6 Flows

Let us view a graph $G = (V, E)$ as a network: its edges carry some kind of flow – of water, electricity, data or similar. How could we model this precisely?

For a start, we ought to know how much flow passes through each edge $e = xy$, and in which direction. In our model, we could assign a positive integer k to the pair (x, y) to express that a flow of k units passes through e from x to y, or assign $-k$ to (x, y) to express that k units of flow pass through e the other way, from y to x. For such an assignment $f: V^2 \to \mathbb{Z}$ we would thus have $f(x, y) = -f(y, x)$ whenever x and y are adjacent vertices.

Typically, a network will have only a few nodes where flow enters or leaves the network; at all other nodes, the total amount of flow into that node will equal the total amount of flow out of it. For our model this means that, at most nodes x, the function f will satisfy *Kirchhoff's law*

Kirchhoff's law

$$\sum_{y \in N(x)} f(x, y) = 0 \,.$$

In this chapter, we call any map $f: V^2 \to \mathbb{Z}$ with the above two properties a 'flow' on G. Sometimes, we shall replace $\mathbb{Z}$ with another group, and as a rule we consider multigraphs rather than graphs.[1] As it turns out, the theory of those 'flows' is not only useful as a model for real flows: it blends so well with other parts of graph theory that some deep and surprising connections become visible, connections particularly with connectivity and colouring problems.

[1] For consistency, we shall phrase some of our proposition for graphs only: those whose proofs rely on assertions proved (for graphs) earlier in the book. However, all those results remain true for multigraphs.

6.1 Circulations

$G = (V, E)$

Let $G = (V, E)$ be an (undirected) multigraph. Every edge $e = xy$ of G has two *directions*, (x, y) and (y, x). (These coincide if e is a loop, so loops have only one direction.) A triple (e, x, y) consisting of an edge together with one of its directions is an *oriented edge*. The oriented edges corresponding to e are its *orientations*, denoted by $\vec{e}$ and $\overleftarrow{e}$. Thus, $\{\vec{e}, \overleftarrow{e}\} = \{(e, x, y), (e, y, x)\}$, but we cannot generally say which is which. We write

edge orientations

$\vec{E}$

$$\vec{E} := \{\, (e, x, y) \mid e \in E;\ x, y \in V;\ e = xy \,\}.$$

for the set of all oriented edges. We shall denote elements of $\vec{E}$ as $\vec{e}$, $\overleftarrow{e}$, etc. even if there is no previously defined edge e, and then use 'e' to refer to its underlying edge.

For an arbitrary set $\vec{F} \subseteq \vec{E}$ of oriented edges we put

$\overleftarrow{F}$

$$\overleftarrow{F} := \{\, \overleftarrow{e} \mid \vec{e} \in \vec{F} \,\}.$$

Note that $\vec{E}$ itself is symmetrical: $\overleftarrow{E} = \vec{E}$. For two sets $X, Y \subseteq V$ of vertices, not necessarily disjoint, and $\vec{F} \subseteq \vec{E}$, we define

$\vec{F}(X, Y)$

$$\vec{F}(X, Y) := \{\, (e, x, y) \in \vec{F} \mid x \in X;\ y \in Y;\ x \neq y \,\},$$

$\vec{F}(x, Y)$

abbreviate $\vec{F}(\{x\}, Y)$ to $\vec{F}(x, Y)$ etc., and write

$\vec{F}(x)$

$$\vec{F}(x) := \vec{F}(x, V) = \vec{F}(\{x\}, \overline{\{x\}}).$$

$\overline{X}$

Here, as below, $\overline{X}$ denotes the complement $V \setminus X$ of a vertex set $X \subseteq V$. Note that any loops at vertices $x \in X \cap Y$ are disregarded in the definitions of $\vec{F}(X, Y)$ and $\vec{F}(x)$.

0

f

Let H be an abelian semigroup,[2] written additively with zero 0. Given $X, Y \subseteq V$, not necessarily disjoint, and a function $f \colon \vec{E} \to H$, let

$f(X, Y)$

$$f(X, Y) := \sum_{\vec{e} \in \vec{E}(X, Y)} f(\vec{e}). \tag{1}$$

$f(x, Y)$

Instead of $f(\{x\}, Y)$ we again write $f(x, Y)$, etc.

circulation

From now on, we assume that H is an abelian group. We call f a *circulation* on G (with values in H) if f satisfies the following two conditions:

(F1) $f(e, x, y) = -f(e, y, x)$ for all $(e, x, y) \in \vec{E}$ with $x \neq y$;

(F2) $f(v, V) = 0$ for all $v \in V$.

---

[2] This chapter contains no group theory. The only semigroups we ever consider for H are the natural numbers, the integers, the reals, the cyclic groups $\mathbb{Z}_k$, and their products $\mathbb{Z}_k \times \mathbb{Z}_m$.

If f satisfies (F1), then

$$f(X, X) = 0$$

for all $X \subseteq V$. If f satisfies (F2), then

$$f(X, V) = \sum_{x \in X} f(x, V) = 0\,.$$

Together, these two basic observations imply that, in a circulation, the net flow across any cut is zero:

Proposition 6.1.1. *If f is a circulation, then $f(X, \overline{X}) = 0$ for every set $X \subseteq V$.* [6.3.1] [6.5.2]

Proof. $f(X, \overline{X}) = f(X, V) - f(X, X) = 0 - 0 = 0.$ $\square$

Since bridges form cuts by themselves, Proposition 6.1.1 implies that circulations are always zero on bridges:

Corollary 6.1.2. *If f is a circulation and $e = xy$ is a bridge in G, then $f(e, x, y) = 0$.* $\square$

The following lemma will be useful for induction proofs later.

Lemma 6.1.3. *Given any edge $e_0 = xy$ of G, every circulation on G/e_0 extends to a circulation on G.* [6.3.1] [6.6.1]

Proof. Let f be a circulation on G/e_0, and $\vec{e_0} := (e_0, x, y)$. By the definition of multigraph minors in Chapter 1.10 we have $E(G/e_0) = E \setminus \{e_0\}$. Edges in $E \setminus E(x, y)$ are loops either in both G and G/e_0 or in neither, so the f-values on their orientations from G/e_0 are well-defined in G too. Edges $e \neq e_0$ in $E(x, y)$ are loops in G/e_0 at the contracted vertex v_{e_0}; we take their f-value from G/e_0 as $f(\vec{e})$ for $\vec{e} := (e, x, y)$ in G, and set $f(\overleftarrow{e}) := -f(\vec{e})$ to satisfy (F1). It remains to define f on $\vec{e_0}$ and $\overleftarrow{e_0}$.

Let

$$\begin{aligned} f_x &:= \sum \big\{ f(e, u, x) \in \vec{E} \mid u \notin \{x, y\} \big\} \\ f_y &:= \sum \big\{ f(e, y, v) \in \vec{E} \mid v \notin \{x, y\} \big\} \\ f_{xy} &:= \sum \big\{ f(e, x, y) \in \vec{E} \mid e \neq e_0 \big\}. \end{aligned}$$

By (F2) for f at v_{e_0} in G/e_0 we have $f_x = f_y$. Setting $f(\vec{e_0}) := f_x - f_{xy}$ and $f(\overleftarrow{e_0}) := -f(\vec{e_0})$ is the unique way now to extend f from $\vec{E} \setminus \{\vec{e_0}, \overleftarrow{e_0}\}$ further to a circulation on G. $\square$

We remark that Lemma 6.1.3 has no analogue for *flows*, circulations that are non-zero everywhere (see Section 6.3): even if f is non-zero on all of G/e_0, its extensions to G may all have to be zero on e_0.

6.2 Flows in networks

In this section we give a brief introduction to the kind of network flow theory that is now a standard proof technique in areas such as matching and connectivity. By way of example, we shall prove a classic result of this theory, the so-called *max-flow min-cut* theorem of Ford and Fulkerson. This theorem alone implies Menger's theorem without much difficulty (Exercise 3), which indicates some of the natural power lying in this approach.

Consider the task of modelling a network with one source s and one sink t, in which the amount of flow through a given link between two nodes is subject to a certain capacity of that link. Our aim is to determine the maximum net amount of flow through the network from s to t. Somehow, this will depend both on the structure of the network and on the various capacities of its connections – how exactly, is what we wish to find out.

$G = (V, E)$
s, t, c, N
network
flow

Let $G = (V, E)$ be a multigraph, $s, t \in V$ two fixed vertices, and $c\colon \vec{E} \to \mathbb{N}$ a map; we call c a *capacity function* on G, and the quadruple $N := (G, s, t, c)$ a *network*. Note that c is defined independently for the two orientations of an edge. A function $f\colon \vec{E} \to \mathbb{R}$ is a *flow* in N if it satisfies the following three conditions (Fig. 6.2.1):

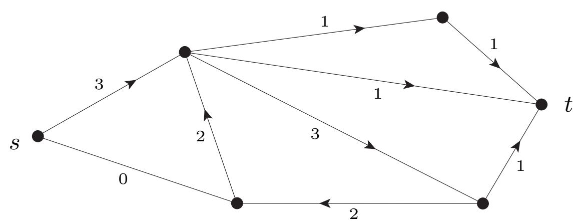


Fig. 6.2.1. A network flow in short notation: all values refer to the direction indicated (capacities are not shown)

(F1) $f(e, x, y) = -f(e, y, x)$ for all $(e, x, y) \in \vec{E}$ with $x \neq y$;

(F2′) $f(v, V) = 0$ for all $v \in V \setminus \{s, t\}$;

(F3) $f(\vec{e}) \leqslant c(\vec{e})$ for all $\vec{e} \in \vec{E}$.

integral
f
cut in N
capacity

We call f *integral* if all its values are integers.

Let f be a flow in N. If $S \subseteq V$ is such that $s \in S$ and $t \in \overline{S}$, we call the pair $(S, \overline{S})$ a *cut in N*, and $c(S, \overline{S})$ the *capacity* of this cut.[3]

Since f now has to satisfy only (F2′) rather than (F2), we no longer have $f(X, \overline{X}) = 0$ for all $X \subseteq V$ (as in Proposition 6.1.1). However, the value is the same for all cuts:

[3] The number $c(S, \overline{S})$ is defined in (1) of Section 6.1.

Proposition 6.2.1. *Every cut* $(S, \overline{S})$ *in* N *satisfies* $f(S, \overline{S}) = f(s, V)$.

Proof. As in the proof of Proposition 6.1.1, we have

$$\begin{aligned} f(S, \overline{S}) &= f(S, V) - f(S, S) \\ &\underset{(F1)}{=} f(s, V) + \sum_{v \in S \smallsetminus \{s\}} f(v, V) - 0 \\ &\underset{(F2')}{=} f(s, V). \end{aligned}$$

□

The common value of $f(S, \overline{S})$ in Proposition 6.2.1 will be called the *total value* of f and denoted by $|f|$;[4] the flow shown in Figure 6.2.1 has total value 3. *total value* $|f|$

By (F3), we have

$$|f| = f(S, \overline{S}) \leqslant c(S, \overline{S})$$

for every cut $(S, \overline{S})$ in N. Hence the total value of a flow in N is never larger than the smallest capacity of a cut. The following *max-flow min-cut* theorem states that this upper bound is always attained by some flow:

Theorem 6.2.2. (Ford & Fulkerson 1956) *max-flow min-cut theorem*
In every network, the maximum total value of a flow equals the minimum capacity of a cut.

Proof. Let $N = (G, s, t, c)$ be a network, and $G =: (V, E)$. We shall define a sequence $f_0, f_1, f_2, \ldots$ of integral flows in N of strictly increasing total value, i.e. with

$$|f_0| < |f_1| < |f_2| < \ldots$$

Clearly, the total value of an integral flow is again an integer, so in fact $|f_{n+1}| \geqslant |f_n| + 1$ for all n. Since all these numbers are bounded above by the capacity of any cut in N, our sequence will terminate with some flow f_n. Corresponding to this flow, we shall find a cut of capacity $c_n = |f_n|$. Since no flow can have a total value greater than c_n, and no cut can have a capacity less than $|f_n|$, this number is simultaneously the maximum and the minimum referred to in the theorem.

For f_0, we set $f_0(\vec{e}) := 0$ for all $\vec{e} \in \vec{E}$. Having defined an integral flow f_n in N for some $n \in \mathbb{N}$, we denote by S_n the set of all vertices v such that G contains an s–v walk $x_0 e_0 \ldots e_{\ell-1} x_\ell$ with S_n

$$f_n(\vec{e_i}) < c(\vec{e_i})$$

[4] Thus, formally, $|f|$ may be negative. In practice, however, we can change the sign of $|f|$ simply by swapping the roles of s and t.

for all $i < \ell$; here, $\vec{e_i} := (e_i, x_i, x_{i+1})$ (and, of course, $x_0 = s$ and $x_\ell = v$).

W If $t \in S_n$, let $W = x_0 e_0 \dots e_{\ell-1} x_\ell$ be the corresponding s–t walk; without loss of generality we may assume that W does not repeat any vertices. Let

ϵ
$$\epsilon := \min \{\, c(\vec{e_i}) - f_n(\vec{e_i}) \mid i < \ell \,\}\,.$$

Then $\epsilon > 0$, and since f_n (like c) is integral by assumption, ϵ is an integer. Let

$$f_{n+1} \colon \vec{e} \mapsto \begin{cases} f_n(\vec{e}) + \epsilon & \text{for } \vec{e} = \vec{e_i},\ \ i = 0, \dots, \ell - 1; \\ f_n(\vec{e}) - \epsilon & \text{for } \vec{e} = \overleftarrow{e_i},\ \ i = 0, \dots, \ell - 1; \\ f_n(\vec{e}) & \text{for } e \notin W. \end{cases}$$

Intuitively, f_{n+1} is obtained from f_n by sending additional flow of value ϵ along W from s to t (Fig. 6.2.2).

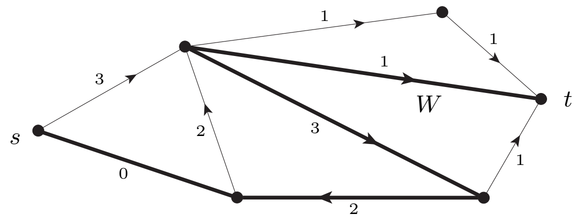


Fig. 6.2.2. An 'augmenting path' W with increment $\epsilon = 2$, for constant flow $f_n = 0$ and capacities $c = 3$

Clearly, f_{n+1} is again an integral flow in N. Let us compute its total value $|f_{n+1}| = f_{n+1}(s, V)$. Since W contains the vertex s only once, $\vec{e_0}$ is the only triple (e, x, y) with $x = s$ and $y \in V$ whose f-value was changed. This value, and hence that of $f_{n+1}(s, V)$ was raised. Therefore $|f_{n+1}| > |f_n|$ as desired.

If $t \notin S_n$, then $(S_n, \overline{S_n})$ is a cut in N. By (F3) for f_n, and the definition of S_n, we have

$$f_n(\vec{e}) = c(\vec{e})$$

for all $\vec{e} \in \vec{E}(S_n, \overline{S_n})$, so

$$|f_n| = f_n(S_n, \overline{S_n}) = c(S_n, \overline{S_n})$$

as desired. □

Since the flow constructed in the proof of Theorem 6.2.2 is integral, we have also proved the following:

Corollary 6.2.3. *In every network (with integral capacity function) there exists an integral flow of maximum total value.* □

6.3 Group-valued flows

Let $G = (V, E)$ be a multigraph. If f and g are two circulations on G with values in the same abelian group H, then $(f+g)\colon \vec{e} \mapsto f(\vec{e}) + g(\vec{e})$ and $-f\colon \vec{e} \mapsto -f(\vec{e})$ are again circulations. The circulations on G with values in H thus form a group in a natural way. $f+g$ $-f$

An *H-flow* in our terminology[5] is a circulation $f\colon \vec{E} \to H$ that is *nowhere zero*, one that satisfies $f(\vec{e}) \neq 0$ for all $\vec{e} \in \vec{E}$. Note that the set of H-flows on G is not closed under addition: if two H-flows add up to zero on some oriented edge $\vec{e}$, then their sum is no longer an H-flow. By Corollary 6.1.2, a graph with an H-flow cannot have a bridge. *H-flow*

For finite groups H, the number of H-flows on G – and, in particular, their existence – surprisingly depends only on the order of H, not on H itself:

Theorem 6.3.1. (Tutte 1954)
For every multigraph G there exists a polynomial P such that, for any finite abelian group H, the number of H-flows on G is $P\big(|H| - 1\big)$.

(6.1.1)
(6.1.3)

Proof. Let $G =: (V, E)$; we use induction on $m := |E|$. Let us assume first that all the edges of G are loops. Then, given any finite abelian group H, every map $\vec{E} \to H \setminus \{0\}$ is an H-flow on G. Since $|\vec{E}| = |E|$ when all edges are loops, there are $\big(|H| - 1\big)^m$ such maps, and $P := x^m$ is the polynomial sought.

Now assume there is an edge $e_0 = xy \in E$ that is not a loop; let $\vec{e}_0 := (e_0, x, y)$ and $E' := E \setminus \{e_0\}$. We consider the multigraphs $e_0 = xy$ E'

$$G_1 := G - e_0 \quad \text{and} \quad G_2 := G/e_0\,.$$

By the induction hypothesis, there are polynomials P_i for $i = 1, 2$ such that, for any finite abelian group H and $k := |H| - 1$, the number of H-flows on G_i is $P_i(k)$. We shall prove that the number of H-flows on G equals $P_2(k) - P_1(k)$; then $P := P_2 - P_1$ is the desired polynomial. P_1, P_2 k

Let H be given, and let F denote the set of all H-flows on G. Our aim is to show that H, F

$$|F| = P_2(k) - P_1(k)\,. \tag{1}$$

The H-flows on G_1 are precisely the restrictions to $\vec{E'}$ of those H-circulations on G that are zero on e_0 but nowhere else. Let us denote the set of these circulations on G by F_1; then F_1

$$|F_1| = P_1(k).$$

[5] To avoid cumbersome repetitions of the phrase 'nowhere zero' before 'H-flow', we deviate slightly here from standard terminology. See the footnote in the notes.

Let F_2 denote the set of H-circulations on G that are non-zero except *possibly* on e_0. The H-flows f on G_2 define circulations $f' \in F_2$ on G as in the proof of Lemma 6.1.3, where $\varphi\colon f \mapsto f'$ is injective. This map φ is also surjective: use Proposition 6.1.1 with $X = \{x, y\}$ to check (F2) at v_{e_0} for the map f induced naturally on $\vec{E}(G/e_0)$ by a given $f' \in F_2$, and note that φ sends f to f' since f' is determined uniquely on $\vec{e_0}$ and $\overleftarrow{e_0}$ by its values on $\vec{E} \smallsetminus \{\vec{e_0}, \overleftarrow{e_0}\}$. Thus,

$$|F_2| = P_2(k).$$

As $F_1 \subseteq F_2$ and $F = F_2 \smallsetminus F_1$, we have (1) as desired. □

flow polynomial

The polynomial P of Theorem 6.3.1 is known as the *flow polynomial* of G.

[6.4.5]
[6.6.1]

Corollary 6.3.2. *If H and H' are two finite abelian groups of equal order, then G has an H-flow if and only if G has an H'-flow.* □

Corollary 6.3.2 has fundamental implications for the theory of algebraic flows: it indicates that crucial difficulties in existence proofs of H-flows are unlikely to be of a group-theoretic nature. On the other hand, being able to choose a convenient group can be quite helpful; we shall see a pretty example for this in Proposition 6.4.5.

k
k-flow

Let $k \geqslant 1$ be an integer and $G = (V, E)$ a multigraph. A $\mathbb{Z}$-flow f on G such that $0 < |f(\vec{e})| < k$ for all $\vec{e} \in \vec{E}$ is called a *k-flow*. Clearly, any k-flow is also an ℓ-flow for all $\ell > k$. Thus, we may ask which is the least integer k such that G admits a k-flow – assuming that such a k exists. We call this least k the *flow number* of G and denote it by $\varphi(G)$; if G has no k-flow for any k, we put $\varphi(G) := \infty$.

flow number $\varphi(G)$

The task of determining flow numbers quickly leads to some of the deepest open problems in graph theory. We shall consider these later in the chapter. First, however, let us see how k-flows are related to the more general concept of H-flows.

σ_k

There is an intimate connection between k-flows and $\mathbb{Z}_k$-flows. Let σ_k denote the natural homomorphism $i \mapsto \bar{i}$ from $\mathbb{Z}$ to $\mathbb{Z}_k$. By composition with σ_k, every k-flow defines a $\mathbb{Z}_k$-flow. As the following theorem shows, the converse holds too: from every $\mathbb{Z}_k$-flow on G we can construct a k-flow on G. In view of Corollary 6.3.2, this means that the general question about the existence of H-flows for arbitrary groups H reduces to the corresponding question for k-flows.

[6.4.1]
[6.4.2]
[6.4.3]
[6.4.5]

Theorem 6.3.3. (Tutte 1950)
A multigraph admits a k-flow if and only if it admits a $\mathbb{Z}_k$-flow.

Proof. Let g be a $\mathbb{Z}_k$-flow on a multigraph $G = (V, E)$; we construct a k-flow f on G. We may assume without loss of generality that G has no loops. Let F be the set of all functions $f\colon \vec{E} \to \mathbb{Z}$ that satisfy (F1), $|f(\vec{e})| < k$ for all $\vec{e} \in \vec{E}$, and $\sigma_k \circ f = g$; note that, like g, any $f \in F$ is nowhere zero.

Let us show first that $F \neq \emptyset$. Since we can express every value $g(\vec{e}) \in \mathbb{Z}_k$ as $\bar{i}$ with $|i| < k$ and then put $f(\vec{e}) := i$, there is clearly a map $f\colon \vec{E} \to \mathbb{Z}$ such that $|f(\vec{e})| < k$ for all $\vec{e} \in \vec{E}$ and $\sigma_k \circ f = g$. For each edge $e \in E$, let us choose one of its two orientations and denote this by $\vec{e}$. We may then define $f'\colon \vec{E} \to \mathbb{Z}$ by setting $f'(\vec{e}) := f(\vec{e})$ and $f'(\overleftarrow{e}) := -f(\vec{e})$ for every $e \in E$. Then f' is a function satisfying (F1) and with values in the desired range; it remains to show that $\sigma_k \circ f'$ and g agree not only on the chosen orientations $\vec{e}$ but also on their inverses $\overleftarrow{e}$. Since σ_k is a homomorphism, this is indeed so:

$$(\sigma_k \circ f')(\overleftarrow{e}) = \sigma_k(-f(\vec{e})) = -(\sigma_k \circ f)(\vec{e}) = -g(\vec{e}) = g(\overleftarrow{e})\,.$$

Hence $f' \in F$, so F is indeed non-empty.

Our aim is to find an $f \in F$ that satisfies Kirchhoff's law (F2), and is thus a k-flow. As a candidate, let us consider an $f \in F$ for which the sum

$$K(f) := \sum_{x \in V} |f(x, V)|$$

of all deviations from Kirchhoff's law is least possible. We shall prove that $K(f) = 0$; then, clearly, $f(x, V) = 0$ for every x, as desired.

Suppose $K(f) \neq 0$. Since f satisfies (F1), and hence $\sum_{x \in V} f(x, V) = f(V, V) = 0$, there exists a vertex x with

$$f(x, V) > 0\,. \tag{1}$$

Let $X \subseteq V$ be the set of all vertices x' for which G contains a walk $x_0 e_0 \dots e_{\ell-1} x_\ell$ from x to x' such that $f(e_i, x_i, x_{i+1}) > 0$ for all $i < \ell$; furthermore, let $X' := X \setminus \{x\}$.

We first show that X' contains a vertex x' with $f(x', V) < 0$. By definition of X, we have $f(e, x', y) \leqslant 0$ for all edges $e = x'y$ such that $x' \in X$ and $y \in \overline{X}$. In particular, this holds for $x' = x$. Thus, (1) implies $f(x, X') > 0$. Then $f(X', x) < 0$ by (F1), as well as $f(X', X') = 0$. Therefore

$$\sum_{x' \in X'} f(x', V) = f(X', V) = f(X', \overline{X}) + f(X', x) + f(X', X') < 0\,,$$

so some $x' \in X'$ must indeed satisfy

$$f(x', V) < 0\,. \tag{2}$$

W As $x' \in X$, there is an x–x' walk $W = x_0 e_0 \dots e_{\ell-1} x_\ell$ such that $f(e_i, x_i, x_{i+1}) > 0$ for all $i < \ell$. We now modify f by sending some flow
f' back along W, letting $f' \colon \vec{E} \to \mathbb{Z}$ be given by

$$f' \colon \vec{e} \mapsto \begin{cases} f(\vec{e}) - k & \text{for } \vec{e} = (e_i, x_i, x_{i+1}),\ \ i = 0, \dots, \ell - 1; \\ f(\vec{e}) + k & \text{for } \vec{e} = (e_i, x_{i+1}, x_i),\ \ i = 0, \dots, \ell - 1; \\ f(\vec{e}) & \text{for } e \notin W. \end{cases}$$

By definition of W, we have $|f'(\vec{e})| < k$ for all $\vec{e} \in \vec{E}$. Hence f', like f, lies in F.

How does the modification of f affect K? At all inner vertices v of W, as well as outside W, the deviation from Kirchhoff's law remains unchanged:

$$f'(v, V) = f(v, V) \qquad \text{for all } v \in V \setminus \{x, x'\}. \tag{3}$$

For x and x', on the other hand, we have

$$f'(x, V) = f(x, V) - k \quad \text{and} \quad f'(x', V) = f(x', V) + k\,. \tag{4}$$

Since g is a $\mathbb{Z}_k$-flow and hence

$$\sigma_k(f(x, V)) = g(x, V) = \bar{0} \in \mathbb{Z}_k$$

and

$$\sigma_k(f(x', V)) = g(x', V) = \bar{0} \in \mathbb{Z}_k\,,$$

$f(x, V)$ and $f(x', V)$ are both multiples of k. Thus $f(x, V) \geqslant k$ and $f(x', V) \leqslant -k$, by (1) and (2). But then (4) implies that

$$|f'(x, V)| < |f(x, V)| \quad \text{and} \quad |f'(x', V)| < |f(x', V)|\,.$$

Together with (3), this gives $K(f') < K(f)$, a contradiction to the choice of f.

Therefore $K(f) = 0$ as claimed, and f is indeed a k-flow. □

Since the sum of two circulations with values in $\mathbb{Z}_k$ is another such circulation, $\mathbb{Z}_k$-flows are often easier to construct (by summing over suitable partial flows) than k-flows. In this way, Theorem 6.3.3 may be of considerable help in determining whether or not some given graph has a k-flow. In the following sections we shall meet a number of examples for this.

Although Theorem 6.3.3 tells us whether a given multigraph admits a k-flow (assuming we know the value of its flow-polynomial for $k - 1$), it does not say anything about the number of such flows. By a recent result of Kochol, this number is also a polynomial in k, whose values can be bounded above and below by the corresponding values of the flow polynomial. See the notes for details.

6.4 k-Flows for small k

Trivially, a graph has a 1-flow (the empty set) if and only if it has no edges. In this section we collect a few simple examples of sufficient conditions under which a graph has a 2-, 3- or 4-flow. More examples can be found in the exercises.

Proposition 6.4.1. *A graph has a 2-flow if and only if all its degrees are even.*

Proof. By Theorem 6.3.3, a graph $G = (V, E)$ has a 2-flow if and only if it has a $\mathbb{Z}_2$-flow, i.e. if and only if the constant map $\vec{E} \to \mathbb{Z}_2$ with value $\overline{1}$ satisfies (F2). This is the case if and only if all degrees are even. $\square$ (6.3.3)

For the remainder of this chapter, let us call a graph *even* if all its vertex degrees are even. *even graph*

Proposition 6.4.2. *A cubic graph has a 3-flow if and only if it is bipartite.*

Proof. Let $G = (V, E)$ be a cubic graph. Let us assume first that G has a 3-flow, and hence also a $\mathbb{Z}_3$-flow f. We show that any cycle $C = x_0 \ldots x_\ell x_0$ in G has even length (cf. Proposition 1.6.1). Consider two consecutive edges on C, say $e_{i-1} := x_{i-1}x_i$ and $e_i := x_i x_{i+1}$. If f assigned the same value to these edges in the direction of the forward orientation of C, i.e. if $f(e_{i-1}, x_{i-1}, x_i) = f(e_i, x_i, x_{i+1})$, then f could not satisfy (F2) at x_i for any non-zero value of the third edge at x_i. Therefore f assigns the values $\overline{1}$ and $\overline{2}$ to the edges of C alternately, and in particular C has even length. (1.6.1) (6.3.3)

Conversely, let G be bipartite, with vertex bipartition $\{X, Y\}$. Since G is cubic, the map $\vec{E} \to \mathbb{Z}_3$ defined by $f(e, x, y) := \overline{1}$ and $f(e, y, x) := \overline{2}$ for all edges $e = xy$ with $x \in X$ and $y \in Y$ is a $\mathbb{Z}_3$-flow on G. By Theorem 6.3.3, then, G has a 3-flow. $\square$

What are the flow numbers of the complete graphs K^n? For odd $n > 1$, we have $\varphi(K^n) = 2$ by Proposition 6.4.1. Moreover, $\varphi(K^2) = \infty$, and $\varphi(K^4) = 4$; this is easy to see directly (and it follows from Propositions 6.4.2 and 6.4.5). Interestingly, K^4 is the only complete graph with flow number 4:

Proposition 6.4.3. *For all even $n > 4$, $\varphi(K^n) = 3$.*

Proof. Proposition 6.4.1 implies that $\varphi(K^n) \geqslant 3$ for even n. We show, by induction on n, that every $G = K^n$ with even $n > 4$ has a 3-flow. (6.3.3)

For the induction start, let $n = 6$. Then G is the edge-disjoint union of three graphs G_1, G_2, G_3, with $G_1, G_2 = K^3$ and $G_3 = K_{3,3}$. Clearly G_1 and G_2 each have a 2-flow, while G_3 has a 3-flow by Proposition 6.4.2. The union of all these flows is a 3-flow on G.

Now let $n > 6$, and assume the assertion holds for $n - 2$. Clearly, G is the edge-disjoint union of a K^{n-2} and a graph $G' = (V', E')$ with $G' = \overline{K^{n-2}} * K^2$. The K^{n-2} has a 3-flow by induction. By Theorem 6.3.3, it thus suffices to find a $\mathbb{Z}_3$-flow on G'. For every vertex z of the $\overline{K^{n-2}} \subseteq G'$, let f_z be a $\mathbb{Z}_3$-flow on the triangle $zxyz \subseteq G'$, where $e = xy$ is the edge of the K^2 in G'. Let $f \colon \vec{E'} \to \mathbb{Z}_3$ be the sum of these flows. Clearly, f is nowhere zero, except possibly in (e, x, y) and (e, y, x). If $f(e, x, y) \neq \overline{0}$, then f is the desired $\mathbb{Z}_3$-flow on G'. If $f(e, x, y) = \overline{0}$, then $f + f_z$ (for any z) is a $\mathbb{Z}_3$-flow on G'. □

Proposition 6.4.4. *Every 4-edge-connected graph has a 4-flow.*

(2.4.2) *Proof.* Let $G = (V, E)$ be a 4-edge-connected graph. By Corollary 2.4.2, G has two edge-disjoint spanning trees T_i, $i = 1, 2$. For each edge $e \notin T_i$ let $C_{i,e}$ be the fundamental cycle with respect to T_i containing e, and let
$f_{1,e}, f_{2,e}$ $f_{i,e}$ be a $\mathbb{Z}_4$-flow of value $\overline{i}$ around $C_{i,e}$ – more precisely: a circulation $\vec{E} \to \mathbb{Z}_4$ with values $\overline{i}$ and $-\overline{i}$ on the edges of $C_{i,e}$ and zero elsewhere.

f_1 Let $f_1 := \sum_{e \notin T_1} f_{1,e}$. Since each $e \notin T_1$ lies on only one cycle $C_{1,e'}$ (namely, for $e = e'$), f_1 takes only the values $\overline{1}$ and $-\overline{1}$ ($= \overline{3}$) outside T_1. Let

$$F := \{\, e \in E(T_1) \mid f_1(e) = \overline{0} \,\}$$

f_2 and $f_2 := \sum_{e \in F} f_{2,e}$. As above, $f_2(e) = \overline{2} = -\overline{2}$ for all $e \in F$. Now
f $f := f_1 + f_2$ is the sum of circulations with values in $\mathbb{Z}_4$, and hence itself a circulation with values in $\mathbb{Z}_4$. Moreover, f is nowhere zero: on edges in F it takes the value $\overline{2}$, on edges of $T_1 - F$ it agrees with f_1 (and is hence non-zero by the choice of F), and on all edges outside T_1 it takes one of the values $\overline{1}$ or $\overline{3}$. Hence, f is a $\mathbb{Z}_4$-flow on G, and the assertion follows by Theorem 6.3.3. □

Our next proposition describes the graphs with a 4-flow in terms
$\mathbb{Z}_m \times \mathbb{Z}_n$ of those with a 2-flow. Given integers $m, n \geqslant 2$, write $\mathbb{Z}_m \times \mathbb{Z}_n$ for the group whose elements are the pairs (a, b) with $a \in \mathbb{Z}_m$ and $b \in \mathbb{Z}_n$ and where $(a, b) + (a', b') := (a + a', b + b')$.

Proposition 6.4.5.

(i) *A graph has a 4-flow if and only if it is the union of two even subgraphs.*

(ii) *A cubic graph has a 4-flow if and only if it is 3-edge-colourable.*

(6.3.2) (6.3.3) *Proof.* By Corollary 6.3.2 and Theorem 6.3.3, a graph has a 4-flow if and only if it has a $\mathbb{Z}_2^2$-flow, where $\mathbb{Z}_2^2 := \mathbb{Z}_2 \times \mathbb{Z}_2$. Assertion (i) now follows from Proposition 6.4.1.

(ii) Let $G = (V, E)$ be a cubic graph. We assume first that G has a $\mathbb{Z}_2^2$-flow f, and define an edge colouring $E \to \mathbb{Z}_2^2 \setminus \{0\}$. As $a = -a$ for all $a \in \mathbb{Z}_2^2$, we have $f(\vec{e}) = f(\overleftarrow{e})$ for every $\vec{e} \in \vec{E}$; let us colour the edge e with this colour $f(\vec{e})$. Now if two edges with a common end v had the same colour, then these two values of f would sum to zero; by (F2), f would then assign zero to the third edge at v. As this contradicts the definition of f, our edge colouring is correct.

Conversely, since the three non-zero elements of $\mathbb{Z}_2^2$ sum to zero, every 3-edge-colouring $c: E \to \mathbb{Z}_2^2 \setminus \{0\}$ defines a $\mathbb{Z}_2^2$-flow on G by letting $f(\vec{e}) = f(\overleftarrow{e}) = c(e)$ for all $\vec{e} \in \vec{E}$. □

Corollary 6.4.6. *Every cubic 3-edge-colourable graph is bridgeless.* □

6.5 Flow-colouring duality

In this section we shall see a surprising connection between flows and colouring: every k-colouring of a plane multigraph gives rise to a k-flow on its dual, and vice versa. In this way, the investigation of k-flows on arbitrary graphs, not necessarily planar, appears as a natural generalization of the familiar map colouring problems in the plane.

Let $G = (V, E)$ and $G^* = (V^*, E^*)$ be dual plane multigraphs. (This implies that G and G^* are connected; see Chapter 4.6.) For simplicity, let us assume that G and G^* have neither bridges nor loops and are non-trivial. For edge sets $F \subseteq E$, let us write $G = (V, E)$ $G^* = (V^*, E^*)$

$$F^* := \{ e^* \in E^* \mid e \in F \}.$$ F^*

Conversely, if a subset of E^* is given, we shall usually write it immediately in the form F^*, and thus let $F \subseteq E$ be defined implicitly via the bijection $e \mapsto e^*$.

Suppose we are given a circulation g on G^*: how can we employ the duality between G and G^* to derive from g some information about G? The most general property of all circulations is Proposition 6.1.1, which says that $g(X, \overline{X}) = 0$ for all $X \subseteq V^*$. By Proposition 4.6.1, the bonds $E^*(X, \overline{X})$ in G^* correspond precisely to the cycles in G. Thus if we take the composition f of the maps $e \mapsto e^*$ and g, and sum its values over the edges of a cycle in G, then this sum should again be zero. Our first aim is to formalize and prove this observation.

Of course, there is still a technical hitch: since g takes its arguments not in E^* but in $\vec{E^*}$, we cannot simply define f as above: we first have to refine the bijection $e \mapsto e^*$ into one from $\vec{E}$ to $\vec{E^*}$, i.e. assign to every $\vec{e} \in \vec{E}$ canonically one of the two orientations of e^*. This will be the purpose of our first lemma. After that, we shall show that f does indeed sum to zero along any cycle in G.

If $C = v_0 \dots v_{\ell-1} v_0$ is a cycle with edges $e_i = v_i v_{i+1}$ (and $v_\ell := v_0$), we shall call

$\vec{C}$

$$\vec{C} := \{ (e_i, v_i, v_{i+1}) \mid i < \ell \}$$

cycle with orientation

a *cycle with orientation*. Note that this definition of $\vec{C}$ depends on the vertex enumeration chosen to denote C: every cycle has two orientations. Conversely, of course, C can be reconstructed from the set $\vec{C}$. In practice, we shall therefore speak about C freely even when, formally, only $\vec{C}$ has been defined.

Lemma 6.5.1. *There exists a bijection* $^*\colon \vec{e} \mapsto \vec{e}^{\,*}$ *from* $\vec{E}$ *to* $\overrightarrow{E^*}$ *with the following properties:*

(i) *The underlying edge of* $\vec{e}^{\,*}$ *is always* e^*, *i.e.* $\vec{e}^{\,*}$ *is one of the two orientations* $\overrightarrow{e^*}, \overleftarrow{e^*}$ *of* e^**;*

(ii) *If* $C \subseteq G$ *is a cycle,* $F := E(C)$, *and if* $X \subseteq V^*$ *is such that* $F^* = E^*(X, \overline{X})$, *then there exists an orientation* $\vec{C}$ *of* C *with* $\{ \vec{e}^{\,*} \mid \vec{e} \in \vec{C} \} = \overrightarrow{E^*}(X, \overline{X})$.

The proof of Lemma 6.5.1 is not entirely trivial: it is based on the so-called *orientability* of the plane, and we cannot give it here. Still, the assertion of the lemma is intuitively plausible. Indeed if we define for $e = vw$ and $e^* = xy$ the assignment $(e, v, w) \mapsto (e, v, w)^* \in \{(e^*, x, y), (e^*, y, x)\}$ simply by turning e and its ends clockwise onto e^* (Fig. 6.5.1), then the resulting map $\vec{e} \mapsto \vec{e}^{\,*}$ satisfies the two assertions of the lemma.

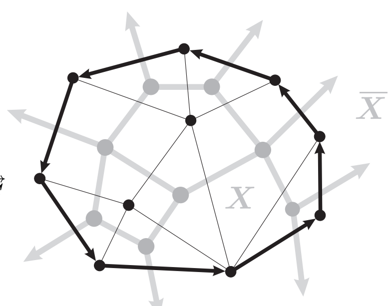


Fig. 6.5.1. Oriented cycle-cut duality

$\vec{e}^{\,*}$

f, g

Consider a fixed bijection $^*\colon \vec{e} \mapsto \vec{e}^{\,*}$ as provided by Lemma 6.5.1. Given an abelian group H, let $f\colon \vec{E} \to H$ and $g\colon \overrightarrow{E^*} \to H$ be two maps such that

$$f(\vec{e}) = g(\vec{e}^{\,*})$$

for all $\vec{e} \in \vec{E}$. For $\vec{F} \subseteq \vec{E}$, we set

$$f(\vec{F}) := \sum_{\vec{e} \in \vec{F}} f(\vec{e}).$$

$f(\vec{C})$ etc.

Lemma 6.5.2.

(i) *The map g satisfies (F1) if and only if f does.*

(ii) *The map g is a circulation on G^* if and only if f satisfies (F1) and $f(\vec{C}) = 0$ for every cycle $\vec{C}$ with orientation.*

(1.9.3)
(4.6.1)
(6.1.1)

Proof. Assertion (i) follows from Lemma 6.5.1 (i) and the fact that $\vec{e} \mapsto \vec{e}^*$ is bijective.

For the forward implication of (ii), let us assume that g is a circulation on G^*, and consider a cycle $C \subseteq G$ with some given orientation. Let $F := E(C)$. By Proposition 4.6.1, $F^* =: E^*(X, \overline{X})$ is a bond of G^*. By definition of f and g, Lemma 6.5.1 (ii) and Proposition 6.1.1 give

$$f(\vec{C}) = \sum_{\vec{e} \in \vec{C}} f(\vec{e}) = \sum_{\vec{d} \in \vec{E^*}(X, \overline{X})} g(\vec{d}) = g(X, \overline{X}) = 0$$

for one of the two orientations $\vec{C}$ of C. Then, by $f(\overleftarrow{C}) = -f(\vec{C})$, also the corresponding value for our given orientation of C must be zero.

For the backward implication it suffices by (i) to show that g satisfies (F2). Let $v \in V^*$ be given. By Lemma 1.9.3, the cut $E^*(v)$ is a disjoint union of bonds $D^* = E^*(X, \overline{X})$; let us name these so that always $v \in X$. Since every edge in these bonds is incident with v, we then have $\vec{E^*}(X, \overline{X}) \subseteq \vec{E^*}(v)$ also for the oriented edges.

By Proposition 4.6.1, each of the sets $D \subseteq E$ is the edge set of a cycle C in G, which by Lemma 6.5.1 (ii) has an orientation $\vec{C}$ such that

$$\{\, \vec{e}^* \mid \vec{e} \in \vec{C} \,\} = \vec{E^*}(X, \overline{X}).$$

Hence $g(X, \overline{X}) = f(\vec{C}) = 0$ by definition of f and g, giving

$$g(v, V^*) = \sum_X g(X, \overline{X}) = 0$$

as desired. $\square$

With the help of Lemma 6.5.2, we can now prove our colouring-flow duality theorem for plane multigraphs. If $P = v_0 \dots v_\ell$ is a path with edges $e_i = v_i v_{i+1}$ ($i < \ell$), we set (depending on our vertex enumeration of P)

$$\vec{P} := \{\, (e_i, v_i, v_{i+1}) \mid i < \ell \,\}$$

$\vec{P}$

and call $\vec{P}$ a $v_0 \to v_\ell$ *path*. Again, P may be given implicitly by $\vec{P}$.

$v_0 \to v_\ell$ *path*

Theorem 6.5.3. (Tutte 1954)
For every dual pair G, G^* of plane multigraphs,

$$\chi(G) = \varphi(G^*)\,.$$

(1.5.5) *Proof.* Let $G =: (V, E)$ and $G^* =: (V^*, E^*)$. For $|G| \in \{1,2\}$ the assertion
V, E is easily checked; we shall assume that $|G| \geqslant 3$, and apply induction on
V^*, E^* the number of bridges in G. If $e \in G$ is a bridge then e^* is a loop, and $G^* - e^*$ is a plane dual of G/e (why?). Hence, by the induction hypothesis,

$$\chi(G) = \chi(G/e) = \varphi(G^* - e^*) = \varphi(G^*)\,;$$

for the first and the last equality we use that, by $|G| \geqslant 3$, e is not the only edge of G.

So all that remains to be checked is the induction start: let us assume that G has no bridge. If G has a loop, then G^* has a bridge, and $\chi(G) = \infty = \varphi(G^*)$ by convention. So we may also assume that G has
k no loop. Then $\chi(G)$ is finite; we shall prove for given $k \geqslant 2$ that G is k-colourable if and only if G^* has a k-flow. As G – and hence G^* – has neither loops nor bridges, we may apply Lemmas 6.5.1 and 6.5.2 to G and G^*. Let $\vec{e} \mapsto \vec{e}^{\,*}$ be a bijection between $\vec{E}$ and $\vec{E^*}$ as in Lemma 6.5.1.

g We first assume that G^* has a k-flow. Then G^* also has a $\mathbb{Z}_k$-flow g.
f As before, let $f\colon \vec{E} \to \mathbb{Z}_k$ be defined by $f(\vec{e}) := g(\vec{e}^{\,*})$. We shall use f to define a vertex colouring $c\colon V \to \mathbb{Z}_k$ of G.

Let T be a normal spanning tree of G, with root r, say. Put $c(r) := \overline{0}$. For every other vertex $v \in V$ let $c(v) := f(\vec{P})$, where $\vec{P}$ is the $r \to v$ path in T. To check that this is a proper colouring, consider an edge $e = vw \in E$. As T is normal, we may assume that $v < w$ in the tree-order of T. If e is an edge of T then $c(w) - c(v) = f(e, v, w)$ by definition of c, so $c(v) \neq c(w)$ since g (and hence f) is nowhere zero. If $e \notin T$, let $\vec{P}$ denote the $v \to w$ path in T. Then

$$c(w) - c(v) = f(\vec{P}) = -f(e, w, v) \neq \overline{0}$$

by Lemma 6.5.2 (ii).

c Conversely, we now assume that G has a k-colouring c. Let us define $f\colon \vec{E} \to \mathbb{Z}$ by

f $$f(e, v, w) := c(w) - c(v)\,,$$

g and $g\colon \vec{E^*} \to \mathbb{Z}$ by $g(\vec{e}^{\,*}) := f(\vec{e})$. Clearly, f satisfies (F1) and takes values in $\{\pm 1, \dots, \pm(k-1)\}$, so by Lemma 6.5.2 (i) the same holds for g. By definition of f, we further have $f(\vec{C}) = 0$ for every cycle $\vec{C}$ with orientation. By Lemma 6.5.2 (ii), therefore, g is a k-flow. $\square$

6.6 Tutte's flow conjectures

How can we determine the flow number of a graph? Indeed, does every (bridgeless) graph have a flow number, a k-flow for some k? Can flow numbers, like chromatic numbers, become arbitrarily large? Can we characterize the graphs admitting a k-flow, for given k?

Of these four questions, we shall answer the second and third in this section: we prove that every bridgeless graph has a 6-flow. In particular, a graph has a flow number if and only if it has no bridge. The question asking for a characterization of the graphs with a k-flow remains interesting for $k = 3, 4, 5$. Partial answers are suggested by the following three conjectures of Tutte, who initiated algebraic flow theory on graphs.

The oldest and best known of the Tutte conjectures is his *5-flow conjecture*:

Five-Flow Conjecture. (Tutte 1954)
Every bridgeless multigraph has a 5-flow.

Which graphs have a 4-flow? By Proposition 6.4.4, the 4-edge-connected graphs are among them. The Petersen graph (Fig. 6.6.1), on the other hand, is an example of a bridgeless graph without a 4-flow: since it is cubic but not 3-edge-colourable, it cannot have a 4-flow by Proposition 6.4.5 (ii).

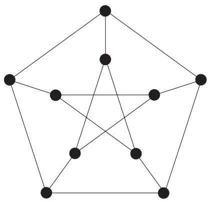


Fig. 6.6.1. The Petersen graph

Tutte's *4-flow conjecture* states that the Petersen graph must be present in every graph without a 4-flow:

Four-Flow Conjecture. (Tutte 1966)
Every bridgeless multigraph not containing the Petersen graph as a minor has a 4-flow.

By Proposition 1.7.3, we may replace the word 'minor' in the 4-flow conjecture by 'topological minor'.

Even if true, the 4-flow conjecture will not be best possible: a K^{11}, for example, contains the Petersen graph as a minor but has a 4-flow, even a 2-flow. The conjecture appears more natural for sparser graphs; a proof for cubic graphs was announced in 1998 by Robertson, Sanders, Seymour and Thomas, which however still has not been published in full.

snark

A cubic bridgeless graph or multigraph without a 4-flow (equivalently, without a 3-edge-colouring) is called a *snark*. The 4-flow conjecture for cubic graphs says that every snark contains the Petersen graph as a minor; in this sense, the Petersen graph has thus been shown to be the smallest snark. Snarks form the hard core both of the four colour theorem and of the 5-flow conjecture: the four colour theorem is equivalent to the assertion that no snark is planar (exercise), and it is not difficult to reduce the 5-flow conjecture to the case of snarks.[6] However, although the snarks form a very special class of graphs, none of the problems mentioned seems to become much easier by this reduction.[7]

Three-Flow Conjecture. (Tutte 1972)
Every multigraph without a cut consisting of exactly one or exactly three edges has a 3-flow.

Again, the 3-flow conjecture will not be best possible: it is easy to construct graphs with three-edge cuts that have a 3-flow (exercise).

By our duality theorem (6.5.3), all three flow conjectures are true for planar graphs and thus motivated: the 3-flow conjecture translates to Grötzsch's theorem (5.1.3), the 4-flow conjecture to the four colour theorem (since the Petersen graph is not planar, it is not a minor of a planar graph), the 5-flow conjecture to the five colour theorem.

We finish this section with the main result of the chapter:

Theorem 6.6.1. (Seymour 1981)
Every bridgeless multigraph has a 6-flow.

(3.3.6)
(6.1.3)
(6.3.2)
(6.3.3)
u

Proof. We prove by induction on $|G|$ that, given any bridgeless multigraph $G = (V, E)$ and a vertex u of G, there exists a $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow $f = f_2 \times f_3$ on G such that f_2 is zero on all the edges at u. By Corollary 6.3.2 and Theorem 6.3.3, then, G will also have a 6-flow.

The induction starts trivially with $|G| = 1$, since loops are ignored in (F1) and (F2). So let G be given for the induction step. This is trivial if G is disconnected or has a cutvertex, so we assume that G is 2-connected. Then $G - u$ is connected.

[6] The same applies to another well-known conjecture, the *cycle double cover conjecture*; see Exercise 17.

[7] That snarks are elusive has been known to mathematicians for some time; cf. Lewis Carroll, *The Hunting of the Snark*, Macmillan 1876.

If $G-u$ has a bridge, e say, let $\{V_1, V_2\}$ be the partition of $V(G-u)$ underlying its cut $\{e\}$. For $i=1,2$ the sets $U_i := V_i \cup \{u\}$ are connected in G; we contract them to form $G_i := G/U_i$ with contracted vertex u_i. By the induction hypothesis, G_i has a $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow $f^i = f_2^i \times f_3^i$ in which f_2^i is zero on u_i. Replacing f_3^1 with $-f_3^1$ if necessary we may assume that it agrees with f_3^2 on $\vec{e}$ and $\overleftarrow{e}$. Now $f := f^1 \cup f^2$ is a $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow on G whose first component f_2 is zero on all the edges at u, as desired.

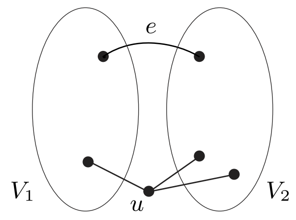


Fig. 6.6.2. Constructing f when $G-u$ has a bridge

It remains to consider the case that $G-u$ is bridgeless. As $G-u$ is connected, this makes it 2-edge-connected. Let H be the union of two H edge-disjoint paths in $G-u$ between distinct neighbours of u; these paths exist by Menger's theorem 3.3.6 (ii). Note that H is an even connected submultigraph of $G-u$ (Figure 6.6.3).

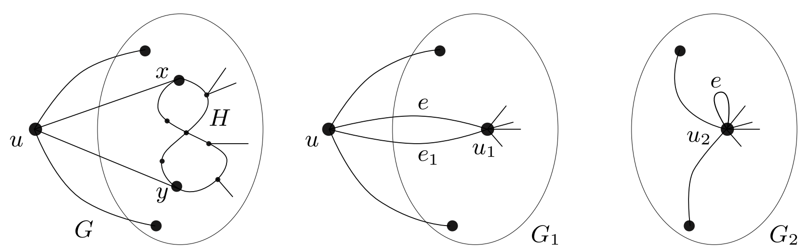


Fig. 6.6.3. Constructing f when $G-u$ has no bridge

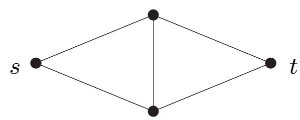
Let G_1 be the minor of G obtained by contracting the edges of H to a vertex u_1, and let G_2 be obtained from G_1 by contracting a u–u_1 edge e_1 to a vertex u_2. Both G_1 and G_2 are bridgeless, because contracting edges never creates a bridge. By the induction hypothesis, G_2 has a $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow $f_2^2 \times f_3^2$ with f_2^2 zero on all the edges at u_2. By Lemma 6.1.3, f_3^2 extends to a $\mathbb{Z}_3$-circulation f_3^1 on G_1 that is non-zero everywhere except possibly on e_1. As G_1 has at least two u–u_1 edges, however, we can modify f_3^1 to make it non-zero on all the edges of G_1, including e_1. Applying Lemma 6.1.3 once more, we extend f_3^1 to a $\mathbb{Z}_3$-circulation f_3 on G that is non-zero on all edges not in H. G_1, u_1 e_1 G_2, u_2

As H is even and f_2^2 maps all the edges at u_2 to zero, we can extend f_2^2 to a $\mathbb{Z}_2$-circulation f_2 on $G-e_1$ that maps all its edges at u to zero, by letting $f_2(\vec{e}) := \bar{1} \in \mathbb{Z}_2$ for all edges e of H. Setting f_2 to zero on e_1 too then turns it into a $\mathbb{Z}_2$-circulation on G. Now $f := f_2 \times f_3$ is a $(\mathbb{Z}_2 \times \mathbb{Z}_3)$-flow on G with f_2 zero on all the edges at u, as desired. $\square$

Exercises

1.$^-$ Prove Proposition 6.2.1 by induction on $|S|$.

2. (i)$^-$ Given $n \in \mathbb{N}$, find a capacity function for the network below such that the algorithm from the proof of the max-flow min-cut theorem will need more than n augmenting paths W if these are badly chosen.



(ii)$^+$ Show that, if all augmenting paths are chosen as short as possible, their number is bounded by a function of the size of the network.

3.$^+$ Derive Menger's Theorem 3.3.5 from the max-flow min-cut theorem.

(Hint. The edge version is easy. For the vertex version, apply the edge version to a suitable auxiliary graph.)

4.$^-$ Let $f\colon \vec{E} \to H$ be a circulation on G and $g\colon H \to H'$ a group homomorphism. Show that $g \circ f$ is a circulation on G. Is $g \circ f$ an H'-flow if f is an H-flow?

5. View the group of circulations on a graph with values in $\mathbb{Z}_2$ as a vector space over $\mathbb{Z}_2$. Find a space in Chapter 1.9 to which it is isomorphic, and write down an explicit isomorphism.

6. Let H be an abelian group, $G = (V, E)$ a connected graph, T a spanning tree, and f a map from the orientations of the edges in $E \setminus E(T)$ to H that satisfies (F1). Show that f extends uniquely to a circulation on G with values in H.

7. (continued)

Let $\mathcal{V}_H = \mathcal{V}_H(G)$ be the group of all maps $V \to H$, and $\mathcal{E}_H = \mathcal{E}_H(G)$ the group of all maps $\vec{E} \to H$ satisfying (F1), both with pointwise addition. Every $\varphi \in \mathcal{V}_H$ defines a $\psi \in \mathcal{E}_H$ by $\psi(e, x, y) := \varphi(y) - \varphi(x)$.

(i) Show that these ψ form a subgroup $\mathcal{B}_H = \mathcal{B}_H(G)$ of $\mathcal{E}_H$ with $\mathcal{B}_H = \{\psi \in \mathcal{E}_H \mid \psi(\vec{C}) = 0$ for every oriented cycle $C \subseteq G\}$, where $\psi(\vec{C}) := \sum_{\vec{e} \in \vec{C}} \psi(\vec{e})$.

(ii) Show that every map $\vec{E}(T) \to H$ satisfying (F1) extends uniquely to a map in $\mathcal{B}_H$.

8.$^+$ (continued)

Let $\mathcal{C}_H$ denote the group of all circulations on G with values in H.

(i) Show that $\mathcal{E}_H/\mathcal{B}_H$ is isomorphic to $\mathcal{C}_H$.

(ii) Show that $\mathcal{E}_H/\mathcal{C}_H$ is isomorphic to $\mathcal{B}_H$.

9.$^-$ Given $k \geqslant 1$, show that a graph has a k-flow if and only if each of its blocks has a k-flow.

10.$^-$ Show that $\varphi(G/e) \leqslant \varphi(G)$ whenever G is a multigraph and e an edge of G. Does this imply that, for every k, the class of all multigraphs admitting a k-flow is closed under taking minors?

11.$^-$ Work out the flow number of K^4 directly, without using any results from the text.

12. Let G be a graph with a k-flow, where $k \geqslant 3$. Find a cubic graph that has a k-flow and from which G can be obtained by contracting edges and identifying nonadjacent vertices. Can you find a construction that does not require the assumption of $k \geqslant 3$?

Do not use the 6-flow Theorem 6.6.1 for the following three exercises.

13. Show that $\varphi(G) < \infty$ for every bridgeless multigraph G.

14. Let G be a bridgeless connected graph with n vertices and m edges. By considering a normal spanning tree of G, show that $\varphi(G) \leqslant m - n + 2$.

15. Assume that a graph G has m spanning trees such that no edge of G lies in all of these trees. Show that $\varphi(G) \leqslant 2^m$.

16. Show that every graph with a Hamilton cycle has a 4-flow. (A *Hamilton cycle* of G is a cycle in G that contains all the vertices of G.)

17. A family of (not necessarily distinct) subgraphs of a graph G is called a *double cover* of G if every edge of G lies on exactly two of these subgraphs. The *cycle double cover conjecture* asserts that every bridgeless multigraph admits a double cover by cycles. Prove the conjecture for graphs with a 4-flow.

18.$^-$ Determine the flow number of $C^5 * K^1$, the wheel with 5 spokes.

19. Find bridgeless graphs G and $H = G - e$ such that $2 < \varphi(G) < \varphi(H)$.

20. Prove Proposition 6.4.1 without using Theorem 6.3.3.

21.$^+$ Prove that a plane triangulation is 3-colourable if and only if all its vertices have even degree.

22. Show that the 3-flow conjecture for planar multigraphs is equivalent to Grötzsch's Theorem 5.1.3.

23. (i)$^-$ Show that the four colour theorem is equivalent to the non-existence of a planar snark, i.e. to the statement that every cubic bridgeless planar multigraph has a 4-flow.

(ii) Can 'bridgeless' in (i) be replaced by '3-connected'?

24.$^{+}$ Show that a graph $G = (V, E)$ has a k-flow if and only if it has an orientation D that directs, for every $X \subseteq V$, at least $1/k$ of the edges in $E(X, \overline{X})$ from X towards $\overline{X}$.

Notes

Network flow theory is an application of graph theory that has had a major and lasting impact on its development over decades. As is illustrated already by the fact that Menger's theorem can be deduced easily from the max-flow min-cut theorem (Exercise 3), the interaction between graphs and networks may go either way: while 'pure' results in areas such as connectivity, matching and random graphs have found applications in network flows, the intuitive power of the latter has boosted the development of proof techniques that have in turn brought about theoretic advances.

The classical reference for network flows is L.R. Ford & D.R. Fulkerson, *Flows in Networks*, Princeton University Press 1962. More recent and comprehensive accounts are given by R.K. Ahuja, T.L. Magnanti & J.B. Orlin, *Network flows*, Prentice-Hall 1993, by A. Frank in his chapter in the *Handbook of Combinatorics* (R.L. Graham, M. Grötschel & L. Lovász, eds.), North-Holland 1995, and by A. Schrijver, *Combinatorial optimization*, Springer 2003. An introduction to graph algorithms in general is given in A. Gibbons, *Algorithmic Graph Theory*, Cambridge University Press 1985.

If one recasts the maximum flow problem in linear programming terms, one can derive the max-flow min-cut theorem from the linear programming duality theorem; see A. Schrijver, *Theory of integer and linear programming*, Wiley 1986.

The more algebraic theory of group-valued flows and k-flows has been developed largely by Tutte; he gives a thorough account in his monograph W.T. Tutte, *Graph Theory*, Addison-Wesley 1984. The fact that the number of k-flows of a multigraph is a polynomial in k, whose values can be bounded in terms of the corresponding values of the flow polynomial, was proved by M. Kochol, Polynomials associated with nowhere-zero[8] flows, *J. Comb. Theory, Ser. B* **84** (2002), 260–269.

Tutte's flow conjectures are covered also in F. Jaeger's survey, Nowhere-zero flow problems, in (L.W. Beineke & R.J. Wilson, eds.) *Selected Topics in Graph Theory 3*, Academic Press 1988. For the flow conjectures, see also T.R. Jensen & B. Toft, *Graph Coloring Problems*, Wiley 1995. The 6-flow theorem is due to P.D. Seymour, Nowhere-zero 6-flows, *J. Comb. Theory, Ser. B* **30** (1981), 130–135. This paper also indicates how Tutte's 5-flow conjecture reduces to snarks. Our proof is from M. Devos and K. Nurse, A short proof of Seymour's 6-flow theorem, arXiv:2307.04768. The proof of the 4-flow conjecture for cubic graphs announced in 1998 Robertson, Sanders, Seymour and Thomas has not yet been entirely written up. C. Thomassen, The weak 3-flow conjecture and the weak circular flow conjecture, *J. Comb. Theory, Ser. B* **102** (2012), 521–529, proved that every graph of large enough edge-connectivity k

[8] In the literature, the term 'flow' is often used to mean what we have called 'circulation', i.e. flows are not required to be nowhere zero unless this is stated explicitly.

has a 3-flow. Thomassen's proof yields this for $k = 8$, and was later improved to give $k = 6$.

Finally, Tutte discovered a 2-variable polynomial associated with a graph, which generalizes both its chromatic polynomial and its flow polynomial. What little is known about this *Tutte polynomial* can hardly be more than the tip of the iceberg: it has far-reaching, and largely unexplored, connections to areas as diverse as knot theory and statistical physics. See D.J.A. Welsh, *Complexity: knots, colourings and counting* (LMS Lecture Notes **186**), Cambridge University Press 1993.