Note

Decomposition Duality

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The purpose of this note is just to point out a duality aspect of graph decompositions that came up unexpectedly in the context of other investigations [1], but which can be presented briefly on its own and may be of independent interest. The duality we shall define generalizes the duality of plane graphs. It thus raises the question whether results based on the latter (such as colouring-flow duality) can be extended or seen in a new light when viewed from a decomposition angle.

Let G, H be graphs. Consider a family $\mathcal{D} = (G_h)_{h \in H}$ of induced subgraphs of G indexed by the vertices of H. Let us call $\mathcal{D}$ an H*-decomposition* of G (into the *parts* G_h) if

(D1) every vertex of G lies in some G_h

(D2) given an edge $e = gg' \in G$, either e lies in some G_h or there exists an edge $hh' \in H$ such that $g \in G_h$ and $g' \in G_{h'}$

and call this decomposition $\mathcal{D}$ *connected* if it also satisfies

(D3) whenever a vertex $g \in G$ lies in $G_{h_1} \cap G_{h_2}$ for some $h_1, h_2 \in H$, there is a path $P = h_1 \dots h_2$ in H such that $g \in G_h$ for every $h \in P$.

When H is a tree, then this kind of decomposition is closely related to standard tree-decompositions; see [1].

For every $g \in G$ define a subgraph H_g of H by setting

$$H_g = H[\{ h \mid g \in G_h \}], \tag{1}$$

and write

$$\mathcal{D}^* := (H_g)_{g \in G} \tag{2}$$

for the family of these subgraphs. We shall call the family $\mathcal{D}^*$ the *dual* of the family $\mathcal{D}$, and its parts H_g the *co-parts* of $\mathcal{D}$.

It is possible to rewrite the conditions (D1)–(D3) more elegantly in terms of these co-parts. Recall that two subgraphs of a graph H are said to *touch* (in H) if they have a vertex in common or H contains an edge between them. Each of the conditions (D1)–(D3) is easily seen to be equivalent to the corresponding following condition:

(C1) every H_g is non-empty;

(C2) for every edge $gg' \in G$, the graphs H_g and $H_{g'}$ touch in H;

(C3) every H_g is connected.

The conditions (C1)–(C3) offer an alternative way to think of a given H-decomposition of G: as a collection of subgraphs of H rather than of G. Indeed, presenting $\mathcal{D}^*$ instead of $\mathcal{D}$ entails no loss of information. For given any two vertices $g \in G$ and $h \in H$, clearly (1) implies that

$$g \in G_h \quad \Leftrightarrow \quad h \in H_g \tag{3}$$

and hence that

$$G_h = G[\{ g \mid h \in H_g \}], \tag{4}$$

the dual statement to (1). So the G_h can be reobtained from the H_g, and they are obtained from them in exactly the same way as the H_g were obtained from the G_h. Thus,

$$\mathcal{D}^{**} = \mathcal{D}.$$

Now it may or may not happen that the family $\mathcal{D}^* = (H_g)_{g \in G}$ obtained from our H-decomposition $\mathcal{D} = (G_h)_{h \in H}$ is in turn a G-decomposition of H. Let us call $\mathcal{D}$ *invertible* if this is the case. By (4), the conditions (C1)–(C3) for when this happens translate as follows:

Proposition 1. *$\mathcal{D}$ is invertible if and only if*

(I1) *every G_h is non-empty;*

(I2) *for every edge $hh' \in H$, the parts G_h and $G_{h'}$ touch in G.*

The dual decomposition $\mathcal{D}^*$ of $\mathcal{D}$ will be connected if and only if

(I3) *every G_h is connected.* □

The 'message' of translating the conditions on $\mathcal{D}^*$ into statements about $\mathcal{D}$ in this way is that $\mathcal{D}$ is invertible as soon as H has no more vertices or edges than are required for $\mathcal{D}$ to satisfy (D1) and (D2). Thus we can make any H-decomposition invertible simply by deleting superfluous vertices and edges of H. (This technique turned out to be surprisingly useful in [2].)

Given a pair of dual decompositions $\mathcal{D}$ and $\mathcal{D}^*$ as above, one might at first be tempted also to think of H as a dual of G, and vice versa. However, given a graph G there is no unique H such that G has an invertible H-decomposition.

For example, every graph $G \neq \emptyset$ has two trivial invertible decompositions: the K_1-decomposition into just one part G, and the G-decomposition into singletons, ie. with $G_g = \{ g \}$. More generally, for every graph H that contains G as a minor, G has the connected H-decomposition $(G_h)_{h \in H}$ into singletons defined by choosing as H_g the subgraph of H induced by the branch set corresponding to g. If G is even an MH, ie. obtained just by contraction without deletion, then (I1) and (I2) hold while (I3) is void, so this decomposition is invertible.

However, what we might ask for is a *canonical* way of obtaining such dual graphs and decompositions:

Problem 2. *For which classes $\mathcal{C}$ of finite graphs can we assign in a non-trivial way to every graph $G \in \mathcal{C}$ a graph $G^* \in \mathcal{C}$ and an invertible G^*-decomposition $\mathcal{D}(G)$ of G so that $G^{**} = G$ and $\mathcal{D}(G^*) = \mathcal{D}^*(G)$?*

Here, $\mathcal{D}^*(G)$ denotes the dual of the decomposition $\mathcal{D}(G)$. The definition of 'non-trivial' will have to be adjusted to need; for example, it should probably preclude most choices of G^* and $\mathcal{D}(G)$ with $G^* = G$.

Abstract though this may seem, there is a well-known instance of this kind of duality: the duality of planar graphs. Indeed, if G is a planar graph (3-connected, say, to make its drawing unique) and G^* is its planar dual, then the obvious G^*-decomposition of G into its face boundaries satisfies all the requirements of Problem 2. A special case of this problem, therefore, would be to ask for $\mathcal{C}$ to extend the class of planar graphs and for G^* and $\mathcal{D}(G)$ to coincide with planar duality and face decompositions when G is planar. For example:

Problem 3. *Can the flow-colouring duality for plane graphs be extended to a larger class based on decomposition duality?*

It should be pointed out that interpreting planar duality as decomposition duality in this way is, so far, no more than a restatement, carrying no 'substance'. The hope, however, is that viewing this duality from a decomposition angle might guide attempts to generalize planar duality in ways that are natural in a decomposition context but might otherwise not be obvious.

References

[1] R. Diestel & D. Kühn, Graph minor hierarchies, preprint 2001 (on this server).

[2] R. Diestel & D. Kühn, Tree hierarchies, manuscripts 2000.