Decomposing Infinite Graphs

Reinhard Diestel

This paper gives an introduction to the theory of simplicial and related decompositions of graphs as developed in [1]. It is intended for the non-specialist, and particular prominence is given to the presentation of open problems.

Introduction

In his classic paper *Über eine Eigenschaft der ebenen Komplexe*, Wagner [19] tackles the following problem. Kuratowski's theorem, in its excluded minor version, states that a finite graph is planar if and only if it has no minor isomorphic to K_5 or to $K_{3,3}$. (A *minor* of G is any graph obtained from some $H \subset G$ by contracting connected subgraphs.) If we exclude only one of these two minors, the graph may no longer be planar—but will it be *very* different from a planar graph? For example, can the non-planarity of an arbitrary finite graph without a K_5 minor be tied down to certain parts of it, the rest of the graph being planar?

Wagner's solution to this problem is based on the following observation. Suppose we take two graphs G_1 and G_2, neither of which has a K_5 minor, and paste them together along a complete subgraph. (Following Wagner, we shall use the term *simplex* for complete graphs. So here we let $G = G_1 \cup G_2$ and assume that $G_1 \cap G_2$ is a simplex.) Then the resulting graph G is again K_5-free (has no K_5 minor). For if $H_1, \ldots, H_5$ are connected subgraphs of some $H \subset G$ whose contraction yields a K_5, then either G_1 or G_2 must also have such subgraphs (Fig. 1), contrary to our assumption that these graphs are K_5-free.

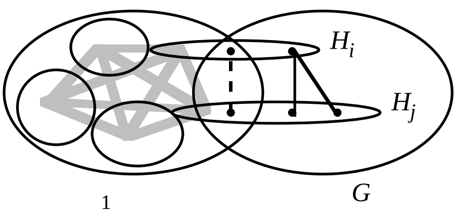

FIGURE 1. Finding a K_5 minor in G_1

Repeating this process, we can easily construct *simplicial decompositions* of arbitrarily large K_5-free graphs: just keep attaching new K_5-free graphs along simplices contained in the graph constructed so far (Fig. 2). And more importantly, the converse is also true: every K_5-free graph, however large, can be constructed in such a simplicial decomposition from *prime* factors, i.e. from 'small' graphs which do not themselves have a simplicial decomposition into more than one factor. (The general question of which graphs admit a prime decomposition is a fundamental problem in simplicial decomposition theory; see Section 1.) Thus, we can characterize the finite K_5-free graphs by their decompositions if we succeed in drawing up a complete list of the prime factors needed to construct all these graphs.

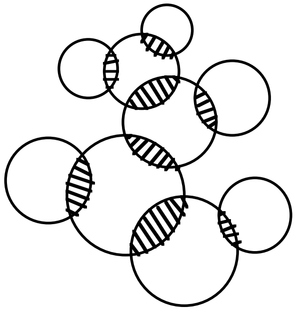

FIGURE 2. A simplicial decomposition

Essentially, this is just what Wagner does in his paper—the only difference being that he keeps the list shorter by including only the factors of *edge-maximal* K_5-free graphs, those in which the addition of any new edge creates a minor isomorphic to K_5. As it turns out, this list, which he calls the 'homomorphism base' of K_5, contains only one non-planar graph W, while all its other graphs are planar (namely, the 4-connected plane triangulations). (More generally, the *homomorphism base* of a finite graph X is the class of all the graphs occuring as factors in prime decompositions of—finite or infinite—edge-maximal X-free graphs.) Our somewhat vague opening question thus has a surprisingly positive answer: the non-planarity of any finite K_5-free graph can be localized within parts of it that are either subgraphs of the one non-planar graph W, or else arise from at least three planar graphs pasted together along a triangle. (Note that pasting planar factors together along a simplex smaller than K_3 still yields a planar graph.)

Since Wagners original paper, homomorphism bases have been determined for several other excluded minors, in each case giving rise to a similar structural characterization of the graphs without this minor (see Wagner [19], [20], [21], Halin [10], [11], [13], [14], or [1, Ch. 6.1] or [7] for a table of all known homomorphism bases.) A typical example: the edge-maximal K_5^--free graphs (where K_5^- denotes a K_5 minus an edge) are precisely the graphs which can be constructed in a simplicial decomposition with attachment simplices of order 2 from factors isomorphic to K_3, $K_{3,3}$, the prism ($K_3 \times K_2$) or wheels.

It is clear that excluded minor theorems in terms of homomorphism bases can be very powerful characterizations. In the case of K_5^-, for instance, the simplicity of the prime factors of the K_5^--free graphs enables us instantly to determine sharp bounds on their chromatic number, minimal degree and so on. Moreover, the graph properties definable by the exclusion of minors are important properties: they are precisely the properties that are closed under subcontraction (= taking minors), and include such natural properties as, say, the embeddability in a given surface.

However, not every homomorphism base offers as much information as does that of K_5^-. In the case of K_5, for example, we know that the base elements, with the exception of the graph W, are maximally planar. But how well do we really *know* an arbitrary maximally planar graph? We can hardly determine its chromatic number!

An important problem, therefore, is to learn to distinguish the minors whose exclusion gives rise to a simple homomorphism base from those where the base elements can be nearly as complicated as the graphs they serve to describe. At first glance, this notion of a 'simple' homomorphism base seems a difficult one to make precise. However, it so happens that as soon as we allow our graphs to be infinite, the simple and the complicated bases seem to fall neatly apart: into bases which are made up of finite graphs only (and are therefore countable, like the homomorphism base of K_5^-) and uncountable bases (like that of K_5, which contains all the—uncountably many—countable maximally planar graphs). By a beautiful theorem of Halin, homomorphism base elements are always themselves countable, whatever the cardinality of the graphs of which they are factors; see [1, Ch. 5].

Calling a homomorphism base *simple* if it is countable—or, alternatively, if all its members are finite—we are thus led to the following problem. (It is unknown whether the two suggested definitions of 'simple' coincide.)

Problem. *For which excluded minors is the corresponding homomorphism base simple?*

Although this problem in its full generality seems to be hard, so little is known about it that even the most basic results would mean progress. For example, if the homomorphism base of X is simple and X' is obtained from X by deleting an edge, is the base of X' again simple? More such conjectures, including some farther reaching ones, can be found in [1, Ch. 6.1] or in [7].

Since their introduction by Wagner for the purpose of investigating the K_5-free graphs, simplicial decompositions have been applied to a wide range of problems, mainly in infinite graph theory. Moreover, the investigation of these and related decompositions has led to an interesting theory in its own right. The aim of this paper is to give an introduction to some of the central aspects of this theory, to state its main results (without proofs, but illustrated by examples), and to present its guiding open problems.

Sections 1 and 2 deal with the problem of the existence of simplicial decompositions into prime or otherwise 'small' factors. Section 3 gives a brief introduction to the problem of when such prime decompositions are unique.

In Section 4 finally, we look at the 'structural essence' of simplicial decompositions, which is neatly captured by another type of decomposition called *tree-decompositions*. For finite graphs, these tree-decompositions reduce to the by now familiar decompositions used by Robertson and Seymour for the proof of their well-quasi-ordering theorem (Wagner's Conjecture).

1. The existence of prime decompositions

The question of which graphs admit a simplicial decomposition into prime factors, already touched upon above, is perhaps the most fundamental and at the same time the most complex problem in simplicial decomposition theory. And while a good deal is now known about prime decompositions, existing results amount to no more than a partial solution of the general problem:

Problem. *Which graphs admit a simplicial decomposition into primes?*

Before we look into this problem further, let us give a precise definition of a simplicial decomposition. In order to make the definition suitable for infinite as well as for finite graphs, we do not follow the intuitive approach of 'decomposing' a graph into smaller and smaller pieces (a process which may never end), but build it up from below, adding one factor at a time.

Thus, let G be a graph, $\sigma > 0$ an ordinal, and let B_λ be an induced subgraph of G for every $\lambda < \sigma$. The family $F = (B_\lambda)_{\lambda<\sigma}$ is called a *simplicial decomposition* of G if the following three conditions hold:

(S1) $G = \bigcup_{\lambda<\sigma} B_\lambda$;

(S2) $\left(\bigcup_{\lambda<\mu} B_\lambda\right) \cap B_\mu =: S_\mu$ is a complete graph for each $\mu \quad (0 < \mu < \sigma)$;

(S3) no S_μ contains B_μ or any other $B_\lambda \quad (0 \leqslant \lambda < \mu < \sigma)$.

(Condition (S3) is of lesser importance; its purpose is to avoid 'redundant' factors.)

The graph $\bigcup_{\lambda<\mu} B_\lambda$ in (S2) will be denoted by $G|_\mu$, and the simplex S_μ in (S2) will be called the *simplex of attachment* of B_μ. A graph is *prime* if it has no simplicial decomposition into more than one factor, and a *prime decomposition* is a simplicial decomposition in which every factor is prime. Furthermore, we shall call a subgraph $H \subset G$ *attached* to another subgraph $H' \subset G \backslash H$ if every vertex of H has a neighbour in H'. (These subgraphs H and H' will usually be induced and connected.) If H is not attached to any component of $G \backslash H$, we call H *unattached* in G.

We remark that a graph is prime if and only if it does not contain a separating simplex. This is not quite as trivial as it may seem, because one has to show that a simplex of attachment continues to separate the graph after the addition of further factors. This, however, is an easy consequence of (S2):

Proposition 1.1. *If $(B_\lambda)_{\lambda<\sigma}$ is a simplicial decomposition of G, and if $x \in G|_\mu \backslash S_\mu$ and $y \in B_\mu \backslash S_\mu$ for some $\mu < \sigma$, then S_μ separates x from y in G.*

Suppose we are given an arbitrary graph G and are asked to find a simplicial decomposition of G into primes. How shall we go about the problem? Clearly, there are two tasks involved: finding the right subgraphs of G to serve as the factors B_λ, and putting them together in accordance with (S1)–(S3).

The problem of which kinds of subgraph may be used as factors in a prime decomposition has been studied thoroughly and may be regarded as fully understood. Essentially, the prime factors of a graph (in any prime decomposition) are its smallest unattached convex subgraphs. (A subgraph $H \subset G$ is *convex* in G if any induced (or 'chordless') path in G connecting vertices of H is contained in H.) In fact, more is true: if F is a simplicial decomposition of G, then any subgraph of G which corresponds to a 'subtree' of the 'decomposition tree' of F (Fig. 2) is convex in G (see [1, Chs. 1.1 and 5.4] for details).

The more challenging part of the prime decomposition problem is the second of the two tasks mentioned: finding the correct order, if it exists, in which the potential prime factors B_λ may be assembled into a simplicial decomposition of G. Starting from a simple example, we shall now look into this problem in some detail. In order to concentrate on the essential, we shall assume that the *set* of potential factors B_λ has already been determined.

1.1. The construction of a prime decomposition

Consider the graph G shown in Figure 3. The potential prime factors of G are the triangles $T_1, \dots, T_4$. Trying to be as short-sighted as possible, let us begin to contstruct a prime decomposition $(B_\lambda)_{\lambda<\sigma}$ of G with the two bottom triangles, setting $B_0 := T_1$ and $B_1 := T_3$, say. Then $S_1 = T_1 \cap T_3$, so S_1 is a simplex (of order 1) as required. For B_2, we can choose between the two remaining factors, the triangles T_2 and T_4. But neither of these choices is feasible: $S_2 = B_2 \cap (B_0 \cup B_1)$ would be a path of length 2 or consist of two isolated vertices—thus in neither case would S_2 be a simplex.

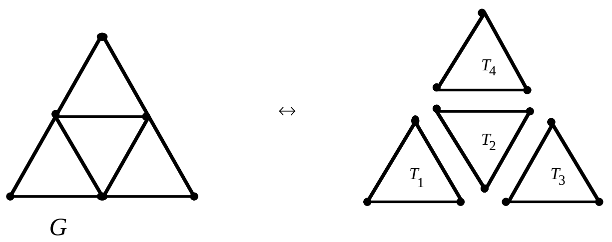

FIGURE 3. Constructing a prime decomposition

What went wrong? Of course, we should not have chosen T_3 as the factor B_1: Proposition 1.1 requires that S_1 separate the vertices of $B_0 \backslash S_1$ from those of $B_1 \backslash S_1$ in G, which is clearly not the case with our choice of T_3 as B_1. And indeed, if we set $B_1 := T_2$ instead, we can easily complete our prime decomposition, e.g. to (T_1, T_2, T_3, T_4).

With the above example in mind, let us now consider a more abstract situation. Let G be a graph containing a simplex S which separates G into two components (making up $G \backslash S$), C and C' say. Let us assume that S is attached to C, and let $S' \subset S$ denote the simplex induced by those vertices of S that have a neighbour in C' (Fig. 4). Suppose we have started to build a prime decomposition of G, having chosen factors B_λ for all λ up to (but excluding) some ordinal μ. Suppose further that the part $G|_\mu$ of G we have covered lies entirely in $G[\,C' \cup S'\,]$ (the subgraph of G induced by the vertices of C' and S'), so that $G|_\mu \cap C = \emptyset$ (but $G|_\mu \cap C' \neq \emptyset$). Let us consider the following question: under what assumptions can we choose B_μ so as to include a vertex from C?

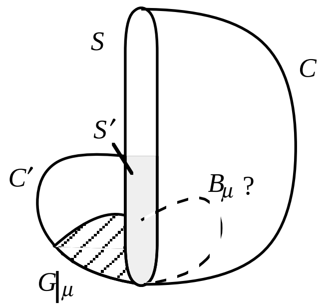

FIGURE 4. Choosing B_μ

Notice first that if B_μ is to include a vertex from C, it must lie entirely within $G[\,C \cup S\,]$: otherwise it would be separated by a subsimplex of S, which contradicts the requirement that it should be prime. Thus S_μ, the intersection of B_μ with $G|_\mu$, will be a part of S'. Now by Proposition 1.1, we can see that S_μ must in fact be equal to S': since S' is attached both to C and to C', S_μ could not otherwise separate $G|_\mu$ from B_μ in G.

This observation has two consequences, one practical, for the construction of concrete prime decompositions, the other more theoretical, for the problem of when such a decomposition exists. The first consequence is that whenever we have $S, C, C', S' \subset G$ as in our example and we have started to construct a prime decomposition of G with factors inside $G[\,C' \cup S'\,]$, we cannot move over into C until we have covered all of S' (because S' has to serve as the simplex of attachment for the first factor including a vertex from C). The impact of this restriction will be the greater the more separating simplices S there are in the graph G to be decomposed.

The other consequence of our observations sounds almost trivial, but it lies at the heart of the existence problem of prime decompositions: since B_μ is to be a prime factor and has to contain S', there must be at least one prime *subgraph* of G which contains S' and a vertex from C. We shall express this by saying that S' has a prime *extension into* C.

In the next section we shall see that such prime extensions do not always exist. This will lead us to the construction of a graph which does not admit a simplicial decomposition into primes, and on to a characterization of the graphs that do have prime decompositions in terms of the existence of prime extensions of separating simplices.

1.2. Prime extensions

When Halin [12] introduced simplicial decomposition for infinite graphs, he also proved the first major theorem on the existence of prime decompositions:

Theorem 1.2. *Every graph not containing an infinite simplex admits a simplicial decomposition into primes.*

The key lemma in the proof of Halin's theorem asserts that a finite simplex has a prime extension into any component to which it is attached.

Halin [12] also gave a construction of a graph that does not admit a prime decomposition. To get the flavour of why finding a prime decomposition may be difficult or impossible, consider the following variation of Halin's example: let x be a single vertex, $S = S[s_1, s_2, \dots]$ an infinite simplex, $C = y_1 y_2 \dots$ a one-way infinite path, and let H_1 be the graph obtained from the disjoint union of x, S and C by joining x to all the vertices of S and drawing the edges $y_i s_j$ for all $i \geqslant j$ (Fig. 5).

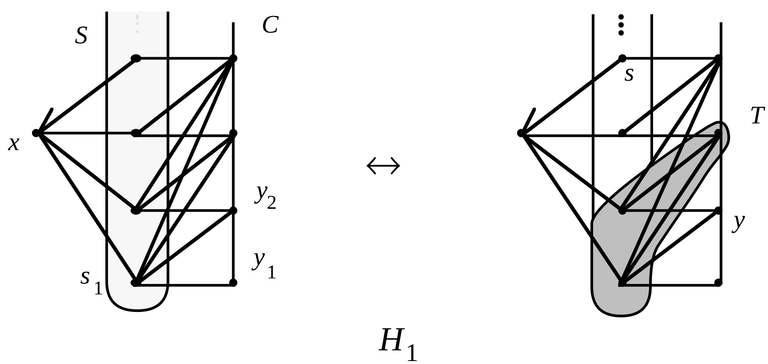

FIGURE 5. S has no prime extension into C

Here, S has no prime extension into C (observe that, in the notation of our earlier example, we would have $C' = \{ x \}$ and hence $S' = S$): since every vertex $y \in C$ is separated in H_1 from some vertex $s \in S$ by a simplex T, there can be no prime subgraph in H_1 which contains both S and a vertex of C.

What does this mean for the construction of a prime decomposition of H_1? Certainly that if we approach S 'from the left', with x being contained in the first factor B_0, then the first factor B_μ containing a vertex from C would have to contain S as well (as its simplex of attachment S_μ), which is impossible if B_μ is to be prime. Hence, we have to approach S from the right: choosing factors from within $H_1 [C \cup S]$ until S is covered, and then move accross to cover x as well. Such a prime decomposition of H_1 does indeed exist; consider, for example, the decomposition

$$F = (Y_1, Y_2, Y_3, \dots , X) ,$$

where

$$\begin{aligned} Y_i &:= H_1 [y_i, y_{i+1}, s_1, \dots, s_i] \\ X &:= H_1 [x, s_1, s_2, s_3 \dots] \end{aligned} \qquad \text{(Fig. 6).}$$

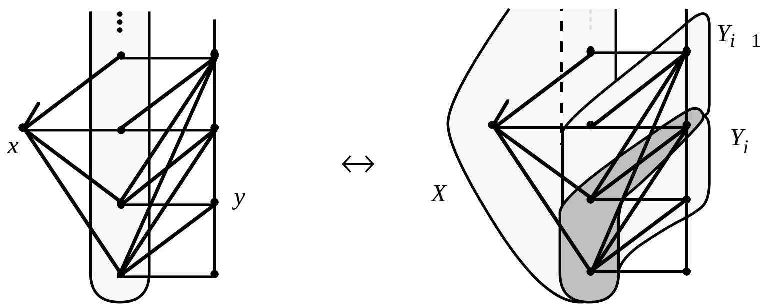

FIGURE 6. A prime decomposition of H_1

However, we only have to alter H_1 slightly to obtain a graph H_0 which does not admit a simplicial decomposition into primes: simply replace x with a copy C' of C (joined to S in the same way as C is). In this graph, which is very similar to Halin's original example, it makes no difference from which side we try to approach S in an attempt to construct a prime decomposition: the moment we try to cross S and move over into the other side we will have to find a prime extension of S (into that side), which does not exist.

Let us reformulate the indecomposability of the graph H_0 within the more general setting of our earlier prototype situation, involving an arbitrary separating simplex S and components C, C' of $G \backslash S$. We have seen that if S' (the part of S whose vertices have a neighbour in C') has no prime extension into C, then any prime decomposition of G has to start with factors containing vertices of C (and hence lying inside $G [C \cup S]$). Now there may be many such configurations of (say) S_i, C_i and C'_i in a graph, including several where S'_i fails to have a prime extension into C_i. As in each case any prime decomposition of G has to begin with vertices from C_i, these requirements are bound to conflict for different i.

One way in which this may happen led to the indecomposability of H_0: in a graph G which contains two critical configurations (S_1, C_1, C'_1) and (S_2, C_2, C'_2), such that $C_1 \subset C'_2$ and $C_2 \subset C'_1$ (Fig. 7), a prime decomposition cannot begin with vertices from C_1 and at the same time with vertices from C_2. In this case, we shall say that G contains *simplices with opposite inaccessible sides*. (In the example of H_0, we have $S_1 = S_2 = S$, and $C_1 = C'_2 = C$ (say), $C_2 = C'_1 = C'$.)

Another way in which critical configurations may impose conflicting requirements on the order of factors in a prime decomposition is that a graph G contains an infinite series of critical configurations (S_i, C_i, C'_i) such that $C_i \supseteq C_{i+1}$ for all i but $\bigcap_{i\in\mathbb{N}} C_i = \emptyset$ (Fig. 7): no matter how we choose the first factor B_0, there will be some $i \in \mathbb{N}$ for which B_0 lies in $G[C'_i \cup S'_i]$, i.e. on the wrong side of S_i. In this case, we shall say that G contains *simplices with an infinite sequence of inaccessible sides*. (See [1, Ch. 2.5] for details.)

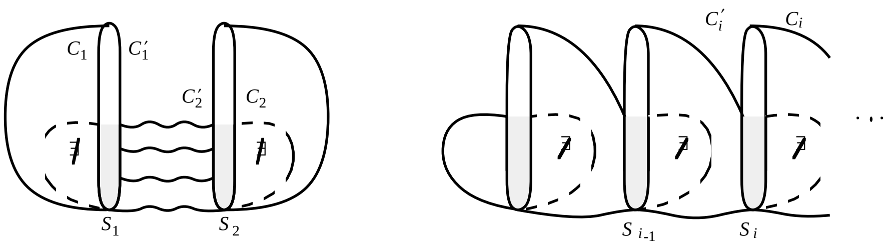

FIGURE 7. The forbidden configurations in Dirac's theorem

The following theorem of Dirac [9] summarizes our observations, providing a necessary condition for the existence of a prime decomposition.

Theorem 1.3. *A graph can only have a simplicial decomposition into primes if it contains neither simplices with opposite inaccessible sides nor simplices with an infinite sequence of inaccessible sides.*

The question of whether or not Theorem 1.3 has a direct converse is still an open problem:

Problem. *Are the conditions in Dirac's theorem also sufficient for the existence of a prime decomposition?*

While it is relatively easy to derive the necessary conditions for the existence of a prime decomposition as stated in Theorem 1.3—the formal proof goes hardly beyond the sketch we have seen above—a sufficiency proof is likely to require deeper arguments. However, it is quite possible that most of the necessary ideas are already contained in the proof of our next theorem, which solves the existence problem of prime decompositions for an important special case.

A simplicial decomposition $F = (B_\lambda)_{\lambda<\sigma}$ of G is called a *simplicial tree-decomposition* if, in addition to (S1)–(S3), it satisfies

(S4) each S_μ is contained in B_λ for some $\lambda < \mu \quad (\mu < \sigma)$.

Condition (S4) and the concept of simplicial tree-decompositions will be introduced properly in Section 4; for the moment it suffices to remark that these decompositions form a natural subclass of all simplicial decompositions, including (for example) all finite simplicial decompositions.

Returning to our prototype configuration of a separating simplex S with components C and C' of $G \backslash S$, notice that (S4) has the simplifying effect of making the situation symmetric: in order for G to admit a simplicial tree-decomposition into primes, S' must have prime extensions into *both* C and C'. For if we start our decomposition in C' (say) and B_μ is the first factor containing a vertex from C, then (S4) requires that S_μ $(= S')$ is also contained in some earlier factor B_λ, which would be a prime extension of S' into C'.

For countable graphs G, it was shown in [5] that the existence of prime extensions in such configurations is not only necessary but also sufficient for the existence of a simplicial tree-decomposition of G into primes (the proof draws on almost all aspects of simplicial decomposition theory; with additional analysis and motivation, it takes up most of a chapter in [1]):

Theorem 1.4. *A countable graph G has a simplicial tree-decomposition into primes if and only if G satisfies the following condition:*

(($\dagger$)) *If $S \subset G$ is a simplex, C and C' are distinct components of $G \backslash S$, and S is attached to C, then the simplex $S' \subset S$ induced by the vertices of S with a neighbour in C' has a prime extension into C.*

Notice that our graph H_1, which has a (general) simplicial decomposition into primes, fails to satisfy ($\dagger$) and therefore has no simplicial tree-decomposition into primes.

To conclude this section, let us see why Theorem 1.4 generally fails for uncountable G.

Let T be a the 'transitive closure' (or comparability graph) of the infinite dyadic tree. (For example, let $V(T) = \{0,1\}^{<\omega}$, the set of all finite 0-1 sequences, and join $(a_0, \dots, a_n)$ to $(b_0, \dots, b_m)$ whenever $n < m$ and $a_i = b_i$ for $i = 0, \dots, n$.) Let T' be obtained from T by adding its 'limit points': for each maximal simplex M in T add a new vertex $v(M)$, joining $v(M)$ to every vertex of M. Thus, $M' := T'[\,M \cup \{v(M)\}\,]$ is a maximal simplex in T'. (In the concrete model of T cited above, the new vertices $v(M)$ may be thought of as representing the 0-1 sequences of length ω.) We shall prove that T' satisfies ($\dagger$) but has no simplicial tree-decomposition into primes.

To see that T' satisfies (†), notice that any separating simplex $S \subset T'$ attached to a component C of $T' \backslash S$ has a prime extension of the form M' into C.

In order to show that T' has no simplicial tree-decomposition into primes, notice first that there are $2^{\aleph_0}$ vertices of the form $v(M)$, and hence $2^{\aleph_0}$ simplices of the form M'. Moreover, these simplices are the only possible prime factors of T'. (Since we skirted round the subject of which subgraphs can occur as prime factors in simplicial decompositions, the reader must be asked to take this on faith. However, we did mention that simplicial factors are always unattached—and the simplices M' are clearly the only unattached and prime subgraphs of T'.)

Now suppose that $F = (B_\lambda)_{\lambda<\sigma}$ is a simplicial tree-decomposition of T' into primes. Consider a maximal simplex M in T, and assume that $M' = B_\mu$. Then $S_\mu \subset M$. Since M' is the only factor in F that contains the entire simplex M, but S_μ must also be contained in some other factor B_λ (by (S4)), S_μ cannot be equal to M; hence, $M \backslash S_\mu \neq \emptyset$. We may therefore associate with the factor $M' = B_\mu$ a vertex $w(M') \in M \backslash S_\mu$, noting that M' is the first factor in F that contains $w(M')$ (because $w(M') \notin S_\mu$ implies that $w(M') \notin G|_\mu$). We have thus obtained an injective map from the uncountable set of factors in F to the vertex set of T, which contradicts the countability of T.

1.3. Simplicial minors

It is a striking phenomenon that long after Halin had published his original example of a graph not admitting a simplicial tree-decomposition into primes, no essentially different such 'counterexample' had been found. Indeed, all graphs G without such a prime decomposition seem to have a structure very much like that of H_1: they all contain an infinite simplex S separating G into components C and C', where S is attached to C, $G[\,C \cup S\,]$ is covered by a family $(Y_1, Y_2, \ldots)$ of convex subgraphs that have the same intersection pattern with S and with each other as in the case of H_1, and S' (defined as earlier) is not contained in any one of the Y_i. It is then only a small step further to notice that contracting C' to a single vertex x and maybe shrinking the subgraphs Y_i a little gives one a contraction of $G[\,C' \cup S \cup C\,]$ onto H_1.

And indeed, it can be shown that

(∗) *Any graph not admitting a simplicial tree-decomposition into primes has a minor isomorphic to H_1.*

How about the converse of (∗)—that is, how far is (∗) from being a characterization of the graphs that admit simplicial tree-decompositions into primes?

Notice that the graph property of not containing H_1 as a minor is closed under taking minors. Thus we can only hope to characterize the prime-decomposable graphs by that property if they share this feature, i.e. if minors of decomposable graphs are again decomposable. Or more intuitively, if we seek to express our observation that H_1 is in a sense the 'simplest' non-decomposable graph by use of the minor relation, then this relation and our notion of 'simpler' should match: a minor of a graph with a prime decomposition should itself admit a prime decomposition, and this should be at most as complex as that of the original graph.

With the usual concept of a minor, however, this is far from true. Recall that a minor of a graph G is obtained in two steps:

(1) taking a subgraph of G, and

(2) contracting connected parts of the subgraph.

Clearly, both these steps are too general to meet our above requirement. That is, if G is a graph that has a relatively simple prime decomposition, and if H is obtained from G by either of the two steps, then H may only admit much more complex prime decompositions, or even none at all.

For example, if G is an infinite simplex and thus admits the trivial prime decomposition consisting only of itself, we can find subgraphs in G with arbitrarily complex or even no prime decompositions, including H_1. An only slightly more complicated example shows that even if we restrict step (1) to taking induced subgraphs it still allows us to obtain H_1 from a prime graph: if we add a new vertex to H_1 and join it to all old vertices by independent paths of length at least 2, the resulting graph is prime (because it has no separating simplex) and contains H_1 as an induced subgraph.

As an example for step (2), consider the graph G obtained by identifying two cycles of order 10 along 3 consecutive vertices x, y, z. Then G is prime, but contracting the edges xy and yz results in a graph that has a separating simplex (the contracted vertex), and therefore only a non-trivial prime decomposition into two factors. Or more extremely, if we subdivide every edge of H_1 once, the resulting graph will again be prime (and therefore admit the trivial prime decomposition), but we can reobtain H_1 from it only by contracting the appropriate edges.

Thus if we want the converse of $(*)$ to hold, we have to restrict the definition of a minor by sharpening both of the above steps.

Let us call two vertices of a graph *simplicially close* if they are not separated by any simplex, and let us call H a *simplicial minor* of G if H is obtained from G by

(1′) taking a convex subgraph of G, and

(2′) contracting connected parts of this convex subgraph in such a way that simplicially close vertices remain simplicially close.

It is easily seen that this restricted concept of a minor no longer admits the examples we considered above. And indeed, simplicial minors satisfy the converse of $(*)$:

(∗∗) *If H_1 is a simplicial minor of G, then G admits no simplicial tree-decomposition into primes.*

(More generally, one can show that if G has a simplicial tree-decomposition into primes and H is a simplicial minor of G, then H has a simplicial tree-decomposition into primes; see [1, Ch. 3.2].)

In order to achieve (∗∗), we had to come down a long way from the most general concept of a minor that formed the basis of (∗). It would therefore not be surprising if we now had to pay for the gain of (∗∗) with the loss of (∗), that is, if not admitting a prime decomposition no longer implied the existence of certain (simplicial) minors like H_1.

The following main theorem of [2] asserts that this is not the case: the graphs admitting a simplicial tree-decomposition into primes are characterized by only two forbidden simplicial minors, H_1 and the graph H_2 obtained from H_1 by filling in all missing edges of the form $y_i y_j$.

Theorem 1.5. *A countable graph G admits a simplicial tree-decomposition into primes if and only if neither H_1 nor H_2 is a simplicial minor of G.*

Given the intuitive appeal of characterizations by forbidden configurations and the apparent suitability of simplicial minors in the context of the prime decomposition problem, it would be interesting to derive a simplicial minor version of Dirac's theorem (Theorem 1.3). Moreover, our example T' of an uncountable graph without a simplicial tree-decomposition into primes (see the end of the previous section) suggests the following conjecture:

Conjecture. *A graph G (of any cardinality) admits a simplicial tree-decomposition into primes if and only if none of H_1, H_2 or T' is a simplicial minor of G.*

2. Decompositions into small factors

The quest for prime decompositions, as discussed in Section 1, arises straight from the very notion of a decomposition: to break down a graph into factors which are as small as possible. In the case of prime decompositions, 'small' is defined locally: a factor is considered small enough (only) if it cannot be replaced with factors decomposing it further.

If we replace this local concept of 'small' with a global one, imposing a fixed bound on the size of each factor, we obtain a problem which is quite different in character but no less interesting: *given any cardinal a, which graphs admit a simplicial decomposition into factors of order $< a$?*

There are two obvious obstructions to the existence of a simplicial decomposition into such uniformly small factors. One is the existence of large complete subgraphs: if $G = K_a$, for example, then G is prime and cannot be decomposed into factors of order $< a$. (Note, however, that if we delete the vertex x from our earlier graph H_1, we obtain a graph H_1' which contains an infinite simplex but still admits a simplicial decomposition into the finite factors Y_i.)

Another property which tends to be incompatible with decomposition into factors of order $< a$ is high connectivity. For suppose that G has a simplicial decomposition into such factors, and that $x, y \in V(G)$ are non-adjacent. If x and y are in the same factor B, they cannot be joined by a or more independent paths. (These paths could without loss of generality be chosen induced, and would thus have to be contained in B, because as a simplicial factor B is a convex subgraph of G.) But if x and y are not in a common factor, they are separated by some simplex of attachment S_μ, so again they cannot be joined by a or more independent paths (because $|S_\mu| < |B_\mu| < a$).

The following theorem of Halin [15] says that if a is regular and uncountable, then these two obstructions to the existence of a simplicial decomposition into factors of order $< a$ are all there are: if we ban K_a subgraphs and large systems of independent paths, the desired decomposition exists.

Theorem 2.1. *Let G be a graph and a a regular uncountable cardinal. Suppose that $G \not\supseteq K_a$, and that for any two non-adjacent vertices $x, y \in V(G)$ there are fewer than a independent x–y paths in G. Then G admits a simplicial decomposition into factors of order $< a$.*

Note that Theorem 2.1 does not extend down to $a = \aleph_0$: the infinite grid, for example, is prime (and thus has no simplicial decomposition into finite factors), but it contains neither an infinite simplex nor an infinite set of independent paths joining any two vertices.

Thus, embarrassingly, the simplest and most intriguing case of our problem remains open:

Problem. *Which graphs admit a simplicial decomposition into finite factors?*

3. The uniqueness of prime decompositions

An aspect of simplicial decomposition theory which has been studied in some detail is that of the uniqueness of prime decompositions:

Problem. *Which graphs have a simplicial decomposition into a unique set of prime factors?*

Recall that, by Halin's theorem (Theorem 1.2), every graph not containing an infinite simplex admits a simplicial decomposition into prime factors. These factors are uniquely determined; they are precisely those prime subgraphs that are convex and unattached. The unique set of primes of a finite graph can be obtained particularly easily, by iteratively splitting the graph along minimal separating simplices.

As soon as we admit infinite simplices as subgraphs, however, prime decompositions need no longer be unique. Consider, for example, our graph $H_1' = H_1 - x$. H_1' admits two different prime decompositions,

$$F = (S, Y_1, Y_2, Y_3, \ldots)$$

and

$$F' = (Y_1, Y_2, Y_3, \ldots)\,.$$

Thus, while F is a perfectly acceptable simplicial decomposition into primes, its factor S is in a sense redundant: if we omit it, the remaining factors still form a prime decomposition of H_1'.

Let us call a simplicial decomposition *reduced* if it has no such redundant factors. Reduced prime decompositions are by far the most 'common' kind; for example, all prime decompositions into finite factors are reduced [6].

The following result was obtained in [1]:

Theorem 3.1. *Any two reduced prime decompositions of a graph have the same set of factors.*

The immediate question arising from this result is whether every graph that has some simplicial decomposition into primes also has a reduced such decomposition—in which case prime decompositions could in practice be taken reduced as a matter of course. However, this is not the case; our graph T, the 'transitive closure' of the infinite dyadic tree, has numerous simplicial decompositions into primes—select maximal simplices M in any order—but no reduced prime decomposition.

It would be interesting to know (and should not be too difficult to decide) whether T is essentially the only example of such a graph:

Problem. *Is T in some sense contained in every graph that admits a simplicial decomposition into primes but no reduced such decomposition?*

The problem of determining the graphs which admit a reduced simplicial decomposition into primes has another fascinating aspect: any solution would, on the basis of other known results, imply a solution to the last problem of the previous section, to determine which graphs admit a simplicial decomposition into finite factors. See [1, Ch. 4.4] for details.

4. Tree-decompositions

Simplicial decomposition theory does not only provide the theoretical basis for applications such as excluded minor theorems by homomorphism bases; it is also valuable for the study of a more general type of decompositions, called tree-decompositions.

The original idea behind the concept of a tree-decomposition, first introduced by Robertson and Seymour [16], was to make precise the apparent tree shape imposed on a graph by a simplicial decomposition; see Fig. 2. How could this be done?

The definition given by Robertson and Seymour refers directly to the tree $T(G)$ which the shape of the graph G is deemed to resemble: they call a family $(X_t)_{t\in T(G)}$ of subsets of $V(G)$ a tree-decomposition of G if

(T1) $\bigcup_{t\in T(G)} X_t = V(G)$;

(T2) $\forall\, xy \in E(G) : \exists\, t \in T(G) : x, y \in X_t$;

(T3) if $t, t', t'' \in T(G)$ and t' lies on the t–t'' path in $T(G)$, then $X_t \cap X_{t''} \subset X_{t'}$.

For compatibility with simplicial decompositions, we shall here take a slightly different approach, closer to our definition of a simplicial decomposition. Given a simplicial decomposition $F = (B_\lambda)_{\lambda<\sigma}$, how should we associate a tree T_F with F, so as to express the tree shape of G imposed by F?

The obvious choice for the vertex set of T_F is the set of factors in F,

$$V(T_F) = \{\, B_\lambda \mid \lambda < \sigma \,\}\,.$$

The edges of T_F should, intuitively, correspond to the simplices of attachment in F, so let us choose them inductively for each $\mu < \sigma$. Having constructed a partial tree on $\{\, B_\lambda \mid \lambda < \mu \,\}$, how shall we join the next 'vertex' B_μ to this tree? In the example of Fig. 2, the obvious solution would be to join B_μ to the unique B_λ (with $\lambda < \mu$) which contains S_μ. Clearly, this can be done whenever F is such that

(S4) each S_μ is contained in B_λ for some $\lambda < \mu \quad (\mu < \sigma)$;

if B_λ is not unique, we simply choose λ to be minimal. We may therefore call F a *simplicial tree-decomposition* of G if, in addition to (S1)–(S3), it also satisfies (S4).

Note that the existence of the decomposition tree T_F, as defined above, is now independent of the conditions which originally gave rise to F as a simplicial decomposition, in particular of (S2). If the tree shape of G is all we are interested in, we may therefore discard (S2) (and, if we so wish, (S3)) and call any family $F = (B_\lambda)_{\lambda<\sigma}$ a *tree-decomposition* of G if it satisfies (S1) and (S4). It is not difficult to show that this definition of a tree-decomposition is equivalent to that given by Robertson and Seymour, except for the additional well-ordering of $V(T_F)$. (The latter is often convenient to have, especially for proofs by induction on μ.)

Tree-decompositions are well worth investigating for their own sake, just as simplicial decomposition are. The most natural questions, however, are different. For example, there is not much point in asking for a tree-decomposition into primes: the only graphs that are prime (i.e. irreducible) with respect to tree-decompositions are the complete graphs, and only chordal graphs can have tree-decompositions into complete factors [3]. On the other hand, the value of a tree-decomposition can meaningfully be measured by the size of its factors. This raises questions like that of determining the *tree-width* of the graphs G with a given property—the tree-width of G is the smallest natural k, if one exists, such that G admits a tree-decomposition into factors of order at most $k+1$—or of how to obtain algorithmically a tree-decomposition of a graph that realizes its tree-width.

The following natural problem has not yet, to my knowledge, been investigated in its own right:

Problem. *Which graphs admit a tree-decomposition into finite factors?*

Robertson, Seymour and Thomas (see [18]) have recently shown that the exclusion of infinite complete minors (even of *subdivisions* of infinite simplices) is sufficient for the existence of such a decomposition: every $K_{\aleph_0}$-free graph has a tree-decomposition into finite factors. The converse of this, however, is false: our decomposition $F' = (Y_1, Y_2, Y_3, \ldots)$ of the graph H_1' in Section 3 is a tree-decomposition into finite factors, but H_1' contains an infinite simplex.

Even if a simplicial decomposition, by the example shown in Fig. 2, has inspired our definition of a tree-decomposition—we have not yet shown that every simplicial decomposition F does indeed satisfy (S4) and thereby defines a decomposition tree T_F. In other words: can a simplicial decomposition fail to satisfy (S4) and thus not be a simplicial tree-decomposition?

At first glance, it seems pretty obvious that (S4) should follow from (S2). For if a simplex of attachment S_μ is not contained in a single earlier factor B_λ, one should expect it to contain vertices that are separated by some other simplex of attachment—which of course is impossible if these vertices, being in the same simplex, are to be adjacent (Fig. 8). And indeed, it is not difficult to show that any simplicial decomposition is in fact a simplicial tree-decomposition—if all its simplices of attachment are finite.

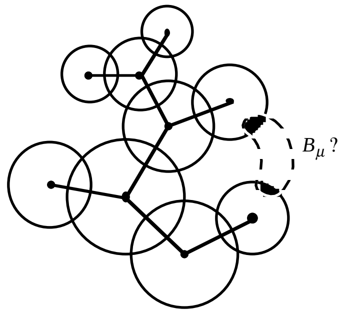

FIGURE 8. How (S2) prevents cycles in the decomposition tree

However, we have already seen an example of a simplicial decomposition which is not a simplicial tree-decomposition: the decomposition of the graph H_1 given in Section 1.2. This decomposition does not satisfy (S4), because the (infinite) simplex of attachment of its last factor X, the simplex S, is not contained in any of the earlier factors Y_i.

Thus if we tried to define a decomposition tree for this decomposition of H_1 in the usual way, we would get stuck at the last factor X: this factor should somehow sit 'above' the infinite path $Y_1 Y_2 \ldots$, but it has no immediate predecessor among the vertices of this path. (Intuitively, we fail to obtain a tree not because the factor X would introduce a cycle—a danger averted effectively by condition (S2); see Fig. 8—but because we would lose the connectedness of the tree.)

Fortunately, such 'trees' with limit points are familiar objects: they are equivalent to *order theoretical trees*, partially ordered sets in which every set of predecessors is linearly ordered. (In our example, $\{ Y_1, Y_2 \ldots \}$ would be the set of predecessors of X.) And indeed, it is not difficult to show that with any simplicial decomposition F which fails to satisfy (S4) one can associate such a (well-founded) order theoretical decomposition tree T_F, much in the same way as one associates a graph theoretical tree with a simplicial tree-decomposition. (The similarity can be made precise: the decomposition tree for general simplicial decompositions is defined in such a way that it coincides with the (natural order of the rooted) decomposition tree induced by (S4) if the decomposition happens to be a simplicial tree-decomposition; see [1, Ch. 5.4] for details.)

Let us return to the original theme of this section. We set out to find a concept of tree-decomposition which makes precise the apparent tree shape of a graph decomposed simplicially. We have partially succeeded in doing so by introducing the condition (S4); this condition, if satisfied, gives rise to a decomposition tree T_F which corresponds in the desired natural way to a given decomposition F. However, we have seen that (S4) is not satisfied by every simplicial decomposition. On the other hand, a general simplicial decomposition still gives rise to a generalized type of decomposition tree. The question we face, therefore, is this: how can (S4) be weakened, to another condition (S5), say, in such a way that every simplicial decomposition satisfies (S5) but (S5) is still strong enough to give rise to an order theoretical decomposition tree as associated with a general simplicial decomposition?

Our discussion of H_1 suggests that the desired condition (S5) should aim to capture the 'acyclicity' in the arrangement of factors in a simplicial decomposition, as shown in Fig. 8. For as we have seen, this acyclicity is precisely the tree-like property which is retained by order theoretical over graph theoretical trees (as opposed to their connectedness), while on the other hand this acyclicity does indeed follow from (S2), and is therefore a feature of general simplicial decompositions. In other words, (S5) should express that a new factor B_μ cannot use vertices of $G|_\mu$ for its simplex of attachment if these vertices are taken from different 'branches' of the decomposition tree of $G|_\mu$. The following condition aims to express this:

$$\text{(S5)}\;\; \mu < \sigma,\; x \in G|_\mu \backslash S_\mu,\; y \in B_\mu \backslash S_\mu \quad \Rightarrow \quad \nexists \lambda < \sigma : x, y \in B_\lambda.$$

And indeed, it is not difficult to show that this condition (S5) has the desired properties: while being a consequence of (S2) and thus common to all simplicial decompositions, it gives rise to the same decomposition tree as that associated with a general simplicial decomposition. We are therefore justified in calling a family $F = (B_\lambda)_{\lambda<\sigma}$ a *generalized tree-decomposition* if it satisfies (S1) and (S5), and the order theoretical tree T_F associated with it its *(generalized) decomposition tree*. Then any tree-decomposition (as well as any simplicial decomposition) is also a generalized tree-decomposition—notice that (S4) implies (S5)—and the (order theoretical) decomposition tree of a generalized tree-decomposition F coincides with the (graph theoretical) decomposition tree of F if F happens to satisfy (S4).

Let us finally see how the concepts of these tree-decompositions can be applied. Recall that with the use of simplicial decompositions for the characterization of minor-closed graph properties (via homomorphism bases) we encountered a serious problem: in many cases the prime factors which were needed to construct all the graphs without a certain given minor were as numerous and as varied as these graphs themselves, a 'characterization' of these graphs by their prime factors therefore both difficult and pointless. It is therefore natural to relax the definition of the decomposition used: this should result in smaller and therefore fewer possible different factors.

The tree-decompositions definied above were inspired by this idea: they still aim to capture the overall tree structure of the graphs with a given property (in our case, without a given excluded minor), while making fewer requirements on the nature of the factors and their attachment graphs. Of course, the graphs with a given property can only be characterized by their tree shape if they all share a common tree structure. While this cannot be expected of an arbitrary minor-closed graph property, it is worth an effort trying to find out for which properties it can be done:

Problem. *Which graph properties can be described in terms of the tree structure of their members?*

For properties defined by the exclusion of a single finite minor X, this question is answered beautifully by a result of Robertson and Seymour [17]: the graphs without an X minor have tree-decompositions into factors of bounded finite order if and only if X is planar.

Graph properties defined by excluding infinite minors have only been studied very recently. For graphs without large infinite simplices as minors, a characterization purely in terms of tree-decompositions was obtained in [8] (or [1, Ch. 5.4]):

Theorem 4.1. *If a is a regular uncountable cardinal, then a graph G has no K_a minor if and only if G admits a generalized tree-decomposition $F = (B_\lambda)_{\lambda<\sigma}$ such that every B_λ and every chain in T_F has order $< a$.*

The proof of this result is based on Theorem 2.1 and uses simplicial decompositions. Applying different methods, Robertson, Seymour and Thomas have recently obtained an equivalent theorem, which moreover has a natural extension to singular a and to $a = \aleph_0$. For a survey of this and other beautiful structure theorems in terms of generalized tree-decompositions see [18] in this volume.

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Reinhard Diestel, Faculty of Mathematics (SFB 343), Bielefeld University