G - X''$ that avoid X'.

Every $C' \in \mathcal{C}'$ lies inside the same component C'' of $G - X''$ as x', because both avoid X'' but G contains an edge from x' to C': otherwise, C' would be in $\mathcal{C}_X$, which it is not since $C - x'$ contains it. Since the components in $\mathcal{C}''$ avoid $X' \ni x'$, we thus have $C' \subseteq C'' \notin \mathcal{C}''$ for every $C' \in \mathcal{C}'$.

Thus, $\bigcup \mathcal{C}'$ and $\bigcup \mathcal{C}''$ are disjoint sets of vertices separated by the finite set X'' (as well as by X'); let $\{A, B\} \in S$ with $A \cap B = X''$ separate them. Both orientations of $\{A, B\}$ must be in τ, by Lemma 3.4 applied with X' and with X'', respectively. This contradicts our assumption that τ is an orientation of S. $\square$

Finally, let us go back and use the maps $g_{X,X'}$ from Lemma 3.2 to prove that the maps $\tau \mapsto U(\tau, X)$ in (4) are essentially surjective; we shall need this in our proof of Theorem 1.

Lemma 3.7. *Let $X \in \mathcal{X}$, and let $U \in \mathcal{U}_X$ be an ultrafilter on $\mathcal{C}_X$ not generated by $\{C\}$ for any finite component C of $G - X$. Then there exists an $\aleph_0$-tangle $\tau \in \Theta$ such that $U = U(\tau, X)$.*

Proof. Assume first that U is non-principal. Our aim is to find a limit point $\upsilon = (\, U_Y \mid Y \in \mathcal{X}\,) \in \mathcal{U}$ such that $U_X = U$. By Lemma 2.3 there will then exist some $\tau \in \Theta$ with $\upsilon = \upsilon_\tau$, for which $U(\tau, X) = U_X = U$ as desired.

$$\begin{tikzcd}
U_{X\cup Y'} \arrow[dr, mapsto, "f"] \arrow[dd, mapsto, dashed, "f?"] & \\
& U_{Y'} \arrow[dd, mapsto, dashed, "f?"] \\
U_{X\cup Y} \arrow[dr, mapsto, "f"] & \\
& U_Y \\
U = U_X \arrow[uu, mapsto, "g"] \arrow[uuuu, mapsto, bend left, "g"] &
\end{tikzcd}$$

FIGURE 5. The known maps are drawn as solid lines, the desired maps as broken lines

For every $X' \in \mathcal{X}$ with $X \subseteq X'$ let $U_{X'} := g_{X,X'}(U)$, and for all other $Y \in \mathcal{X}$ let $U_Y := f_{X',Y}(U_{X'})$ for $X' := X \cup Y$ (Fig. 5). Then, in fact,

$$U_Y = f_{X\cup Y,Y} \circ g_{X,X\cup Y}(U)$$

for every $Y \in \mathcal{X}$, and $U_X = U$. To show that $(\, U_Y \mid Y \in \mathcal{X}\,) \in \mathcal{U}$, we have to show that $f_{Y',Y}(U_{Y'}) = U_Y$ for all $Y \subseteq Y' \in \mathcal{X}$. To see this, note first that

$$\begin{aligned}
U_{X\cup Y} &= g_{X,X\cup Y}(U) \\
&= g_{X,X\cup Y}(f_{X\cup Y',X}(U_{X\cup Y'})) \\
&= g_{X,X\cup Y}(f_{X\cup Y,X}(f_{X\cup Y',X\cup Y}(U_{X\cup Y'}))) \\
&= f_{X\cup Y',X\cup Y}(U_{X\cup Y'}).
\end{aligned}$$

Hence

$$\begin{aligned}
f_{Y',Y}(U_{Y'}) &= f_{Y',Y}(f_{X\cup Y',Y'}(g_{X,X\cup Y'}(U))) \\
&= f_{Y',Y}(f_{X\cup Y',Y'}(U_{X\cup Y'})) \\
&= f_{X\cup Y',Y}(U_{X\cup Y'}) \\
&= f_{X\cup Y,Y}(f_{X\cup Y',\, X\cup Y}(U_{X\cup Y'})) \\
&= f_{X\cup Y,Y}(U_{X\cup Y}) \\
&= f_{X\cup Y,Y}(g_{X,X\cup Y}(U)) \\
&= U_Y,
\end{aligned}$$

as desired.

Suppose now that U is principal, generated by $\{C\}$ with $C \in \mathcal{C}_X$ say, and that C is infinite. If C contains a ray, the end ω of this ray defines τ_ω, for which $U = U(\tau_\omega, X)$ as desired. If not, then C has a finite set Z of vertices such that $C - Z$ has infinitely many components: otherwise we could construct a ray $z_0 z_1 \dots$ in C inductively by choosing each z_n from an infinite component of $C - \{z_0, \dots, z_{n-1}\}$. Pick a non-principal ultrafilter U_Z on the set of components of $C - Z$, notice that these are also components of $G - X'$ for $X' = X \cup Z$, and let U' be the (non-principal) ultrafilter on $\mathcal{C}_{X'}$ generated by U_Z. Then $f_{X',X}(U') = U$, by definition of U' and $f_{X',X}$. By the case already treated, there exists $\tau \in \Theta$ such that $U' = U(\tau, X')$. This τ achieves our aim, since

$$U(\tau, X) = f_{X',X}(U(\tau, X')) = f_{X',X}(U') = U$$

by Lemma 2.2. $\square$

4. Tangles at infinity: compactifying an arbitrary graph

Our aim in this section is to prove Theorem 1: that the $\aleph_0$-tangles can be used as points at infinity to compactify G, in a way that yields its Freudenthal compactification when G is locally finite (and hence all its $\aleph_0$-tangles are end tangles). To make this process more transparent we shall first define a topology on $\mathcal{U}$ itself, which can be done in a rather canonical way. We shall then adapt this to define a topology on all of $G \cup \mathcal{U}$ that induces this topology on $\mathcal{U}$, as well as the usual 1-complex topology on G. This space $|G| = G \cup \mathcal{U}$ or, equivalently by Lemma 2.3, the space $|G| = G \cup \Theta$, will be the desired compactification of G.

For each $X \in \mathcal{X}$, take the topology on $\mathcal{U}_X$ whose basic open sets are those of the form

$$\mathcal{O}(\mathcal{C}) := \{\, U \in \mathcal{U}_X \mid \mathcal{C} \in U \,\},$$

one for each $\mathcal{C} \subseteq \mathcal{C}_X$. This topology, the Stone topology on $\mathcal{C}_X$, makes $\mathcal{U}_X$ into a compact topological space[3] in which the principal ultrafilters on $\mathcal{C}_X$ are dense. The space $\mathcal{U}_X$ is clearly Hausdorff, indeed totally disconnected.

Our inverse system $(\mathcal{U}_X \mid X \in \mathcal{X})$ is compatible with these topologies:

Lemma 4.1. *The maps $f_{X',X}\colon \mathcal{U}_{X'} \to \mathcal{U}_X$ are continuous.* □

In fact, for $X \subseteq X' \in \mathcal{X}$ the open sets $f^{-1}_{X',X}(\mathcal{O}(\mathcal{C})) \subseteq \mathcal{U}_{X'}$ are themselves basic:

$$f^{-1}_{X',X}(\mathcal{O}(\mathcal{C})) = \mathcal{O}(\mathcal{C}') \quad \text{for} \quad \mathcal{C}' = \{\, C' \in \mathcal{C}_{X'} \mid \exists C \in \mathcal{C} : C \supseteq C' \,\}. \tag{7}$$

Topologizing the $\mathcal{U}_X$ has an interesting windfall for graphs without ends:

Proposition 4.2. *Every infinite graph has an $\aleph_0$-tangle.*

Proof. Inverse limits of non-empty compact spaces are non-empty [14]. Hence $\mathcal{U}$ is not empty, and so by Lemma 2.3 neither is Θ. □

Let us give the set $\Theta = \mathcal{U} = \varprojlim\, (\mathcal{U}_X \mid X \in \mathcal{X})$ of $\aleph_0$-tangles in G the subspace topology from $\prod_{X \in \mathcal{X}} \mathcal{U}_X$ endowed with the product topology of the $\mathcal{U}_X$.

Proposition 4.3. *The topological space of all $\aleph_0$-tangles of an infinite graph is compact and totally disconnected.*

Proof. Since the $\mathcal{U}_X$ are Hausdorff and the $f_{X',X}$ are continuous, the space $\mathcal{U} = \varprojlim\, (\mathcal{U}_X \mid X \in \mathcal{X})$ is closed in $\prod_{X \in \mathcal{X}} \mathcal{U}_X$, which inherits its own compactness from that of the $\mathcal{U}_X$ by Tychonov's theorem. It is totally disconnected, because the $\mathcal{U}_X$ are. □

To describe this topology more explicitly, consider any $X \in \mathcal{X}$ and $\mathcal{C} \subseteq \mathcal{C}_X$. Let

$$\mathcal{O}(X, \mathcal{C}) := \{\, \upsilon \in \mathcal{U} \mid \mathcal{C} \in U_X \,\} = f_X^{-1}(\mathcal{O}(\mathcal{C})),$$

where $\upsilon = (\, U_X \mid X \in \mathcal{X}\,)$ and f_X is the (continuous) projection $\mathcal{U} \to \mathcal{U}_X$.

Lemma 4.4. *The sets $\mathcal{O}(X, \mathcal{C})$ form a basis of open sets in $\mathcal{U}$.*

Proof. By definition of the topology on $\mathcal{U}$, these sets form a subbasis. By (7) they even form a basis, because every finite intersection of such sets $\mathcal{O}(X, \mathcal{C})$ can be rewritten as the union of sets $\mathcal{O}(X', \mathcal{C}')$ with X' the (finite) union of these X. □

[3] The Stone-Čech compactification of $\mathcal{C}_X$. Its compactness is immediate from Tychonov's theorem when we view $\mathcal{U}_X$ as a subspace of $2^{2^{\mathcal{C}_X}}$: the set $\mathcal{U}_X$ is closed in this compact space, since any violation of the ultrafilter axioms involves only finitely many subsets of $\mathcal{C}_X$.

When G is locally finite and connected, it is compactified by its ends in the so-called *Freudenthal compactification* [8, 9; 5]. The following definition for our arbitrary G defaults to this when G is locally finite and connected, and hence all its $\aleph_0$-tangles are end tangles.

Let us view G as a 1-complex with the usual topology. Its edges are copies of the real interval $[0,1]$, and choosing for every edge $e = vw$ at some vertex v any half-open partial edge $e_x = [v, x) \subseteq e$ with $x \in \mathring{e}$, makes $\bigcup_e e_x$ into an open neighbourhood of v. Let us extend G to a topological space

$$|G| = G \cup \mathcal{U} = G \cup \Theta$$

(cf. Lemma 2.3) by also declaring as open, for all $X \in \mathcal{X}$ and all $\mathcal{C} \subseteq \mathcal{C}_X$, the sets

$$\mathcal{O}_G(X, \mathcal{C}) := \bigcup \mathcal{C} \cup \mathring{E}(X, \bigcup \mathcal{C}) \cup \mathcal{O}(X, \mathcal{C}) \tag{8}$$

and taking the topology on $|G|$ that this generates. Here, $\mathring{E}(X, \bigcup \mathcal{C})$ is the set of all inner points of edges between X and components of $G - X$ in $\mathcal{C}$. Note that the subspace topology on $\mathcal{U} \subseteq |G|$ is our original topology on $\mathcal{U}$, and that the subspace topology on G is its orginal 1-complex topology.

Let us prove that $|G| = G \cup \mathcal{U} = G \cup \Theta$ is a compact space, the *tangle compactification* of G:

Theorem 1. *Let G be any graph.*

(i) *$|G|$ is a compact space in which G is dense and $|G| \smallsetminus G$ is totally disconnected.*

(ii) *If G is locally finite and connected, then all its $\aleph_0$-tangles are ends, and $|G|$ coincides with the Freudenthal compactification of G.*

Proof. (i) Consider any cover $\mathcal{O}$ of $|G|$ by open subsets of G and basic open sets of the form $\mathcal{O}_G(X, \mathcal{C})$. Since $\mathcal{U}$ is compact in the subspace topology of $|G|$, this has a finite subset of the form

$$\mathcal{F} = \{ \mathcal{O}_G(X, \mathcal{D}_X) \mid X \in \mathcal{X}' \}$$

(with $\mathcal{X}' \subseteq \mathcal{X}$ finite) that covers $\mathcal{U}$. Our aim is to show that, for $X' := \bigcup \mathcal{X}'$, the sets in $\mathcal{F}$ and $G[X']$ together cover $|G|$: since $G[X']$ is a finite graph and hence compact, there will then also be a finite subcover of $\mathcal{O}$ for all of $|G|$.

For every $X \in \mathcal{X}'$, let

$$\mathcal{D}'_X := \{ C' \in \mathcal{C}_{X'} \mid \exists C \in \mathcal{D}_X \colon C \supseteq C' \}.$$

Then $f^{-1}_{X',X}(\mathcal{O}(\mathcal{D}_X)) = \mathcal{O}(\mathcal{D}'_X)$ by (7). As $f_X = f_{X',X} \circ f_{X'}$, this implies

$$\mathcal{O}(X, \mathcal{D}_X) = f^{-1}_X(\mathcal{O}(\mathcal{D}_X)) = f^{-1}_{X'}(\mathcal{O}(\mathcal{D}'_X)) = \mathcal{O}(X', \mathcal{D}'_X) \tag{9}$$

and hence $\mathcal{O}_G(X, \mathcal{D}_X) \supseteq \mathcal{O}_G(X', \mathcal{D}'_X)$, since $\bigcup \mathcal{D}_X \supseteq \bigcup \mathcal{D}'_X$ by definition of $\mathcal{D}'_X$.

It thus suffices to show that the $\mathcal{O}_G(X', \mathcal{D}'_X)$ cover $|G| \setminus G[X']$, i.e., that every component of $G - X'$ is an element of some $\mathcal{D}'_X$ with $X \in \mathcal{X}'$. Suppose not, and let

$$\mathcal{C}' := \mathcal{C}_{X'} \setminus \bigcup_{X \in \mathcal{X}'} \mathcal{D}'_X.$$

If $\bigcup \mathcal{C}'$ is a finite graph, we add this to $G[X']$ and achieve $\mathcal{C}' = \emptyset$ as desired. We may therefore assume that $\bigcup \mathcal{C}'$ is infinite.

If $\mathcal{C}'$ contains an infinite component C', let $U' \in \mathcal{U}_{X'}$ be the ultrafilter on $\mathcal{C}_{X'}$ generated by $\{C'\}$. If not, then $\mathcal{C}'$ is infinite; pick a non-principal ultrafilter on $\mathcal{C}'$ and let $U' \in \mathcal{U}_{X'}$ be the (non-principal) ultrafilter it generates on all of $\mathcal{C}_{X'}$. By Lemmas 3.7 and 2.3, there exists $\upsilon = (\,U_X \mid X \in \mathcal{X}\,) \in \mathcal{U}$ such that $U_{X'} = U'$. By (9) and the definition of $\mathcal{C}'$, this υ does not lie in $\bigcup \mathcal{F}$, a contradiction.

(ii) This is easy; see [5] for the definition of the Freudenthal compactification. To see that the topology for $|G|$ defined there coincides with ours here, remember that if G is locally finite then any finite $X \subseteq V$ sends only finitely many edges to $G - X$. $\square$

We remark that, as defined above, $|G|$ is not in general Hausdorff: the centre of an infinite star, for example, cannot be topologically separated from any open set containing a tangle of that star. However, every two points of $V \cup \Theta \subseteq |G|$ have neighbourhoods in $|G|$ that meet only in inner points of edges. If we delete the set $\mathring{E}$ of all inner edge points from $|G|$, the resulting space $|G| \setminus \mathring{E}$ will be a Hausdorff compactification of V that still reflects the structure of G.

Conversely, if we are prepared to give up the compactness of $|G|$ (while keeping $\mathcal{U} = |G| \setminus G$ and $V \cup \Theta = |G| \setminus \mathring{E}$ compact, which may be more crucial), we can make $|G|$ itself Hausdorff: we just have to allow more open sets in (8) by replacing $\mathring{E}(X, \bigcup \mathcal{C})$ with unions of either arbitrary half-edges $(y, z) \subseteq (x, z)$ for each $e = (x, z) \in \mathring{E}(X, \bigcup \mathcal{C})$, or by uniformly chosen such half-edges (where all y have distance some fixed positive $\epsilon < 1$ from $z \in \bigcup \mathcal{C}$ when the edge $e = [x, z]$ is viewed as a copy of the real interval $[0, 1]$.

5. Closed tangles

When Robertson and Seymour introduced tangles for finite graphs, their intended key feature was that they point to parts of the graph that are in some sense highly connected. Any large enough grid, for example, defines a k-tangle for any fixed k, even though it is not highly connected as a subgraph: since every separation $\{A, B\}$ of order $< k$ leaves most of the grid on one side, it can be oriented 'towards' that side, and these orientations satisfy the tangle axioms.

Our $\aleph_0$-tangles do not all point to a highly connected part of G. Indeed, G could be a locally finite tree, but it would still have end tangles pointing to its ends – which can hardly be seen as highly connected structures in any sense. On the other hand, an infinite complete subgraph also defines an $\aleph_0$-tangle, for which a better case could be made. Ultrafilter tangles, however, have no connected – let alone highly connected – focus at all.

Some attempts have been made to at least identify those kinds of ends that tell us where our graph is highly connected. Candidates included the Halin ends that are not Freudental (or *topological*) ends [2], which are those that have one or more vertices send an infinite fan to each of their rays [6]. The earliest attempt, perhaps, was to consider 'thick' ends [11, 18], those of infinite (vertex) degree: these are the ends that contain an infinite set of disjoint rays [5, 4]. Halin [11] showed that these are precisely the ends (whose $\aleph_0$-tangle is) defined by a half grid minor. An obvious analogue would be to consider the ends defined by a full grid minor – these have been characterized by Heuer [12] – or infinite clique minors or subdivision as in [16, 17].

I would like to propose a new alternative: that an $\aleph_0$-tangle is deemed to signify a highly connected part of G if and only if it is closed in a certain natural topology on $\vec{S} = \vec{S}_{\aleph_0}$. We shall be able to characterize those tangles in graph-theoretical terms. They will all be end tangles, including those defined by an infinite complete subgraph but not, for example, the end tangles of a tree.

The topology on $\vec{S}$ has the following basic open sets. Pick a finite set $Z \subseteq V$ and an oriented separation (A_Z, B_Z) of $G[Z]$. Then declare as open the set $O(A_Z, B_Z)$ of all $(A, B) \in \vec{S}$ such that $A \cap Z = A_Z$ and $B \cap Z = B_Z$. We shall say that these (A, B) *induce* (A_Z, B_Z) on Z, writing $(A_Z, B_Z) =: (A, B) \restriction Z$, and that (A, B) and (A', B') *agree on* Z if $(A, B) \restriction Z = (A', B') \restriction Z$.

It is easy to see that the sets $O(A_Z, B_Z)$ do indeed form the basis of a topology on $\vec{S}$. Indeed, $(A, B) \in \vec{S}$ induces (A_1, B_1) on Z_1 and (A_2, B_2) on Z_2 if and only if it induces on $Z = Z_1 \cup Z_2$ some separation (A_Z, B_Z) which in turn induces (A_i, B_i) on Z_i for both i. Hence $O(A_1, B_1) \cap O(A_2, B_2)$ is the union of all these $O(A_Z, B_Z)$.

Example 5.1. If G is a single ray $v_0 v_1 \ldots$ with end ω, say, then $\tau = \tau_\omega$ is not closed in $\vec{S}$. Indeed, τ contains $(\emptyset, V)$ by Lemma 1.1, and hence does not contain $(V, \emptyset)$. But for every finite $Z \subseteq V$ the restriction $(Z, \emptyset)$ of $(V, \emptyset)$ to Z is also induced by $(\{v_0, \ldots, v_n\}, \{v_n, v_{n+1}, \ldots\}) \in \tau$ for every n large enough that $Z \subseteq \{v_0, \ldots, v_{n-1}\}$. So $(V, \emptyset) \in \vec{S} \setminus \tau$ has no open neighbourhood in $\vec{S} \setminus \tau$. □

Example 5.2. Ultrafilter tangles $\tau \in \Theta$ are never closed in $\vec{S}$. Indeed, let $X \in \mathcal{X}$ witness that τ is an ultrafilter tangle, pick $\mathcal{C} \in U(\tau, X)$, and consider $(A, B) \in \vec{S}$ for $A = X \cup \bigcup \mathcal{C}$ and $B = V \setminus \bigcup \mathcal{C}$. This is a separation in $\vec{S} \setminus \tau$ (cf. Lemma 3.4), but every open neighbourhood of (A, B) meets τ: for every finite $Z \subseteq V$ we can find a separation $(A', B') \in \tau$ such that $(A', B') \restriction Z = (A, B) \restriction Z$. Such a separation (A', B') can be obtained from (A, B) by moving all the components of $\mathcal{C}$ that lie in $A \setminus Z$ to B. Then $(A', B') \in \tau$, again by Lemma 3.4 (and Lemma 1.10). The details are left to the reader; Theorem 4 below includes a formal proof. $\square$

Are any $\aleph_0$-tangles closed in $\vec{S}$? As we have seen, they must be end tangles. But such end tangles do exist. Here is the example promised earlier:

Example 5.3. If $K \subseteq V$ spans an infinite complete graph in G, then the $\aleph_0$-tangle

$$\tau = \{ (A, B) \in \vec{S} \mid K \subseteq B \} \tag{10}$$

is closed in $\vec{S}$. We omit the easy proof. $\square$

Perhaps surprisingly, it is not hard to characterize the $\aleph_0$-tangles that are closed. They are all essentially like Example 5.3: we just have to generalize the infinite complete subgraph used appropriately. Of the two obvious generalizations, infinite complete minors [17] or subdivisions of infinite complete graphs [16], the latter turns out to be the right one.

Let κ be any cardinal. A *κ-block* in G is a maximal set of at least κ vertices no two of which can be separated in G by fewer than κ vertices. For example, the set of branch vertices of a $TK_{\aleph_0}$ is an $\aleph_0$-block. The converse is also true:

Lemma 5.4. *When κ is infinite, $K \subseteq V$ is a κ-block in G if and only if it is the set of branch vertices of some $TK_\kappa \subseteq G$.*

Proof. For the forward implication, we find the κ independent paths corresponding to the edges of the K_κ in κ steps, viewed as an ordinal. At each step fewer than κ vertices have been used, so these do not separate the two vertices to be joined next. $\square$

We can now prove our last remaining theorem. Let us say that a set $K \subseteq V$ *defines* an $\aleph_0$-tangle τ if τ satisfies (10).

Theorem 4. *Let G be any graph.*

(i) *The $\aleph_0$-tangles in G that are not end tangles are never closed in $\vec{S}$.*

(ii) *An end tangle in G is closed in $\vec{S}$ if and only if it is defined by an $\aleph_0$-block.*

Proof. For every $TK_{\aleph_0} = H \subseteq G$ there is a unique end ω of G containing all the rays in H. Then $\tau \in \Theta$ is defined by the set of branch vertices of this $TK_{\aleph_0}$ if and only if $\tau = \tau_\omega$. In view of Lemma 5.4 it thus suffices to show that τ is closed in $\vec{S}$ if and only if it is defined by an $\aleph_0$-block.

Suppose first that τ is defined by an $\aleph_0$-block K. To show that τ is closed, we have to find for every $(A, B) \in \vec{S} \setminus \tau$ a finite set $Z \subseteq V$ such that no $(A', B') \in \vec{S}$ that agrees with (A, B) on Z lies in τ. As $(A, B) \notin \tau$, we have $K \subseteq A$; pick $z \in K \setminus B$. Then every $(A', B') \in \vec{S}$ that agrees with (A, B) on $Z := \{z\}$ also also lies in $\vec{S} \setminus \tau$, since $z \in A' \setminus B'$ and this implies $K \notin B'$.

Conversely, consider any $\tau \in \Theta$ and let

$$K := \bigcap \{ B \mid (A, B) \in \tau \}.$$

No two vertices in K can be separated by in G by a finite-order separation: one orientation (A, B) of this separation would be in τ, which would contradict the definition of K since $A \setminus B$ also meets K. If K is infinite, it will clearly be maximal with this property, and hence be an $\aleph_0$-block. This $\aleph_0$-block K will define τ: by definition of K we have $K \subseteq B$ for every $(A, B) \in \tau$, while also every $(A, B) \in \vec{S}$ with $K \subseteq B$ must be in τ: otherwise $(B, A) \in \tau$ and hence $K \subseteq A$ by definition of K, but $K \nsubseteq A \cap B$ because this is finite. Hence τ will be defined by an $\aleph_0$-block, as desired for the forward implication.[4]

It thus suffices to show that if K is finite then τ is not closed in $\vec{S}$, which we shall do next.

Assume that K is finite. We have to find some $(A, B) \in \vec{S} \setminus \tau$ that is a limit point of τ, i.e., which agrees on every finite $Z \subseteq V$ with some $(A', B') \in \tau$. We choose $(A, B) := (V, K) \in \vec{S} \setminus \tau$ (Lemma 1.1).

To complete our proof as outlined, let any finite set $Z \subseteq V$ be given. For every $z \in Z \setminus K$ choose $(A_z, B_z) \in \tau$ with $z \in A_z \setminus B_z$: this exists, because $z \notin K$. By Lemma 1.2, the supremum of all these elements of τ and $(K, V) \in \tau$ is again in τ: we have $(A', B') \in \tau$ for

$$A' := K \cup \bigcup_{z \in Z \setminus K} A_z \quad \text{and} \quad B' := V \cap \bigcap_{z \in Z \setminus K} B_z\,.$$

As desired, $(A', B') \restriction Z = (A, B) \restriction Z$ (which is $(Z, Z \cap K)$, since $(A, B) = (V, K)$): every $z \in Z \setminus K$ lies in some A_z and outside that B_z, so $z \in A' \setminus B'$, while every $z \in Z \cap K$ lies in $K \subseteq A'$ and also, by definition of K, in every B_z (and hence in B'), since $(A_z, B_z) \in \tau$. □

6. Outlook

There are some obvious leads the reader may like to follow up, as well as one not so obvious one.

The most obvious is to study the space $|G|$ more closely. There are plenty of basic questions about $|G|$ that we have not even addressed. For example, how is $|G|$ related to the Stone-Čech compactification of G? For which G is $|G|$ the coarsest compactification in which its ends appear as distinct points? If it is not, is there a unique such topology, and is there a canonical way to obtain it from $|G|$?

More important, and probably a good guidance also for which of these basic questions to address, is the potential of $|G|$ for applications in graph theory. For locally finite graphs, the study of its end compactification $|G|$ has proved very

[4] Whether or not τ is closed in $\vec{S}$ is immaterial; we just did not use this assumption.

enlightening indeed, and has led to some considerable advances even for purely graph-theoretic problems not originally involving ends [4]. Might considering our tangle compactification $|G|$ lead to similar advances for arbitrary infinite graphs G?

Another obvious lead is to consider κ-tangles for cardinals $\kappa > \aleph_0$. Do the κ-tangles that are closed in the space $\vec{S}_\kappa$ of all oriented separations of G of order $< \kappa$ form interesting highly connected substructures that do not coincide with classical such structures such as TK_κ subgraphs?

Finally, there is an intriguing way to generalize $\aleph_0$-tangles to separation systems of much more general discrete structures than graphs, introduced in [7]. Essentially, all we need to remember of $\vec{S}$ is that it is a poset with an order-reversing involution. One can then define stars of 'oriented separations' (elements of $\vec{S}$) as earlier in Section 1, and for a set $\mathcal{F}$ of such stars one can consider $\mathcal{F}$-tangles. Perhaps there is a natural (submodular) 'order' function on $\vec{S}$, as is the case, for example, for separations in matroids. But even if not, there is a way of expressing $\aleph_0$-tangles in this framework without any reference to an order function – or, indeed, to the cardinality of $\bigcap_i B_i$ as in the definition of $\mathcal{T}_{<\aleph_0}$.

We need one more definition to express this. Call an oriented separation $\vec{s} \in \vec{S}$ *small* if $\vec{s} \leqslant \overleftarrow{s}$, where $\overleftarrow{s}$ denotes the image of $\vec{s}$ under the involution.[5] Using this term, we can rephrase the definition of a $\mathcal{T}_{<\aleph_0}$-tangle of S without mentioning cardinalities:

Observation. *The $\mathcal{T}_{<\aleph_0}$-tangles of S are the consistent orientations τ of S such that no finite star $\sigma \subseteq \tau$ has a supremum in $\vec{S}$ whose inverse ist small.*

Proof. A $\mathcal{T}_{<\aleph_0}$-tangle cannot contain such a star $\sigma = \{\, (A_i, B_i) \mid i = 1, \dots, n \,\}$: since the supremum of σ is $(\bigcup_i A_i, \bigcap_i B_i)$, the inverse of this can be small only if $\bigcap_i B_i$ is finite, which would place σ in $\mathcal{T}_{<\aleph_0}$.

Conversely, let us show that if τ is a consistent orientation of S such that no star $\sigma \subseteq \tau$ has a supremum in $\vec{S}$ with a small inverse, then τ has no subset in $\mathcal{T}_{<\aleph_0}$. For let $\sigma = \{\, (A_i, B_i) \mid i = 1, \dots, n \,\} \subseteq \tau$ be such a subset. Then $X := \bigcap_i B_i$ is finite, and the separations $(A'_i, B_i) \geqslant (A_i, B_i)$ with $A'_i := A_i \cup X$ still lie in $\vec{S}$. In fact, they must also lie in τ. For if $(B_i, A'_i) \in \tau$ for some i, then $\{(A_i, B_i), (B_i, A'_i)\} \subseteq \tau$ is a star whose supremum $(A_i \cup B_i, B_i \cap A'_i) = (V, X)$ has a small inverse. But the supremum of $\sigma' := \{\, (A'_i, B_i) \mid i = 1, \dots, n \,\} \subseteq \tau$ is $(\bigcup A'_i, \bigcap B_i) = (V, X)$, which has a small inverse – a contradiction to the choice of τ. $\square$

If our characterization of the $\aleph_0$-tangles of G in terms of $\mathcal{U}$ can be redone in this abstract setting, it may become meaningful to consider ultrafilter tangles in more general structures than graphs, such as ends in matroids, that have been sought for some time.

[5] For $\vec{s} = (A, B)$, this would be $\overleftarrow{s} = (B, A)$. The small separations in our $\vec{S}$ are those of the form (A, V): they satisfy $(A, V) \leqslant (V, A)$, and are the only separations with this property.

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