On Infinite Cycles I

To the memory of C.St.J.A. Nash-Williams

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Abstract

We adapt the cycle space of a finite graph to locally finite infinite graphs, using as infinite cycles the homeomorphic images of the unit circle S^1 in the graph compactified by its ends. We prove that this cycle space consists of precisely the sets of edges that meet every finite cut evenly, and that the spanning trees whose fundamental cycles generate this cycle space are precisely the end-faithful spanning trees. We also generalize Euler's theorem by showing that a locally finite connected graph with ends contains a closed topological curve traversing every edge exactly once if and only if its entire edge set lies in this cycle space.

1 Introduction

One of the basic and well-known facts about finite graphs is that their *fundamental* cycles C_e (those consisting of a chord $e = xy$ on some fixed spanning tree T together with the path xTy joining the endvertices of e in T) generate their entire cycle space: every cycle of the graph can be written as a sum mod 2 of fundamental cycles. Richter [11] asked if and how this fact might generalize appropriately to locally finite infinite graphs. We show that this question, if viewed in the right way, admits a surprisingly elegant positive answer involving 'infinite cycles'. As a spin-off, we obtain infinite generalizations of some other properties of the cycle space of a finite graph, such as cycle-cut orthogonality and Euler's theorem.

Of course, the finite fundamental cycle theorem transfers verbatim to infinite graphs as long as we consider only the usual finite cycles, and stick to the usual definition of the cycle space as the subspace of the edge space generated by these cycles. Our motivation to introduce infinite cycles was, originally, just to make the problem more natural and more interesting. But it has since turned out that the cycle space we propose here appears to be the 'right' notion for locally finite graphs in a more technical sense too.[1] (For an overview, as well as a non-technical introduction to the subject, we refer the reader to [5].)

To motivate our ideas, let us look at an informal example. Let L be the 2-way infinite ladder viewed as a 1-complex, and compactify it by adding two points ω, ω' at infinity, one for each end of the ladder (Figure 1). Let a *circle* in the resulting topological space $\overline{L}$ be any homeomorphic image of the unit circle S^1 in the Euclidean plane. Then every cycle of L is a circle in $\overline{L}$, but there are more circles than these. For example, the two sides of the ladder (each a 2-way infinite path) form a circle C_1 together with the points ω and ω', and for every rung vw the two horizontal 1-way infinite paths from v or w towards ω form a circle C_2 together with ω and the edge vw. Both these circles contain infinitely many edges, and they are determined by these edges as the closure of their union.

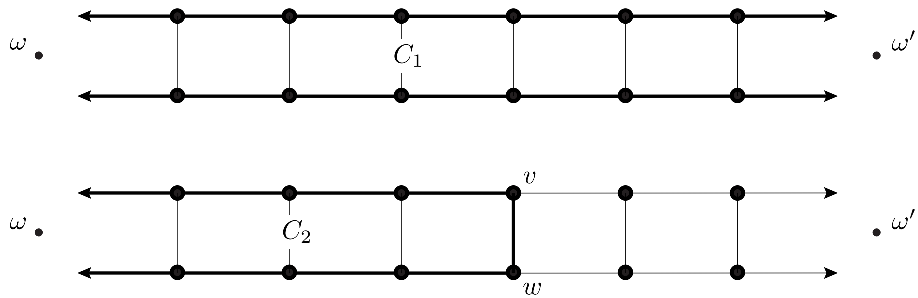

Figure 1: Two infinite circles in the double ladder plus ends

Now consider a spanning tree T of L. If T consists of the bottom side of L and all the rungs, then every edge e of the top side of L induces a fundamental cycle C_e. The sum (mod 2) of all these fundamental cycles is precisely the edge set of C_1, the set of horizontal edges of L. Similarly, the edge set of C_2 is the sum all the fundamental cycles C_e with e left of v (Figure 2).

However, for the spanning tree T' consisting of the two sides of L and the one rung vw, neither C_1 nor C_2 can be expressed as a sum of fundamental cycles. Indeed, as every fundamental cycle contains the edge vw, any sum of

[1]Since we wrote this paper, our notion of cycle space has been shown to permit the extension to locally finite graphs of several classical finite theorems which, unlike the fundamental cycle theorem, do not extend naïvely. Our extension of Eulers theorem is one example; a more comprehensive list can be found in [5]. Graphs with infinite degrees are treated in [6] and [7].

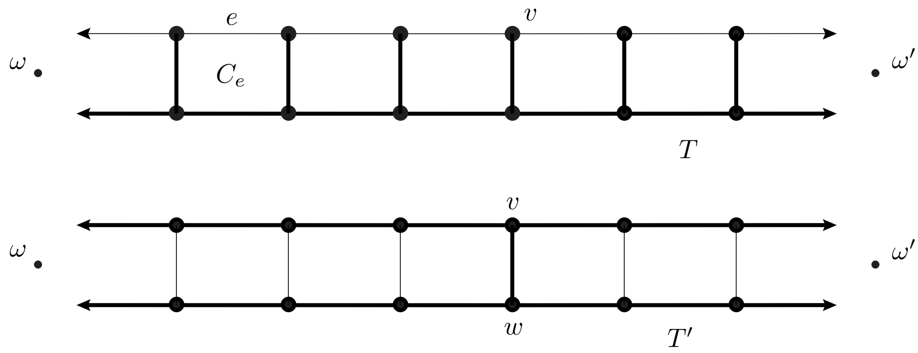

Figure 2: Two spanning trees of the double ladder

infinitely many fundamental cycles will be ill-defined: it is not clear whether the edge vw should belong to this sum or not.

So even this simple example shows that our task is interesting: while it is possible and natural to extend the usual cycle space of a finite graph to infinite graphs in a way that allows for both infinite (topological) cycles and infinite sums generating such cycles, the answer to the question of whether all infinite cycles and their sums can be generated from fundamental cycles is by no means clear and will, among other things, depend on the spanning tree considered.

Here is an overview of the layout of our paper and its main results. In Section 2 we identify some minimum requirements which any topology on an infinite graph with its ends—in the ladder example, these are the points ω and ω'— should satisfy in order to reflect our intuitive geometric picture of ends as distinct points at infinity. We then define the cycle space of a locally finite graph more formally.

In Section 3 we introduce *end-faithful* spanning trees. We show that, in a locally finite graph, these are precisely the spanning trees for which infinite sums of fundamental cycles are always well-defined.[2]

In Section 4 we consider the question of how best to choose the topology on an infinite graph with ends to obtain the most natural notion of a circle

[2]In [7] we extend the notion of end-faithful spanning trees fully to our new topological setting. The *topological spanning trees* defined there are the path-connected subspaces of the graph with ends that are made up of entire edges and contain all the graph's vertices and ends but no circle. The end-faithful spanning trees of a locally finite graph are precisely those of its ordinary spanning trees whose closure is a topological spanning tree. In general, however, a topological spanning tree need not induce a connected subgraph; its path-connectedness may result from the existence of topological paths that contain ends.

and the strongest possible infinite version of the fundamental cycle theorem. For locally finite graphs we prove that this topology is essentially unique.

As our first main result we prove in Section 5 that, for the topology chosen in Section 4 and for precisely the spanning trees identified in Section 3, every infinite cycle of a locally finite graph is the (infinite) sum of fundamental cycles, and so are the other elements of its cycle space.

In Section 6 we ask to what extent this result depends on the concrete topology assumed. We find an abstract condition on the topology of a locally finite graph such that our theorem holds, for any end-faithful spanning tree, if and only if this condition is met.

As our second main result we show in Section 7 that the usual cycle-cut orthogonality in finite graphs extends as follows: a set of edges in a locally finite graph lies in its cycle space if and only if it meets every finite cut in an even number of edges. As a corollary, we obtain an extension of Nash-Williams's theorem that a graph is an edge-disjoint union of cycles if and only if all its cuts are even or infinite.[3] Finally, we show that a connected locally finite graph with ends admits a *topological Euler tour*, a closed topological curve traversing every edge once, if and only if its entire edge set lies in its cycle space.

2 Basic facts and concepts

The terminology we use is that of [2]. We shall freely view a graph either as a combinatorial object or as the topological space of a 1-complex. (So every edge is homeomorphic to the real interval $[0,1]$, the basic open sets around an inner point being just the open intervals on the edge. The basic open neighbourhoods of a vertex x are the unions of half-open intervals $[x,z)$, one from every edge $[x,y]$ at x; note that we do not require local finiteness here.) When E is a set of edges we let $\mathring{E}$ denote the union of their interiors, i.e. the set of all inner points of edges in E.

Given a spanning tree T in a graph G, every edge $e \in E(G) \setminus E(T)$ is a *chord* of T, and the unique cycle C_e in $T+e$ is a *fundamental cycle* with respect to T.

A family $(A_i)_{i\in I}$ of subsets of a set A will be called *thin* if no element of A lies in A_i for infinitely many i, and the *sum* $\sum_{i\in I} A_i$ of this family is the set of those elements of A that lie in A_i for an odd number of indices i.

[3] In [6] we will prove more generally that an arbitrary set of edges (not just the entire edge set) is a disjoint union of the edge sets of cycles as soon as it lies in the cycle space.

A homeomorphic image (in the subspace topology) of $[0, 1]$ in a topological space X will be called an *arc* in X; a homeomorphic image of S^1 in X is a *circle* in X.

We shall frequently use the following well-known fact [1, Thm. 3.7]:

Lemma 2.1 *Every continuous injective map from a compact space X to a Hausdorff space Y is a topological embedding, i.e. a homeomorphism between X and its image in Y under the subspace topology.* □

We refer to 1-way infinite paths as *rays*, to 2-way infinite paths as *double rays*, and to the subrays of rays or double rays as their *tails*. If we consider two rays in a graph G as *equivalent* if no finite set of vertices separates them in G, then the equivalence classes of rays are known as the *ends* of G. (The ladder, for example, has two ends, the grid has one, and the binary tree has continuum many; see [3] for more background.) We shall write $\overline{G}$ for the union of G (viewed as a space, i.e. a set of points) and the set of its ends. Topologically, ends may be thought of as additional 'vertices at infinity'. (The standard topology on $\overline{G}$ that we shall consider will bear this out in the technical compactification sense of the word, but all we try to convey at this point is the intended geometric intuition.)

We shall consider various topologies on $\overline{G}$ in this paper. But they will all satisfy the following two minimum requirements, without which we feel the resulting notion of a circle would seem unnatural and contrived.

The topology on $\overline{G}$ is Hausdorff, and it induces on G the given topology of G as a 1-complex. (1)

Moreover, every ray should converge to the end it belongs to:

If $R \subseteq G$ is a ray and ω is the end of G containing R, then every neighbourhood of ω contains a tail of R. (2)

Together, conditions (1) and (2) imply that a subset of G is open in $\overline{G}$ if and only if it is open in G; we shall use this fact freely throughout the paper. It can be used to show—by elementary topological arguments, but not completely trivially—that every arc in $\overline{G}$ whose endpoints are vertices or ends, and similarly every circle in $\overline{G}$, includes every edge of G of which it contains an inner point. Thus in particular, every circle in $\overline{G}$ 'has' a unique set of edges, and we may define the *circuits* of G as the edge sets of its circles. Note that these include the edge sets of the usual finite cycles in G, and in particular of its fundamental cycles (with respect to any given spanning tree); we shall call these latter the *fundamental circuits* of G.

Based on the concept of a circuit and our earlier definition of 'sum', we may now define the *cycle space* $\mathcal{C}(G)$ of a locally finite graph G as the set of sums of circuits in $\overline{G}$. Then, just as for finite graphs, $\mathcal{C}(G)$ is a subspace of the edge space of G, and the definition reduces to the standard one for finite G. As a consequence of our main result (and hence assuming a concrete topology for $\overline{G}$) we shall see later that $\mathcal{C}(G)$ is closed also under taking infinite sums. This does not appear to be obvious from the definition, and we have not pursued the question of whether the definition implies it (independently of the topology assumed).

Another condition that seems natural in a context where the circles of a graph are to be represented by their edge sets is that every circle is uniquely determined by its edges, as the closure of their union. Equivalently:

$$\text{For every circle } C \subseteq \overline{G}, \text{ the set } C \cap G \text{ is dense in } C. \tag{3}$$

Although we shall not formally require (3), the topologies we shall consider for $\overline{G}$ will turn out to satisfy this condition, too. However, (3) does not follow from (1) and (2): in Section 4 we shall construct a graph with a topology satisfying (1) and (2) that contains a circle consisting entirely of ends.

3 Choosing the spanning tree for a locally finite graph

Our ladder example seemed to suggest that choosing the right spanning tree might be an essential and difficult part of our problem. Fortunately, this is not the case: as we shall see, there is a canonical kind of spanning tree that will always do the job, and none other will. Before we define these spanning trees, however, let us recall a standard lemma about locally finite graphs; the proof is not difficult and is included in [4, Lemma 1.2].

Lemma 3.1 *Let U be an infinite set of vertices in a connected locally finite graph G. Then there exists a ray $R \subseteq G$ for which G contains an infinite set of disjoint U–R paths.* □

If T is a spanning tree of a graph G, then clearly every end ω of T is a subset of a unique end ω' of G. The tree T is called *end-faithful* in G if this *canonical projection* $\omega \mapsto \omega'$ is 1–1 and onto, i.e. if after fixing an arbitrary root $t_0 \in T$ we have exactly one ray in T starting at t_0 in every end of G. It is not difficult to show that every connected countable graph has an end-faithful spanning tree; for example, the *normal* spanning trees constructed in [8] or [2, Ch. 1.5.5] are end-faithful when the graph is infinite. See [3] for further details.

The following observation shows that we shall want to restrict our attention to end-faithful spanning trees: the edge set of any other spanning tree T would always contain a non-empty circuit, which is not only counterintuitive but would also put an end to our hopes of showing that all circuits are sums of fundamental circuits. (Clearly, in any such sum each fundamental circuit present could be taken to occur exactly once, but then the sum would contain its chord and hence not lie in $E(T)$.)

Lemma 3.2 *Let T be a spanning tree of a locally finite graph G, and assume that $E(T)$ contains no non-empty circuit of G. Then T is end-faithful in G.*

Proof. If T is not end-faithful then the map $\omega \mapsto \omega'$ above fails to be 1–1, since by Lemma 3.1 it cannot fail to be onto. (Given any ray R in G, apply the lemma to T with $U = V(R)$ to obtain a ray in T equivalent to R.) Then we can find rays $R_1, R_2 \subseteq T$ that start at the same vertex and are otherwise disjoint, but belong to the same end ω' of G. It is now straightforward to check that, by (1) and (2), $R_1 \cup R_2 \cup \{\omega'\}$ is a circle in G. So $E(R_1 \cup R_2)$ is a circuit of G, as desired. □

The converse of Lemma 3.2 will follow, for a concrete topology on $\overline{G}$ we shall consider, from our main result that whenever T is end-faithful in G all the circuits in G are sums of fundamental circuits (and hence, in particular, not contained in T). Conditions (1) and (2) do not imply the converse of Lemma 3.2 for arbitrary topologies, though: in Section 4 we shall construct a topology for the binary tree which satisfies (1) and (2), but under which the edge set of a double ray in the tree occurs as a circuit. A more complicated example at the end of Section 6 will show that the converse of Lemma 3.2 does not even follow from (1), (2) and (3).

End-faithful spanning trees have the pleasant property that every sum of fundamental circuits is well-defined:

Lemma 3.3 *Let T be an end-faithful spanning tree of a locally finite graph G. Then the fundamental circuits of G with respect to T form a thin family.*

Proof. Suppose there are infinitely many fundamental cycles $C_1, C_2, \dots$ all containing the same edge $e = xy$. Then xy is an edge of T; let T_x and T_y be the components of $T - e$ containing x and y, respectively. For $i = 1, 2, \dots$ let $e_i = x_i y_i$ be the edge of C_i not on T; since $e \in C_i$, we may assume that $x_i \in T_x$ and $y_i \in T_y$.

Applying Lemma 3.1 to T_x with $U = \{x_1, x_2, \dots\}$, we obtain a ray $R_x \subseteq T_x$ and an infinite index set $I \subset \mathbb{N}$ such that the paths $P_i \subseteq T_x$ from x_i to R_x are disjoint for different $i \in I$. Applying Lemma 3.1 to T_y with $U = \{y_i \mid i \in I\}$, we likewise obtain a ray $R_y \subseteq T_y$ and an infinite index set $I' \subseteq I$ such that the paths $Q_i \subseteq T_y$ from y_i to R_y are disjoint for different $i \in I'$. As the rays R_x and R_y are disjoint, they belong to different ends of T. But each of the paths $P_i e_i Q_i$ with $i \in I'$ links R_x to R_y in G, and these are infinitely many disjoint paths. Therefore R_x and R_y belong to a common end of G, so T is not end-faithful. □

We remark that the converse of Lemma 3.3 holds too: if T is not end-faithful, we can always find a family of fundamental circuits that is not thin.

4 Choosing the topology on $\overline{G}$

Since the meaning of our intended result (that the fundamental circuits of a graph generate its cycle space) depends on the notion of a circle and hence on the topology considered for $\overline{G}$, we have to fix some such topology at some point. But which topology should we choose? In Section 2 we laid down two minimum requirements for any topology on $\overline{G}$ that we might consider as natural, conditions (1) and (2). However, these two conditions do not determine the topology on $\overline{G}$.

For example, the following topology satisfies (1) and (2) and would not seem unnatural. Given an end ω and a finite set S of vertices of G, there is exactly one component $C = C_G(S, \omega)$ of $G - S$ which contains a tail of every ray in ω. We say that ω *belongs* to C. Writing $E_G(S, \omega)$ for the set of all the S–C edges in G, let us consider the topology on $\overline{G}$ that is generated by the open sets of G (as a 1-complex) and all sets of the form $\{\omega\} \cup C_G(S, \omega) \cup \mathring{E}'_G(S, \omega)$, where $\mathring{E}'_G(S, \omega)$ is any union of half-edges $(x, y] \subset e$, one for every $e \in E_G(S, \omega)$, with $x \in \mathring{e}$ and $y \in C$. Then the circles in this topology resemble those of our ladder example:

Proposition 4.1 *Let G be any infinite graph. Under the above topology, every circle in $\overline{G}$ is either a finite cycle or the union of finitely many double rays with their ends.*

Proposition 4.1 is not difficult to prove, and it easily implies that every circuit is the sum of finite circuits. If G is locally finite then this implies by Lemma 3.3 that, given any end-faithful spanning tree, the fundamental circuits do indeed generate the cycle space.

Although the topology for $\overline{G}$ on which Proposition 4.1 is based may be a natural one to consider, it does not yield the strongest possible theorem. For note that if $\textsc{Top}_1$ and $\textsc{Top}_2$ are topologies on $\overline{G}$ such that $\textsc{Top}_2$ is Hausdorff and coarser than $\textsc{Top}_1$, then every $\textsc{Top}_1$-circle is also a $\textsc{Top}_2$-circle (cf. Lemma 2.1). Thus reducing the collection of open sets increases the set of circles, and so we can strengthen our theorem by proving it for a coarser topology. Our next aim, therefore, is to introduce a topology on $\overline{G}$ that is coarser than that considered above. For locally finite G, this topology will turn out to be coarsest possible with (1), and therefore yield the best possible result.

Given an end ω and a finite set S of vertices of G, let $\overline{C}_G(S,\omega)$ denote the union of $C := C_G(S,\omega)$ with the set of all ends belonging to C. There is an obvious correspondence between these ends of G and those of C, and we shall not normally distinguish between them. (Thus, $\overline{C}$ will be treated as a subset of $\overline{G}$ when this simplifies the notation.) Let $\textsc{Top}$ denote the topology on $\overline{G}$ generated by the open sets of the 1-complex G and all sets of the form

$$\widehat{C}_G(S,\omega) := \overline{C}_G(S,\omega) \cup \mathring{E}'_G(S,\omega)\,,$$

where again $\mathring{E}'_G(S,\omega)$ is any union of half-edges $(x,y] \subset e$, one for every $e \in E_G(S,\omega)$, with $x \in \mathring{e}$ and $y \in C$. So for each end ω, the sets $\widehat{C}_G(S,\omega)$ with S varying over the finite subsets of $V(G)$ are the basic open neighbourhoods of ω. The topology which $\textsc{Top}$ induces on the end space $\overline{G} \setminus G$ of G is the standard topology there as studied in the literature.

The following observation is not difficult to prove; see e.g. [3].

Lemma 4.2 *If G is connected and locally finite, then $\overline{G}$ is compact in* $\textsc{Top}$.

□

By Lemma 2.1, the topology of a compact Hausdorff space cannot be made coarser without loss of the Hausdorff property. So for G locally finite, there is no Hausdorff topology on $\overline{G}$ which is strictly coarser than $\textsc{Top}$, and in this sense proving the theorem for $\textsc{Top}$ (as we shall do in the next section) will be best possible.

Lemma 4.3 *For every infinite graph G, the topology* $\textsc{Top}$ *satisfies (1), (2) and (3).*

Proof. Conditions (1) and (2) hold trivially; we prove (3). Suppose there is a circle C in $\overline{G}$ such that $C \cap G$ is not dense in C. Then some point on C has a neighbourhood N in C that consists entirely of ends. We may assume that

$N \not\supseteq C$, and that $N = O \cap C$ for some basic open set O in $\overline{G}$. Then $O = \widehat{D}$ for some component D of $G - S$ with $S \subseteq V(G)$ finite, and $N = (\overline{D} \setminus G) \cap C$ is the intersection of two closed sets in $\overline{G}$. So N is closed in $\overline{G}$ and hence in C. Since $N = O \cap C$ is also open in C, the homeomorphism between C and S^1 takes N to an open and closed proper subset of S^1, contradicting its connectedness. □

Let us return to the case when G is locally finite. Since Top is best possible for our purposes among all topologies comparable with it, the question arises whether there are topologies which satisfy our minimum requirements (1) and (2) but are incomparable with Top. In the remainder of this section, we first construct such an example. However we then show that a slight strengthening of (2) will rule out such (pathological) examples and imply that every topology on $\overline{G}$ satisfying this condition and (1) is indeed comparable with Top, making our theorem best possible also in a more global sense.

So we are looking for a locally finite graph G with a topology that satisfies (1) and (2) but is incomparable with Top. Since every Hausdorff topology comparable with Top refines Top and hence inherits (3) from it (Lemma 2.1), it suffices to construct a topology for $\overline{G}$ that violates (3). As a spin-off, we thus obtain that conditions (1) and (2) do not imply (3):

Proposition 4.4 *There exists a locally finite graph G with a topology that satisfies (1) and (2) but not (3), and hence is incomparable with* Top.

Proof. Our graph G will be the infinite binary tree T. We label its vertices with finite 0–1 sequences in the obvious way: the root (which is considered as the *lowest* point in T) is labelled with the empty sequence, and if a vertex has label ℓ then its two successors are labelled $\ell 0$ and $\ell 1$. Then the rays from the root (and hence the ends of T) correspond bijectively to the infinite 0–1 sequences and may be thought of as elements of the real interval $[0, 1]$ in their binary expansion. Let J be the set of all rationals in $(0, 1)$ with a finite binary expansion, and let $J' := [0, 1] \setminus J$. Then each $r \in J'$ comes from exactly one ray R_r, while every $q \in J$ comes from two: a ray R_q labelled eventually 0, and a ray R'_q labelled eventually 1. (For example, the rays $1011000\ldots$ and $10101111\ldots$ both correspond to $11/16$.) Let ω_r, ω_q and ω'_q denote the ends containing R_r, R_q and R'_q, respectively. Let M' be the set of the ends ω'_q, and let M be the set of all the other ends.

We now define the topology on $\overline{T}$ so as to turn the bijection between $[0, 1]$ and M into a topological embedding. Every point in T will have the same basic open neighbourhoods as it does in T viewed as a 1-complex. The basic open neighbourhoods of an end $\omega_p \in M$ are constructed as follows. Choose an open neighbourhood I of p in $[0, 1]$. For each $s \in I$ choose a point z on R_s, and let N_s be the set consisting of ω_s and all the points of T (not of $\overline{T}$) above z. Now take the union of the N_s over all $s \in I$ to be a basic open neighbourhood of ω_p. The basic open neighbourhoods of the ends $\omega'_q \in M'$ will be as small as possible given (1) and (2), consisting just of the 'open final segments' of R'_q: pick a point $z \in R'_q$, choose for every vertex t_i on R'_q above z a point z_i in the interior of the edge at t_i that does not lie on R'_q, and take the union of the set of points on R'_q above z with all the partial edges $[t_i, z_i)$ to be a basic open neighbourhood of ω'_q.

It is straightforward to check that the topology generated in this way satisfies (1) and (2). But it violates (3), since the union of M and the rays $R_0 = 000\dots$ and $R_1 = 111\dots$ forms a circle. □

With only a slight modification, the above example will even contain a circle consisting entirely of ends. Indeed, if we identify 0 and 1 in $[0, 1]$ to form a circle (and put ω_1 in M' rather than M), then in the analogously defined topology the set M of ends becomes a circle in $\overline{T}$.

In Section 6 we shall construct a graph G with a topology that satisfies (3) as well as (1) and (2) but is still incomparable with Top.

The topology on the binary tree constructed in the proof of Proposition 4.4 will hardly be considered as natural. But how unnatural did it *have* to be? Our next Proposition answers this question in an unexpectedly clear-cut way: any topology that witnesses Proposition 4.4 has to violate an only slight and still pretty natural sharpening of condition (2). Or put another way, every topology that satisfies (1) and this new condition is comparable with Top.

In order to state the new condition we need a definition. A *comb* C with *back* R is obtained from a ray R and a sequence $x_1, x_2, \dots$ of distinct vertices by adding for each $i = 1, 2, \dots$ a (possibly trivial) x_i–R path P_i so that all the P_i are disjoint and R meets P_{i+1} after P_i. The vertices x_i will be called the *teeth* of C. When we speak of a comb in G, however, we wish to admit inner points of edges as teeth. We therefore call C a *comb in* G if C is a comb in some subdivision of G (in which every edge may be assumed to be subdivided at most once).

Clearly, condition (2) is a special case of the more general requirement that the teeth of every comb converge to the end of its back:

Every neighbourhood of an end ω contains all but finitely many of the teeth of every comb in G whose back lies in ω. (4)

Note that when G is locally finite, (4) holds in TOP and (1) and (4) imply (3). Indeed, we have the following stronger assertion:

Proposition 4.5 *Let G be a locally finite connected graph, and let* TOP′ *be any topology on $\overline{G}$ that satisfies (1) and (4). Then* TOP′ *is a refinement of* TOP.

Proof. For a proof that the open sets of TOP are open in TOP′, it suffices to show that for every end ω of G and every finite $S \subseteq V(G)$ there exists a TOP′-neighbourhood of ω contained in $\overline{C}$, where $C := C_G(S, \omega)$. Suppose not, let $s_1, s_2, \ldots$ be an enumeration of the vertices outside S and C, and put $S_i := S \cup \{s_1, \ldots, s_i\}$. Now consider any i. By (1), the vertices of S_i together with all their incident edges in G form a compact set $\overline{S}_i = G[S_i \cup N(S_i)]$ in TOP′. So every TOP′-neighbourhood of ω contains a neighbourhood that avoids $\overline{S}_i$, and hence meets one of the components of $G - S_i$ other than C. (Recall that no such neighbourhood is contained in $\overline{C}$, and apply (2).) As $G - S_i$ has only finitely many components, this implies that there is a component $D_i \neq C$ of $G - S_i$ met by every TOP′-neighbourhood of ω.

Choosing these components $D_1, D_2, \ldots$ in turn, we can ensure that $D_1 \supset D_2 \supset \ldots$. Now pick a sequence of distinct points $x_i \in D_i$ so that no two of them lie inside the same edge. By Lemma 3.1 (applied to the subdivision of G obtained by making every x_i a vertex) there is a comb C in G with teeth among the x_i and back R, say. Since each D_i contains all but finitely many of the x_j and is separated from the rest of G by the finite set S_i, this ray R has a tail in every D_i. But any two rays with this property are equivalent (because the S_i eventually contain any finite separator of D_1), so the end ω' of R is independent of the choice of $x_1, x_2, \ldots$ but depends only on the sequence $D_1 \supset D_2 \supset \ldots$.

Since TOP′ is Hausdorff, there are disjoint TOP′-neighbourhoods N of ω and N' of ω'. As N meets every D_i but no vertex or edge lies in more than finitely many D_i, we may choose all our points x_i inside N and hence outside N'. Hence our comb C contradicts (4). □

5 The generating theorem for locally finite graphs

Let us now prove our main theorem: under TOP, the fundamental circuits with respect to any end-faithful spanning tree generate the entire cycle space of a locally finite graph.

Theorem 5.1 *Let G be a locally finite connected graph, let $\overline{G}$ be endowed with* Top*, and let T be any spanning tree of G. Then the following are equivalent:*

(i) *Every circuit of G is a sum of fundamental circuits.*

(ii) *Every element of the cycle space $\mathcal{C}(G)$ of G is a sum of fundamental circuits.*

(iii) *T is end-faithful.*

Proof. Clearly, (ii) implies (i). Lemma 3.2 and the remark preceding it show that (i) implies (iii). To show that (iii) implies (ii) it suffices to prove the following:

If T is end-faithful, then every circuit C of G is equal to the sum of all the fundamental circuits C_e with $e \in C \setminus E(T)$. $(*)$

Indeed, every element Z of $\mathcal{C}(G)$ is by definition the sum of a thin family $\mathcal{F}$ of circuits. By $(*)$, each $C \in \mathcal{F}$ is the sum of fundamental circuits C_e with $e \in C$. Since $\mathcal{F}$ is thin, none of these edges e lies on more than finitely many circuits in $\mathcal{F}$, so all these fundamental circuits together form a family in which none of them occurs infinitely often. By Lemma 3.3 this is again a thin family, so it has a well-defined sum. Clearly, this sum equals Z.

To prove $(*)$ let C be given, and let C' be its defining circle in $\overline{G}$. (Since Top satisfies (3), C' is the closure of $\bigcup C$, so in particular C' is uniquely determined.) Pick a homeomorphism $\sigma : S^1 \to C'$. We have to show that an edge f of G lies in C if and only if it lies in an odd number of the circuits C_e in $(*)$. This is clear when f is a chord of T, as in that case f lies on C_e only if $f = e$.

So consider an edge $f \in T$. Let G_1 and G_2 denote the subgraphs of G induced by the two components of $T - f$, and let E_f be the set of G_1–G_2 edges in G (including f). Note that the edges $e \neq f$ in E_f are precisely the chords e of T with $f \in C_e$. Since the family of these C_e is thin by Lemma 3.3, the set E_f is finite. Hence for $i = 1, 2$, G_i is a component of $G - S$ for the finite set $S = N(G_i)$ of its neighbours outside, every set of the form $\widehat{G}_i$ is open in Top , and $\overline{G}_i = \widehat{G}_i \setminus \mathring{E}_f$ is open in $\overline{G} \setminus \mathring{E}_f$.

As σ is a homeomorphism, $S^1 \setminus \sigma^{-1}(\mathring{E}_f \cap C')$ consists of finitely many intervals, $I_1, \ldots, I_k$ say. Each $\sigma(I_i)$ is a connected subset of $C' \setminus \mathring{E}_f$ and hence cannot meet both of the disjoint open subsets $\overline{G}_1$ and $\overline{G}_2$ of $\overline{G} \setminus \mathring{E}_f$. Our circle C' therefore contains an even number of edges from E_f. Hence, C contains f if and only if it contains an odd number of other edges from E_f, which it does if and only if f lies on an odd number of the circuits C_e with $e \in C$ and hence in the sum of $(*)$. □

By the argument that showed $(*)$ to be sufficient for a proof of Theorem 5.1, the theorem and Lemma 3.3 imply the following:

Corollary 5.2 *If G is locally finite, then its cycle space in* Top *is closed under taking sums.* □

We conclude this section with an example illustrating how unlike our initial ladder examples the circuits covered by Theorem 5.1 can become. Adding just a few edges to the binary tree, we obtain a graph in which all the fundamental circuits sum up to a single circle containing continuum many ends and a 'dense' set of double rays (so that between any two double rays there lies another).

Consider again the infinite binary tree T, and let J, J', R_q, R'_q, ω_q and ω'_q be defined as in the proof of Proposition 4.4. Let D_0 be the double ray formed by the rays $R_0 = 000\dots$ and $R_1 = 111\dots$. For every $q \in J$ add an edge $e_q = t_q t'_q$ between disjoint tails of R_q and R'_q, so that if ℓ is the label of the last common vertex of R_q and R'_q, the vertex t_q is labelled $\ell 011$ and t'_q is labelled $\ell 100$. Then the double rays $D_q = (t_q R_q \cup t'_q R'_q) + e_q$ are disjoint from D_0 and from each other, and T is an end-faithful spanning tree of the resulting graph G.

Let us show that the union C of all the D_q (for $q \in J \cup \{0\}$) and the set of ends of G is a circle in Top. Let I_0 be a closed interval on S^1. Let $\sigma : I_0 \mapsto D_0 \cup \{\omega_0, \omega_1\}$ be a homeomorphism, and put $x_0 := \sigma^{-1}(\omega_0)$ and $x_1 := \sigma^{-1}(\omega_1)$. Our aim is to extend σ to a homeomorphism between S^1 and C.

Let $I := S^1 \setminus \mathring{I}_0$, and think of x_0 as the *left* and x_1 as the *right* endpoint of I. Assign to the points $q \in J$ disjoint closed subintervals $I_q = [x_q, x'_q]$ of $\mathring{I}$, so that I_{q_1} lies left of I_{q_2} whenever $q_1 < q_2$, and I is the closure of $U := \bigcup_q I_q$. (For example, this could be done inductively in ω steps.) Then the points of $\mathring{I} \setminus U$ correspond bijectively to the points in $J' \cap (0,1)$ of the completion $[0,1]$ of J; let x_r be the point of $\mathring{I} \setminus U$ corresponding to $r \in J' \cap (0,1)$. Finally, let $\sigma : S^1 \to C$ map each I_q continuously onto $D_q \cup \{\omega_q, \omega'_q\}$ so that $\sigma(x_q) = \omega_q$ and $\sigma(x'_q) = \omega'_q$, and put $\sigma(x_r) = \omega_r$ for all $r \in J' \cap (0,1)$. Then $\sigma : S^1 \to C$ is a homeomorphism, so C is indeed a circle.

Theorem 5.1 now says that all the fundamental ciruits in G together sum to an infinite circuit: the edge set of our circle C. Once observed, this can also easily be checked directly.

6 A topological condition equivalent to the generating theorem

Let G be a locally finite connected graph. In this section we shall identify a condition just in terms of the topology on $\overline{G}$ that is equivalent to the validity of Theorem 5.1 (ii). This has some interesting consequences.

First, since the elements of the cycle space of G are the same—the sums of fundamental circuits of any end-faithful spanning tree—whenever this condition holds, we find that the cycle space is independent of the topology used as long as it satisfies this condition. In particular, refinements of Top (which will all satisfy the condition) may have fewer circuits than Top (recall Prop. 4.1) but will have the same cycle space. Second, since the new condition will not follow from (1), (2) and (3), we also obtain a negative answer to the question of whether these three (rather natural) conditions alone can guarantee the validity of Theorem 5.1.

We will need the following lemma from elementary topology [9, p. 208]. A continuous image of $[0, 1]$ in a topological space X is a (topological) *path* in X; the images of 0 and 1 are its *endpoints*. By Lemma 2.1, a path in a Hausdorff space is an arc if and only if the corresponding map $[0, 1] \to X$ is injective.

Lemma 6.1 *Every path with distinct endpoints x, y in a Hausdorff space X contains an arc in X between x and y.* □

As always, we consider only topologies on $\overline{G}$ that satisfy our two minimum requirements (1) and (2). Then if E is any finite set of edges and C is a component of $G - E$, the subspace $\overline{C}$ of $\overline{G}$ is path-connected. Our new condition says that these $\overline{C}$ are in fact the whole path components of $\overline{G} \setminus \mathring{E}$:

Whenever E is a finite set of edges of G, every path component of the topological space $\overline{G} \setminus \mathring{E}$ is of the form $\overline{C}$, for some component C of the graph $G - E$. (5)

Note that (5) implies (3). Indeed if (3) fails, then $\overline{G}$ contains an arc $[\omega, \omega']$ consisting entirely of ends. Let S be a finite set of vertices separating a ray in ω from a ray in ω', and let E be the set of edges incident with S. Then ω and ω' do not both lie in $\overline{C}$ for any component C of $G - E$, although they do lie in the same path component of $\overline{G} \setminus \mathring{E}$.

Although Top clearly satisfies (5), it is not difficult to construct locally finite graphs with topologies that satisfy (1), (2), (3) and (5) but are incomparable with Top. (By Proposition 4.5, these topologies must violate (4).)

The following result may thus be viewed as a topologically best-possible strengthening of Theorem 5.1:

Theorem 6.2 *Let G be a locally finite connected graph, let $\overline{G}$ carry any topology satisfying (1) and (2), and let T be an end-faithful spanning tree in G. Then the following two assertions are equivalent:*

(i) *$\overline{G}$ satisfies (5);*

(ii) *every element of the cycle space of G is the sum of fundamental circuits.*

Proof. The proof of Theorem 5.1 shows that (i) implies (ii). Indeed, as $\overline{G}$ satisfies (5), both $\overline{G}_1$ and $\overline{G}_2$ as considered there are path components of $\overline{G} \setminus \mathring{E}_f$ and thus again each $\sigma(I_i)$ lies in either $\overline{G}_1$ or $\overline{G}_2$.

So let us prove the converse implication. If (5) fails, then for some finite set $E \subseteq E(G)$ there are components $D_1 \neq D_2$ of $G - E$ such that $\overline{D}_1$ and $\overline{D}_2$ are contained in the same path component of $\overline{G} \setminus \mathring{E}$. By making E smaller, we may assume that D_1 and D_2 are the only components of $G - E$. Let $f_1, \dots, f_k$ be the D_1–D_2 edges contained in T.

For each $i = 1, \dots, k$, let E_i be the set of the edges of G between different components of $T - f_i$. Since the edges $e \neq f_i$ in E_i are precisely the chords e of T with $f_i \in C_e$, Lemma 3.3 implies that each E_i is finite.

By definition, $\overline{D}_1$ and $\overline{D}_2$ are joined in $\overline{G} \setminus \mathring{E}$ by a topological path π; since they are path-connected, we may assume that the endpoints of π are vertices, and by Lemma 6.1 we may assume that π is an arc.

Since an arc between two vertices includes every edge of which it contains an inner point, and since the E_i are finite, $\pi \setminus (\mathring{E}_1 \cup \dots \cup \mathring{E}_k)$ consists of finitely many closed segments whose endpoints are vertices. One of these, π' say, is again an arc from a vertex $v_1 \in D_1$ to a vertex $v_2 \in D_2$. (For since π contains no D_1–D_2 edge, the endpoints of every missing edge lie in the same D_j.)

Pick an edge $f_i \in \{f_1, \dots, f_k\}$ from the path $v_1 T v_2$, let P be the segment of $v_1 T v_2$ that includes f_i and meets π' only in its endpoints, and let π'' be the segment of π' between these points. Then $P \cup \pi''$ is a circle in $\overline{G}$ that contains f_i but no other edge from E_i. Its circuit is therefore not a sum of fundamental circuits, so (ii) fails as required. □

Corollary 6.3 *The cycle space of a locally finite graph G is independent of the topology chosen for $\overline{G}$, as long as the topology satisfies (1), (2) and (5).*

In particular, $\mathcal{C}(G)$ is the same for all refinements of Top *(with (1) and (2)), and thus uniquely determined for all topologies that satisfy (1) and (4).*

Proof. For the second statement, let us first verify (5) for an arbitrary refinement Top' of Top that satisfies (1) and (2). Since rays converge to their ends by (2), every $\overline{C}$ as in (5) is path-connected. So any path component D of $\overline{G} \setminus \mathring{E}$ is a disjoint union of such $\overline{C}$. But every $\overline{C}$ is open in $\overline{G} \setminus \mathring{E}$, in Top and hence also in Top'. Hence D, being connected, consists of a single $\overline{C}$.

For the last statement, recall that every topology satisfying (1) and (4) is a refinement of Top (Prop. 4.5). □

In the remainder of the section we construct an example which shows that (5) does not follow from (1), (2) and (3). Hence, by Theorem 6.2, these three conditions alone cannot guarantee the validity of Theorem 5.1.

Consider again the infinite binary tree T, and let J, J', R_q, R'_q, ω_q and ω'_q be as in the proof of Proposition 4.4. Add a new double ray D with ends $\tau \neq \tau'$ which meets T exactly in its root t. Moreover for each $q \in J$ add a new double ray D_q with ends $\tau_q \neq \tau'_q$, together with an edge $e_q = v_q v'_q$ joining a vertex v_q on D_q to a vertex v'_q on D; choose these edges e_q independent. Denote the graph thus obtained by G.

Let us now define the topology on $\overline{G}$. G itself will carry the topology of a 1-complex. The basic open neighbourhoods of an end ν of the form τ, τ', ω_q or ω'_q with $q \in J$ will consist of an 'open final segment' of the ray $R \in \nu$ starting at t: pick a point z on R, as well as an inner point z_e of every edge $e \notin R$ incident with a vertex $v \in \mathring{z}R$; then take the union of $\mathring{z}R$ with all the partial edges $[v, z_e) \subset e$ to be a basic open neighbourhood of ν. The basic open neighbourhoods of an end ω_r for $r \in J'$ are constructed as follows. Choose an open neighbourhood I of r in $[0, 1]$. For each $s \in I \cap J'$ choose a point z on R_s, and let N_s be the set consisting of ω_s and all the points of T above z. For each $q \in I \cap J$ pick an inner point z_q of e_q. Take the union of the N_s over all $s \in I \cap J'$ together with the union of $\{\tau_q, \tau'_q\} \cup D_q \cup [v_q, z_q)$ over all $q \in I \cap J$ to be a basic open neighbourhood $N(I)$ of ω_r. To construct a basic open neighbourhood of an end τ_q (respectively τ'_q) for $q \in J$, choose an open neighbourhood I of q in $(0, q]$ (respectively $[q, 1)$) and take the union of $N(I)$ (defined as before) together with τ_q (respectively τ'_q) and an open final segment of the subray of $D_q - v_q$ contained in τ_q (respectively τ'_q) to be a basic open set.

Clearly, the topology generated in this way satisfies (1) and (2). As in the proof of Lemma 4.3 one can show (3). (Indeed, the open set N considered there can again be chosen so that it is also closed.) Furthermore, using similar arguments as in the example in Section 5 one can show that all the ends ω_r ($r \in J'$) and all the sets $\{\tau_q\} \cup D_q \cup \{\tau'_q\}$ with $q \in J$ together form an arc π whose endpoints belong to different components of $G - t$. Hence, this topology does not satisfy (5).

Finally, since G is a tree and π forms a circle together with the rays R_0 and R_1, our example shows also that the converse of Lemma 3.2 can fail even for topologies satisfying (1), (2) and (3).

7 Cycle-cut orthogonality, cycle decompositions, and Euler tours

Let G be a locally finite connected graph, and let $\overline{G}$ be endowed with Top.[4]

Recall that a *cut* in G is the set of all the edges of G between the two classes of some bipartition of $V(G)$. When G is finite, the elements of its cycle space are precisely those sets of edges that are orthogonal to every cut in G, ie. contain an even number of edges from every cut [2]. This generalizes as follows:[5]

Theorem 7.1 *The following statements are equivalent for every $E \subseteq E(G)$:*

(i) $E \in \mathcal{C}(G)$*;*

(ii) $|E \cap F|$ *is even for every finite cut F of G.*

Proof. The fact that every circuit in G meets every finite cut in an even number of edges is proved as in the proof of Theorem 5.1, using (5) instead of the definition of Top if desired. Since sums (mod 2) of even sets are even, this implies (i)→(ii).

For the converse implication, assume without loss of generality that G is connected, and let T be an end-faithful spanning tree of G. Assuming (ii), we show that E is equal to the sum $Z \in \mathcal{C}(G)$ of all the fundamental circuits C_e with $e \in E \setminus E(T)$. For every chord e of T in G, clearly $e \in E$ if and only if $e \in Z$. So consider an edge $f \in T$. Let E_f be the set of edges $e \neq f$ of G between the two components of $T - f$. Since T is end-faithful and $f \in C_e$ for precisely those chords e of T that lie in E_f, Lemma 3.3 implies that E_f is finite, and $f \in Z$ if and only if $|E_f \cap E|$ is odd. By (ii), the latter holds if and only if $f \in E$, as desired. □

[4]We shall use Top explicitly only in the proof of Theorem 7.2. For Thm 7.1 it is enough to assume that G satisfies the generating theorem, ie. satisfies (1), (2) and (5).

[5]In [6] we extend Theorem 7.1 to arbitrary infinite graphs, with appropriate adaptations of the notions of cycle space and finite cut.

We point out that an infinite circuit and an infinite cut of G may well meet in an odd number of edges, because rays on either side of the cut may belong (and hence converge) to the same end. The circuit C_2 in the double ladder (Fig. 1) is a simple example.

Let us now look at the special case of $E = E(G)$. A classical theorem of Nash-Williams [10] says that a connected graph (of any cardinality) is an edge-disjoint union of finite cycles if and only if each of its cuts is either infinite or even. By Theorem 7.1, a further property of G equivalent to these is that $E \in \mathcal{C}(G)$.

When G is finite, this property is equivalent to yet another: that G contains an *Euler tour*, a closed walk that traverses every edge exactly once. For our locally finite graph G, we have the following infinite analogue. Call a continuous (but not necessarily injective) map $\sigma\colon S^1 \to \overline{G}$ a *topological Euler tour* of $\overline{G}$ if every inner point of an edge of G is the image of exactly one point of S^1. (Thus, every edge is traversed exactly once, and in a 'straight' manner.)

Theorem 7.2 *The following statements are equivalent for $E = E(G)$:*

(i) $E \in \mathcal{C}(G)$;

(ii) *every cut in G is either infinite or even;*

(iii) *E is a disjoint union of finite circuits;*

(iv) *E is a disjoint union of circuits;*

(v) *$\overline{G}$ admits a topological Euler tour.*

Proof. The equivalence (i)↔(ii) follows from Theorem 7.1, the implication (ii)→(iii) from Nash-Williams's theorem. (In fact, as Nash-Williams observed, it is easy for countable graphs.) As (iii)→(iv) is trivial and (v)→(ii) again follows as in the proof of Theorem 5.1, it remains to prove (iv)→(v).

Let $\mathcal{D}$ be a set of disjoint circuits in G whose union is E. We shall define a topological Euler tour $\sigma\colon S^1 \to \overline{G}$ as a limit of a sequence of continuous maps $\sigma_1, \sigma_2, \ldots$ from S^1 to $\overline{G}$, each with the property that every inner point of an edge of G is the image of at most one point of S^1. The image of σ_n will be the closure in $\overline{G}$ of $\bigcup\bigcup \mathcal{D}_n$ for some subset $\mathcal{D}_n \subseteq \mathcal{D}$ of circuits, where $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \ldots$ and $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \ldots = \mathcal{D}$.

Let σ_1 be a homeomorphism between S^1 and the defining circle of any circuit $D \in \mathcal{D}$, and put $\mathcal{D}_1 := \{D\}$. Given $n \in \mathbb{N}$, assume inductively that σ_n has been defined as stated. For the vertices $v \in \sigma_n(S^1)$ pick disjoint closed intervals I_v on S^1 so that σ_n maps an inner point of I_v to v and $\sigma_n(I_v)$ is contained in the union of two edges at v. Let $\mathcal{D}^{n+1}$ denote the set of circuits in $\mathcal{D} \setminus \mathcal{D}_n$ that contain an edge incident with a vertex in $\sigma_n(S^1)$, and put $\mathcal{D}_{n+1} := \mathcal{D}_n \cup \mathcal{D}^{n+1}$. For every $D \in \mathcal{D}^{n+1}$ pick such a vertex $v(D) \in \sigma_n(S^1)$. Let σ_{n+1} be obtained from σ_n by the following local changes inside some of the intervals I_v with $v \in \sigma_n(S^1) \setminus \sigma_{n-1}(S^1)$ (where $\sigma_0(S^1) := \emptyset$). Since G is locally finite, v serves as $v(D)$ for only finitely many $D \in \mathcal{D}^{n+1}$, say for $D_1, \dots, D_k$. Write I_v as the union of $k+2$ closed subintervals of equal length (overlapping pairwise in at most one point), and choose as $\sigma_{n+1} \restriction I_v$ a topological path which maps all the intersection points between adjacent subintervals to v, maps the union of the first and last subinterval onto $\sigma_n(I_v)$, and maps the ith inner subinterval onto the defining circle of D_i, $i = 1, \dots, k$.

Note that as soon as σ_n and σ_{n+1} agree on a point $x \in S^1$ for some n, we also have $\sigma_m(x) = \sigma_n(x)$ for all $m > n$. For such a point x we let $\sigma(x) := \sigma_n(x)$. Now let $x \in S^1$ be such that its images $y_n := \sigma_n(x)$ are all distinct for different n. Then for each n there exists a circuit $D_{n+1} \in \mathcal{D}^{n+1}$ such that y_{n+1} lies on an edge in D_{n+1} (possibly at its endpoint, if y_{n+1} is a vertex). By Lemma 3.1, the set $\{y_1, y_2, \dots\}$ has an end ω in its closure, which we choose as $\sigma(x)$.

It remains to check that σ is well defined and continuous at these points x. Let $\widehat{C}_G(S, \omega)$ be a basic open neighbourhood of ω. Choose n large enough that all the edges at vertices in S lie in $\bigcup \mathcal{D}_{n-1}$. (Since G is connected, the definition of the sets $\mathcal{D}^{n+1}$ implies that every edge lies in $\bigcup \mathcal{D}_n$ for some n.) Then the closure of $D_{n+1} \cup D_{n+2} \cup \dots$ is path-connected, contains $y_{n+1}, y_{n+2}, \dots$, and avoids S, and hence lies in $\widehat{C}_G(S, \omega)$. Thus, $\omega = \sigma(x)$ is the only end in the closure of $\{y_1, y_2, \dots\}$, and similarly σ maps the entire interval $I_v \ni x$ considered for the definition of σ_{n+2} to $\widehat{C}_G(S, \omega)$.

Since G is connected, σ traverses every edge of G. Moreover, if we chose the maps $\sigma_{n+1} \restriction I_v$ in the obvious 'minimal' way, it does so only once and without repeating inner points of the edge. □

In [6] we extend the equivalence between (i) and (iv) in Theorem 7.2 to arbitrary subsets E of $E(G)$. This is much harder than Theorem 7.2 and cannot easily be reduced to it, because a set $E \in \mathcal{C}(G)$ does not normally lie in the cycle space of the subgraph of G it induces. (For example, a single infinite circuit C in G is, as a graph by itself, merely a disjoint union of double rays containing no cycles at all).

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