4 Planar Graphs

When we draw a graph on a piece of paper, we naturally try to do this as transparently as possible. One obvious way to limit the mess created by all the lines is to avoid intersections. For example, we may ask if we can draw the graph in such a way that no two edges meet in a point other than a common end.

Graphs drawn in this way are called *plane graphs*; abstract graphs that *can* be drawn in this way are called *planar*. Due to their natural appeal, plane graphs have been studied right from the outset of graph theory; the most prominent example is the four colour problem for maps, which we shall meet in Chapter 5.

The seemingly unrelated question of how we can characterize planar graphs combinatorially, and conversely of which role they play elsewhere in graph theory, is not only equally interesting, but has been found to be quite unexpectedly relevant. Indeed, we shall see in Chapter 12 that the planar graphs – in fact, the graphs embeddable in any given surface – play a central role in one of the deepest theorems of all of graph theory: the structure theorem that describes all graphs without any fixed minor. (Note that there is nothing topological in the definition of that class.) Thus, planar graphs are not just for colouring puzzles: they are important, and they deserve rigorous study.

In this chapter we study both plane and planar graphs, as well as the relationship between the two: the question of how an abstract planar graph might be drawn in fundamentally different ways. After collecting together in Section 4.1 the few basic topological facts that will enable us later to prove all results rigorously without too much technical ado, we begin in Section 4.2 by studying the structural properties of plane graphs. In Section 4.3, we investigate how two drawings of the same graph can differ. The main result of that section is that 3-connected planar graphs have essentially only one drawing, in some very strong and natural topological sense. The next two sections are devoted to the proofs of all the classical planarity criteria, conditions telling us when an abstract graph is planar. We complete the chapter with a section on *plane duality*, a notion with fascinating links to algebraic, colouring, and flow properties of graphs (Chapters 1.9 and 6.5).

The traditional notion of a graph drawing is that its vertices are represented by points in the Euclidean plane, its edges are represented by curves between these points, and different curves meet only in common endpoints. To avoid unnecessary topological complication, however, we shall only consider curves that are piecewise linear; it is not difficult to show that any drawing can be straightened out in this way, so the two notions come to the same thing.

4.1 Topological prerequisites

In this section we briefly review some basic topological definitions and facts needed later. All these facts have (by now) easy and well-known proofs; see the notes for sources. Since those proofs contain no graph theory, we do not repeat them here: indeed our aim is to collect precisely those topological facts that we need but do *not* want to prove. Later, all proofs will follow strictly from the definitions and facts stated here (and be guided by but not rely on geometric intuition), so the material presented now will help to keep elementary topological arguments in those proofs to a minimum.

A *straight line segment* in the Euclidean plane is a subset of $\mathbb{R}^2$ that has the form $\{\, p + \lambda(q-p) \mid 0 \leqslant \lambda \leqslant 1 \,\}$ for distinct points $p, q \in \mathbb{R}^2$. A *polygon* is a subset of $\mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the unit circle S^1, the set of points in $\mathbb{R}^2$ at distance 1 from the origin. Here, as later, any subset of a topological space is assumed to carry the subspace topology. A *polygonal arc* is a subset of $\mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the closed unit interval $[0,1]$. The images of 0 and of 1 under such a homeomorphism are the *endpoints* of this polygonal arc, which *links* them and runs *between* them. Instead of 'polygonal arc' we shall simply say *arc* in this chapter. If P is an arc between x and y, we denote the point set $P \smallsetminus \{x, y\}$, the *interior* of P, by $\mathring{P}$. As continuous images of $[0,1]$, arcs, and finite unions of arcs, are compact, and hence closed in $\mathbb{R}^2$. Their complements in $\mathbb{R}^2$, therefore, are open.

polygon

arc

$\mathring{P}$, $\mathring{e}$

Let $O \subseteq \mathbb{R}^2$ be any open set. Being linked by an arc in O defines an equivalence relation on O. The corresponding equivalence classes are again open; they are the *regions* of O. A closed set $X \subseteq \mathbb{R}^2$ is said to *separate* a region O' of O if $O' \smallsetminus X$ has more than one region. The *frontier* of a set $X \subseteq \mathbb{R}^2$ is the set Y of all points $y \in \mathbb{R}^2$ such that every neighbourhood of y meets both X and $\mathbb{R}^2 \smallsetminus X$. Note that if X is open then its frontier lies in $\mathbb{R}^2 \smallsetminus X$.

region
separate
frontier

The frontier of a region O of $\mathbb{R}^2 \setminus X$, where X is a finite union of points and arcs, has two important properties. The first is accessibility: if $x \in X$ lies on the frontier of O, then x can be linked to some point in O by a straight line segment whose interior lies wholly inside O. As a consequence, any two points on the frontier of O can be linked by an arc whose interior lies in O (why?). The second notable property of the frontier of O is that it separates O from the rest of $\mathbb{R}^2$. Indeed, if $\varphi\colon [0,1] \to P \subseteq \mathbb{R}^2$ is continuous, with $\varphi(0) \in O$ and $\varphi(1) \notin O$, then P meets the frontier of O at least in the point $\varphi(y)$ for $y := \inf \{ x \mid \varphi(x) \notin O \}$, the *first point* of P in $\mathbb{R}^2 \setminus O$.

Theorem 4.1.1. (Jordan Curve Theorem for Polygons) [4.2.2] [4.2.5] [4.2.6] [4.2.7] [4.3.1] [4.5.1] [4.6.1] [5.1.2]
For every polygon $P \subseteq \mathbb{R}^2$, the set $\mathbb{R}^2 \setminus P$ has exactly two regions. Each of these has the entire polygon P as its frontier.

With the help of Theorem 4.1.1, it is not difficult to prove the following lemma.

Lemma 4.1.2. *Let P_1, P_2, P_3 be three arcs, between the same two endpoints but otherwise disjoint.* [4.2.6] [4.2.7] [4.2.8] [12.7.4]

(i) *$\mathbb{R}^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly three regions, with frontiers $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_1 \cup P_3$.*

(ii) *If P is an arc between a point in $\mathring{P}_1$ and a point in $\mathring{P}_3$ whose interior lies in the region of $\mathbb{R}^2 \setminus (P_1 \cup P_3)$ that contains $\mathring{P}_2$, then $\mathring{P} \cap \mathring{P}_2 \neq \emptyset$.*

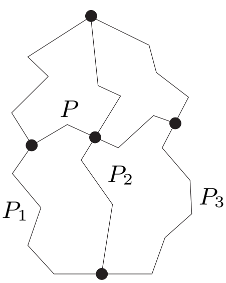

Fig. 4.1.1. The arcs in Lemma 4.1.2 (ii)

Our next lemma complements the Jordan curve theorem by saying that an arc does *not* separate the plane. For easier application later, we phrase this a little more generally:

Lemma 4.1.3. *Let $X_1, X_2 \subseteq \mathbb{R}^2$ be disjoint sets, each the union of finitely many points and arcs, and let P be an arc between a point in X_1 and one in X_2 whose interior lies in a region O of $\mathbb{R}^2 \setminus (X_1 \cup X_2)$. Then $O \setminus P$ is a region of $\mathbb{R}^2 \setminus (X_1 \cup P \cup X_2)$.* [4.2.2] [4.2.4]

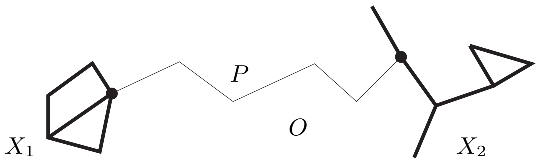

Fig. 4.1.2. P does not separate the region O of $\mathbb{R}^2 \setminus (X_1 \cup X_2)$

It remains to introduce a few terms and facts that will be used only once, when we consider notions of equivalence for graph drawings in Chapter 4.3.

S^n As usual, we denote by S^n the n-dimensional sphere, the set of points in $\mathbb{R}^{n+1}$ at distance 1 from the origin. The 2-sphere minus its 'north pole' $(0,0,1)$ is homeomorphic to the plane; let us choose a fixed

π such homeomorphism $\pi\colon S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ (for example, stereographic projection). If $P \subseteq \mathbb{R}^2$ is a polygon and O is the bounded region of $\mathbb{R}^2 \setminus P$, let us call $C := \pi^{-1}(P)$ a *circle on* S^2, and the sets $\pi^{-1}(O)$ and $S^2 \setminus \pi^{-1}(P \cup O)$ the *regions* of $S^2 \setminus C$.

Our last tool is the theorem of Jordan and Schoenflies, again adapted slightly for our purposes:

[4.3.1] **Theorem 4.1.4.** *Let $\varphi\colon C_1 \to C_2$ be a homeomorphism between two circles on S^2, let O_1 be a region of $S^2 \setminus C_1$, and let O_2 be a region of $S^2 \setminus C_2$. Then φ can be extended to a homeomorphism $C_1 \cup O_1 \to C_2 \cup O_2$.*

4.2 Plane graphs

plane graph A *plane graph* is a pair (V, E) of finite sets with the following properties (the elements of V are again called *vertices*, those of E *edges*):

(i) $V \subseteq \mathbb{R}^2$;

(ii) every edge is an arc between two vertices;

(iii) different edges have different sets of endpoints;

(iv) the interior of an edge contains no vertex and no point of any other edge.

A plane graph (V, E) defines a graph G on V in a natural way. As long as no confusion can arise, we shall use the name G of this abstract graph also for the plane graph (V, E), or for the point set $V \cup \bigcup E$; similar notational conventions will be used for abstract versus plane edges, for subgraphs, and so on.[1]

[1] However, we shall continue to use $\setminus$ for differences of point sets and $-$ for graph differences – which may help a little to keep the two apart.

When G is a plane graph, we call the regions of $\mathbb{R}^2 \setminus G$ the *faces* of G. These are open subsets of $\mathbb{R}^2$ and hence have their frontiers in G. Since G is bounded – i.e., lies inside some sufficiently large disc D – exactly one of its faces is unbounded, the face that contains $\mathbb{R}^2 \setminus D$. This face is the *outer face* of G; the other faces are its *inner faces*. We denote the set of faces of G by $F(G)$. *faces* $F(G)$

The faces of plane graphs and their subgraphs are related in the obvious way:

Lemma 4.2.1. *Let G be a plane graph, $f \in F(G)$ a face, and $H \subseteq G$ a subgraph.* [4.4.3]

(i) *H has a face f' containing f.*

(ii) *If the frontier of f lies in H, then $f' = f$.*

Proof. (i) Clearly, the points in f are equivalent also in $\mathbb{R}^2 \setminus H$; let f' be the equivalence class of $\mathbb{R}^2 \setminus H$ containing them.

(ii) Recall from Section 4.1 that any arc between f and $f' \setminus f$ meets the frontier X of f. If $f' \setminus f \neq \emptyset$ then there is such an arc inside f', whose points in X do not lie in H. Hence $X \not\subseteq H$. □

In order to lay the foundations for the (easy but) rigorous introduction to plane graphs that this section aims to provide, let us descend once now into the realm of truly elementary topology of the plane, and prove what seems entirely obvious:[2] that the frontier of a face of a plane graph G is always a subgraph of G – not, say, half an edge.

The following lemma states this formally, together with two similarly 'obvious' properties of plane graphs:

Lemma 4.2.2. *Let G be a plane graph and e an edge of G.* [4.5.1] [4.5.2] [12.7.4]

(i) *If X is the frontier of a face of G, then either $e \subseteq X$ or $X \cap \mathring{e} = \emptyset$.*

(ii) *If e lies on a cycle $C \subseteq G$, then e lies on the frontier of exactly two faces of G, and these are contained in distinct faces of C.*

(iii) *If e lies on no cycle, then e lies on the frontier of exactly one face of G.*

Proof. We prove all three assertions together. Let us start by considering one point $x_0 \in \mathring{e}$. We show that x_0 lies on the frontier of either exactly two faces or exactly one, according as e lies on a cycle in G or not. We then show that every other point in $\mathring{e}$ lies on the frontier of exactly the same faces as x_0. Then the endpoints of e will also lie on the frontier of (4.1.1) (4.1.3)

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[2] Note that even the best intuition can only ever be 'accurate', i.e., coincide with what the technical definitions imply, inasmuch as those definitions do indeed formalize what is intuitively intended. Given the complexity of definitions in elementary topology, this can hardly be taken for granted.

these faces – simply because every neighbourhood of an endpoint of e is also the neighbourhood of an inner point of e.

Since $G \setminus \mathring{e}$ is compact, we can find around every point $x \in \mathring{e}$ an open
D_x disc D_x that meets G only in those (one or two) straight line segments that contain x.

x_0, S Let us pick an inner point x_0 from a straight line segment $S \subseteq e$. Then $D_{x_0} \cap G = D_{x_0} \cap S$, so $D_{x_0} \setminus G$ is the union of two open half-discs. Since these half-discs do not meet G, they each lie in a face of G. Let
f_1, f_2 us denote these faces by f_1 and f_2; they are the only faces of G with x_0 on their frontier, and they may coincide (Fig. 4.2.1).

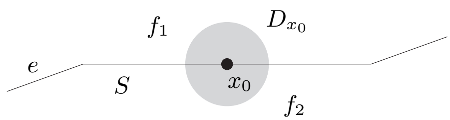

Fig. 4.2.1. Faces f_1, f_2 of G in the proof of Lemma 4.2.2

If e lies on a cycle $C \subseteq G$, then D_{x_0} meets both faces of C (Theorem 4.1.1). Since f_1 and f_2 are contained in faces of C by Lemma 4.2.1, this implies $f_1 \neq f_2$. If e does not lie on any cycle, then e is a bridge and thus links two disjoint point sets X_1, X_2 as in Lemma 4.1.3, with $X_1 \cup X_2 = G \setminus \mathring{e}$. Clearly, $f_1 \cup \mathring{e} \cup f_2$ is the subset of a face f of $G - e$. By Lemma 4.1.3, $f \setminus \mathring{e}$ is a face of G, while $f_1, f_2 \subseteq f \setminus \mathring{e}$ by definition of f. Since f_1 and f_2 are also faces of G, this implies $f_1 = f \setminus \mathring{e} = f_2$.

x_1 Now consider any other point $x_1 \in \mathring{e}$. Let P be the arc from x_0 to
P x_1 contained in e. Since P is compact, finitely many of the discs D_x
$D_0, \dots, D_n$ with $x \in P$ cover P. Let us enumerate these discs as $D_0, \dots, D_n$ in the natural order of their centres along P; adding D_{x_0} or D_{x_1} as necessary, we may assume that $D_0 = D_{x_0}$ and $D_n = D_{x_1}$. By induction on n, one
y easily proves that every point $y \in D_n \setminus e$ can be linked by an arc inside
z $(D_0 \cup \ldots \cup D_n) \setminus e$ to a point $z \in D_0 \setminus e$ (Fig. 4.2.2); then y and z are equivalent in $\mathbb{R}^2 \setminus G$. Hence, every point of $D_n \setminus e$ lies in f_1 or in f_2, so x_1 cannot lie on the frontier of any other face of G. Since both half-discs of $D_0 \setminus e$ can be linked to $D_n \setminus e$ in this way (swap the roles of D_0 and D_n), we find that x_1 lies on the frontier of both f_1 and f_2. □

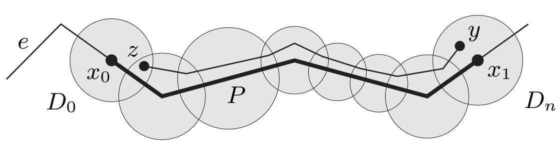

Fig. 4.2.2. An arc from y to D_0, close to P

Corollary 4.2.3. *The frontier of a face is always the point set of a subgraph.* □

The subgraph of G whose point set is the frontier of a face f is said to *bound* f and is called its *boundary*; we denote it by $G[f]$. A face is said to be *incident* with the vertices and edges of its boundary. By Lemma 4.2.1 (ii), every face of G is also a face of its boundary; we shall use this fact frequently in the proofs to come. *boundary* $G[f]$

Proposition 4.2.4. *A plane forest has exactly one face.* [4.6.1]

Proof. Use induction on the number of edges and Lemma 4.1.3. □ (4.1.3)

With just one exception, different faces of a plane graph have different boundaries:

Lemma 4.2.5. *If a plane graph has different faces with the same boundary, then the graph is a cycle.* [4.3.1]

Proof. Let G be a plane graph, and let $H \subseteq G$ be the boundary of distinct faces f_1, f_2 of G. Since f_1 and f_2 are also faces of H, Proposition 4.2.4 implies that H contains a cycle C. By Lemma 4.2.2 (ii), f_1 and f_2 are contained in different faces of C. Since f_1 and f_2 both have all of H as boundary, this implies that $H = C$: any further vertex or edge of H would lie in one of the faces of C and hence not on the boundary of the other. Thus, f_1 and f_2 are distinct faces of C. As C has only two faces, it follows that $f_1 \cup C \cup f_2 = \mathbb{R}^2$ and hence $G = C$. □ (4.1.1)

Proposition 4.2.6. *In a 2-connected plane graph, every face is bounded by a cycle.* [4.3.1] [4.4.3] [4.5.1] [4.5.2]

Proof. Let f be a face in a 2-connected plane graph G. We show by induction on $\|G\|$ that $G[f]$ is a cycle. If G is itself a cycle, this holds by Theorem 4.1.1; we therefore assume that G is not a cycle. (3.1.1) (4.1.1) (4.1.2)

By Proposition 3.1.1, there exist a 2-connected plane graph $H \subseteq G$ and a plane H-path P such that $G = H \cup P$. The interior of P lies in a face f' of H, which by the induction hypothesis is bounded by a cycle C. H P f', C

If $G[f] \subseteq H$, then f is also a face of H (Lemma 4.2.1 (ii)), and we are home by the induction hypothesis. If $G[f] \nsubseteq H$, then $G[f]$ meets $P \setminus H$, so $f \subseteq f'$ and $G[f] \subseteq C \cup P$ (why?). By Lemma 4.2.1 (ii), then, f is a face of $C \cup P$ and hence bounded by a cycle (Lemma 4.1.2 (i)). □

In a 3-connected graph, we can identify the face boundaries among the other cycles in purely combinatorial terms:

Proposition 4.2.7. *The face boundaries in a 3-connected plane graph are precisely its non-separating induced cycles.* [4.3.2] [4.5.2]

(3.3.6) (4.1.1) (4.1.2)

Proof. Let G be a 3-connected plane graph, and let $C \subseteq G$. If C is a non-separating induced cycle, then by the Jordan curve theorem its two faces cannot both contain points of $G \setminus C$. Therefore it bounds a face of G.

C, f Conversely, suppose that C bounds a face f. By Proposition 4.2.6, C is a cycle. If C has a chord $e = xy$, then the components of $C - \{x, y\}$ are linked by a C-path in G, because G is 3-connected. This path and e both run through the other face of C (not f) but do not intersect, a contradiction to Lemma 4.1.2 (ii).

It remains to show that C does not separate any two vertices $x, y \in G - C$. By Menger's theorem (3.3.6), x and y are linked in G by three independent paths. By Lemma 4.2.1 (i), f lies inside a face of their union, and by Lemma 4.1.2 (i) this face is bounded by only two of the paths. The third therefore avoids f and its boundary C. $\square$

maximal plane graph A plane graph G is called *maximally plane*, or just *maximal*, if we cannot add a new edge to form a plane graph $G' \supsetneq G$ with $V(G') = V(G)$. *plane triangulation* We call G a *plane triangulation* if every face of G (including the outer face) is bounded by a triangle.

[4.4.1] [5.4.2]

Proposition 4.2.8. *A plane graph of order at least* 3 *is maximally plane if and only if it is a plane triangulation.*

(4.1.2)

Proof. Let G be a plane graph of order at least 3. It is easy to see that if every face of G is bounded by a triangle, then G is maximally plane. Indeed, any additional edge e would have its interior inside a face of G and its ends on the boundary of that face. Hence these ends are already adjacent in G, so $G \cup e$ cannot satisfy condition (iii) in the definition of a plane graph.

f Conversely, assume that G is maximally plane and let $f \in F(G)$ be
H a face; let us write $H := G[f]$. Since G is maximal as a plane graph, $G[H]$ is complete: any two vertices of H that are not already adjacent in G could be linked by an arc through f, extending G to a larger plane
n graph. Thus $G[H] = K^n$ for some n – but we do not know yet which edges of $G[H]$ lie in H.

Let us show first that H contains a cycle. If not, then $G \setminus H \neq \emptyset$: by $G \supseteq K^n$ if $n \geqslant 3$, or else by $|G| \geqslant 3$. On the other hand we have $f \cup H = \mathbb{R}^2$ by Proposition 4.2.4 and hence $G = H$, a contradiction.

Since H contains a cycle, it suffices to show that $n \leqslant 3$: then $H = K^3$
C, v_i as claimed. Suppose $n \geqslant 4$, and let $C = v_1 v_2 v_3 v_4 v_1$ be a cycle in $G[H]$ $(= K^n)$. By $C \subseteq G$, our face f is contained in a face f_C of C; let f'_C
f_C, f'_C be the other face of C. Since the vertices v_1 and v_3 lie on the boundary of f, they can be linked by an arc whose interior lies in f_C and avoids G. Hence by Lemma 4.1.2 (ii), the plane edge $v_2 v_4$ of $G[H]$ runs through f'_C rather than f_C (Fig. 4.2.3). Analogously, since $v_2, v_4 \in G[f]$, the edge $v_1 v_3$ runs through f'_C. But the edges $v_1 v_3$ and $v_2 v_4$ are disjoint, so this contradicts Lemma 4.1.2 (ii). $\square$

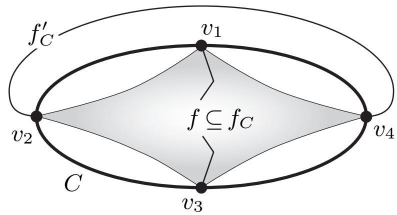

Fig. 4.2.3. The edge v_2v_4 of G runs through the face f'_C

The following classic result of Euler (1752) – here stated in its simplest form, for the plane – marks one of the common origins of graph theory and topology. The theorem relates the number of vertices, edges and faces in a plane graph: taken with the correct signs, these numbers always add up to 2. The general form of Euler's theorem asserts the same for graphs suitably embedded in other surfaces, too: the sum obtained is always a fixed number depending only on the surface, not on the graph, and this number differs for distinct (orientable closed) surfaces. Hence, any two such surfaces can be distinguished by a simple arithmetic invariant of the graphs embedded in them![3]

Let us then prove Euler's theorem in its simplest form:

Theorem 4.2.9. (Euler's Formula)
Let G be a connected plane graph with n vertices, m edges, and ℓ faces. Then

$$n - m + \ell = 2\,.$$

Proof. We fix n and apply induction on m. For $m \leqslant n-1$, G is a tree and $m = n-1$ (why?), so the assertion follows from Proposition 4.2.4. (1.5.1) (1.5.2)

Now let $m \geqslant n$. Then G has an edge e that lies on a cycle; let $G' := G - e$. By Lemma 4.2.2 (ii), e lies on the boundary of exactly two faces f_1, f_2 of G, and as the points in $\mathring{e}$ are all equivalent in $\mathbb{R}^2 \setminus G'$, there is a face f_e of G' containing $\mathring{e}$. We show that e G' f_1, f_2 f_e

$$F(G) \setminus \{f_1, f_2\} = F(G') \setminus \{f_e\}\,; \qquad (*)$$

then G' has exactly one face and one edge less than G, and so the assertion follows from the induction hypothesis for G'.

For a proof of $(*)$ let first $f \in F(G) \setminus \{f_1, f_2\}$ be given. By Lemma 4.2.2 (i) we have $G[f] \subseteq G \setminus \mathring{e} = G'$, and hence $f \in F(G')$ by Lemma 4.2.1 (ii). As clearly $f \neq f_e$, this establishes the forward inclusion in $(*)$.

Conversely, consider any face $f' \in F(G') \setminus \{f_e\}$. Clearly $f' \neq f_1, f_2$, and $f' \cap \mathring{e} = \emptyset$. Hence every two points of f' lie in $\mathbb{R}^2 \setminus G$ and are

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[3] This fundamental connection between graphs and surfaces lies at the heart of the proof of the famous Robertson-Seymour *graph minor theorem*; see Chapter 12.7.

equivalent there, so G has a face f containing f'. By Lemma 4.2.1 (i), however, f lies inside a face f'' of G'. Thus $f' \subseteq f \subseteq f''$ and hence $f' = f = f''$, since both f' and f'' are faces of G'. □

[4.4.1]
[5.1.2]
[7.3.5]

Corollary 4.2.10. *A plane graph with $n \geqslant 3$ vertices has at most $3n-6$ edges. Every plane triangulation with n vertices has $3n-6$ edges.*

Proof. By Proposition 4.2.8 it suffices to prove the second assertion. In a plane triangulation G, every face boundary contains exactly three edges, and every edge lies on the boundary of exactly two faces (Lemma 4.2.2). The bipartite graph on $E(G) \cup F(G)$ with edge set $\{\, ef \mid e \subseteq G[f] \,\}$ thus has exactly $2\,|E(G)| = 3\,|F(G)|$ edges. According to this identity we may replace ℓ with $2m/3$ in Euler's formula, and obtain $m = 3n-6$. □

Euler's formula can be useful for showing that certain graphs cannot occur as plane graphs. The graph K^5, for example, has $10 > 3 \cdot 5 - 6$ edges, more than allowed by Corollary 4.2.10. Similarly, $K_{3,3}$ cannot be a plane graph. For since $K_{3,3}$ is 2-connected but contains no triangle, every face of a plane $K_{3,3}$ would be bounded by a cycle of length $\geqslant 4$ (Proposition 4.2.6). As in the proof of Corollary 4.2.10 this implies $2m \geqslant 4\ell$, which yields $m \leqslant 2n-4$ when substituted in Euler's formula. But $K_{3,3}$ has $9 > 2 \cdot 6 - 4$ edges.

Clearly, along with K^5 and $K_{3,3}$ themselves, their subdivisions cannot occur as plane graphs either:

[4.4.5]
[4.4.6]
[5.1.2]

Corollary 4.2.11. *A plane graph contains neither K^5 nor $K_{3,3}$ as a topological minor.* □

Surprisingly, it turns out that this simple property of plane graphs identifies them among all other graphs: as Section 4.4 will show, an arbitrary graph can be drawn in the plane if and only if it has no (topological) K^5 or $K_{3,3}$ minor.

4.3 Drawings

planar embedding

An embedding in the plane, or *planar embedding*, of an (abstract) graph G is an isomorphism between G and a plane graph H. The latter will be called a *drawing* of G. We shall not always distinguish notationally between the vertices and edges of G and of H. In this section we investigate how two planar embeddings of a graph can differ.

drawing

How should we measure the likeness of two embeddings $\varrho\colon G \to H$ and $\varrho'\colon G \to H'$ of a planar graph G? An obvious way to do this is to consider the canonical isomorphism $\sigma := \varrho' \circ \varrho^{-1}$ between H and H' as abstract graphs, and ask how much of their position in the plane this isomorphism respects or preserves. For example, if σ is induced by a simple rotation of the plane, we would hardly consider ϱ and ϱ' as genuinely different ways of drawing G.

So let us begin by considering any abstract isomorphism $\sigma\colon V \to V'$ between two plane graphs $H = (V, E)$ and $H' = (V', E')$, with face sets $F(H) =: F$ and $F(H') =: F'$ say, and try to measure to what degree σ respects or preserves the features of H and H' as plane graphs. In what follows we shall propose three criteria for this in decreasing order of strictness (and increasing order of ease of handling), and then prove that for most graphs these three criteria turn out to agree. In particular, applied to the isomorphism $\sigma = \varrho' \circ \varrho^{-1}$ considered earlier, all three criteria will say that there is essentially only one way to draw a 3-connected graph.

σ

$H;\ V, E, F$

$H';V',E',F'$

Our first criterion for measuring how well our abstract isomorphism σ preserves the plane features of H and H' is perhaps the most natural one. Intuitively, we would like to call σ 'topological' if it is induced by a homeomorphism from the plane $\mathbb{R}^2$ to itself. To avoid having to grant the outer faces of H and H' a special status, however, we take a detour via the homeomorphism $\pi\colon S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$ chosen in Section 4.1: we call σ a *topological isomorphism* between the plane graphs H and H' if there exists a homeomorphism $\varphi\colon S^2 \to S^2$ such that $\psi := \pi \circ \varphi \circ \pi^{-1}$ induces σ on $V \cup E$. (More formally: we ask that ψ agree with σ on V, and that it map every plane edge $xy \in H$ onto the plane edge $\sigma(x)\sigma(y) \in H'$. Unless φ fixes the point $(0,0,1)$, the map ψ will be undefined at $\pi(\varphi^{-1}(0,0,1))$.)

π

topological isomorphism

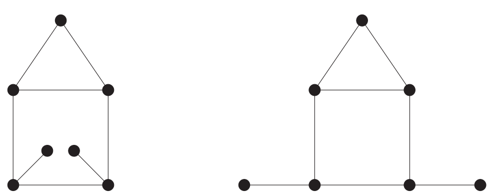

Fig. 4.3.1. Two drawings of a graph that are not topologically isomorphic – why not?

It can be shown that, up to topological isomorphism, inner and outer faces are indeed no longer different: if we choose as φ a rotation of S^2 mapping the π^{-1}-image of a point of some inner face of H to the north pole $(0,0,1)$ of S^2, then ψ maps the rest of this face to the outer face of $\psi(H)$. (To ensure that the edges of $\psi(H)$ are again piecewise linear, however, one may have to adjust φ a little.)

If σ is a topological isomorphism as above, then – except possibly for a pair of missing points where ψ or ψ^{-1} is undefined – ψ maps the faces of H onto those of H' (proof?). In this way, σ extends naturally to a bijection $\sigma: V \cup E \cup F \to V' \cup E' \cup F'$ which preserves incidence of vertices, edges and faces.

Let us single out this last property of a topological isomorphism as the second criterion for how well an abstract isomorphism between plane graphs respects their position in the plane: let us call σ a *combinatorial isomorphism* of the plane graphs H and H' if it can be extended to a bijection $\sigma: V \cup E \cup F \to V' \cup E' \cup F'$ that preserves incidence not only of vertices with edges but also of vertices and edges with faces. (Formally: we require that a vertex or edge $x \in H$ shall lie on the boundary of a face $f \in F$ if and only if $\sigma(x)$ lies on the boundary of the face $\sigma(f)$.)

combinatorial isomorphism

Fig. 4.3.2. Two drawings of a graph that are combinatorially isomorphic but not topologically – why not?

If σ is a combinatorial isomorphism of the plane graphs H and H', it maps the face boundaries of H to those of H'. Let us pick out this property as our third criterion, and call σ a *graph-theoretical isomorphism* of the plane graphs H and H' if

graph-theoretical isomorphism

$$\{\, \sigma(H[f]) : f \in F \,\} = \{\, H'[f'] : f' \in F' \,\}.$$

Thus, we no longer keep track of *which* face is bounded by a given subgraph: the only information we keep is whether a subgraph bounds some face or not, and we require that σ map the subgraphs that do onto each other. At first glance, this third criterion may appear a little less natural than the previous two. However, it has the practical advantage of being formally weaker and hence easier to verify, and moreover, it will turn out to be equivalent to the other two in most cases.

As we have seen, every topological isomorphism between two plane graphs is also combinatorial, and every combinatorial isomorphism is also graph-theoretical. The following theorem shows that, for most graphs, the converse is true as well:

Theorem 4.3.1.

(i) *Every graph-theoretical isomorphism between two plane graphs is combinatorial. Its extension to a face bijection is unique if and only if the graph is not a cycle.*

(ii) *Every combinatorial isomorphism between two 2-connected plane graphs is topological.*

(4.1.1) (4.1.4) (4.2.5) (4.2.6)

Proof. Let $H = (V, E)$ and $H' = (V', E')$ be two plane graphs, put $F(H) =: F$ and $F(H') =: F'$, and let $\sigma\colon V \to V'$ be an isomorphism between the underlying abstract graphs. Extend σ to a map $V \cup E \to V' \cup E'$ by letting $\sigma(xy) := \sigma(x)\sigma(y)$.

(i) If H is a cycle, the assertion follows from the Jordan curve theorem. We now assume that H is not a cycle. Let $\mathcal{B}$ and $\mathcal{B}'$ be the sets of all face boundaries in H and H', respectively. If σ is a graph-theoretical isomorphism, then the map $B \mapsto \sigma(B)$ is a bijection between $\mathcal{B}$ and $\mathcal{B}'$. By Lemma 4.2.5, the map $f \mapsto H[f]$ is a bijection between F and $\mathcal{B}$, and likewise for F' and $\mathcal{B}'$. The composition of these three bijections is a bijection between F and F', which we choose as $\sigma\colon F \to F'$. By construction, this extension of σ to $V \cup E \cup F$ preserves incidences (and is unique with this property), so σ is indeed a combinatorial isomorphism.

(ii) Let us assume that H is 2-connected, and that σ is a combinatorial isomorphism. We have to construct a homeomorphism $\varphi\colon S^2 \to S^2$ which, for every vertex or plane edge $x \in H$, maps $\pi^{-1}(x)$ to $\pi^{-1}(\sigma(x))$. Since σ is a combinatorial isomorphism, $\tilde{\sigma}: \pi^{-1} \circ \sigma \circ \pi$ is an incidence preserving bijection from the vertices, edges and faces[4] of $\tilde{H} := \pi^{-1}(H)$ to the vertices, edges and faces of $\tilde{H}' := \pi^{-1}(H')$. σ $\tilde{\sigma}$ $\tilde{H}, \tilde{H}'$

$$\begin{array}{ccc}
S^2 \supseteq \tilde{H} & \xrightarrow{\tilde{\sigma}} & \tilde{H}' \subseteq S^2 \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
\mathbb{R}^2 \supseteq H & \xrightarrow[\sigma]{} & H' \subseteq \mathbb{R}^2
\end{array}$$

Fig. 4.3.3. Defining $\tilde{\sigma}$ via σ

We construct φ in three steps. Let us first define φ on the vertex set of $\tilde{H}$, setting $\varphi(x) := \tilde{\sigma}(x)$ for all $x \in V(\tilde{H})$. This is trivially a homeomorphism between $V(\tilde{H})$ and $V(\tilde{H}')$.

As the second step, we now extend φ to a homeomorphism between $\tilde{H}$ and $\tilde{H}'$ that induces $\tilde{\sigma}$ on $V(\tilde{H}) \cup E(\tilde{H})$. We may do this edge by edge, as follows. Every edge xy of $\tilde{H}$ is homeomorphic to the edge $\tilde{\sigma}(xy) = \varphi(x)\varphi(y)$ of $\tilde{H}'$, by a homeomorphism mapping x to $\varphi(x)$ and y to $\varphi(y)$. Then the union of all these homeomorphisms, one for every edge of $\tilde{H}$, is indeed a homeomorphism between $\tilde{H}$ and $\tilde{H}'$ – our desired extension of φ to $\tilde{H}$: all we have to check is continuity at the vertices (where the edge homeomorphisms overlap), and this follows at once from our assumption that the two graphs and their individual edges all carry the subspace topology in $\mathbb{R}^3$.

[4] By the 'vertices, edges and faces' of $\tilde{H}$ and $\tilde{H}'$ we mean the images under π^{-1} of the vertices, edges and faces of H and H' (plus $(0, 0, 1)$ in the case of the outer face). Their sets will be denoted by $V(\tilde{H})$, $E(\tilde{H})$, $F(\tilde{H})$ and $V(\tilde{H}')$, $E(\tilde{H}')$, $F(\tilde{H}')$, and incidence is defined as inherited from H and H'.

In the third step we now extend our homeomorphism $\varphi\colon \tilde{H} \to \tilde{H}'$ to all of S^2. This can be done analogously to the second step, face by face. By Proposition 4.2.6, all face boundaries in $\tilde{H}$ and $\tilde{H}'$ are cycles. Now if f is a face of $\tilde{H}$ and C its boundary, then $\tilde{\sigma}(C) := \bigcup\{\, \tilde{\sigma}(e) \mid e \in E(C)\,\}$ bounds the face $\tilde{\sigma}(f)$ of $\tilde{H}'$. By Theorem 4.1.4, we may therefore extend the homeomorphism $\varphi\colon C \to \tilde{\sigma}(C)$ defined so far to a homeomorphism from $C \cup f$ to $\tilde{\sigma}(C) \cup \tilde{\sigma}(f)$. We finally take the union of all these homeomorphisms, one for every face f of $\tilde{H}$, as our desired homeomorphism $\varphi\colon S^2 \to S^2$; as before, continuity is easily checked. □

Let us return now to our original goal, the definition of equivalence for planar embeddings. Let us call two planar embeddings ϱ, ϱ' of a graph *equivalent embeddings* G *topologically* (respectively, *combinatorially*) *equivalent* if $\varrho' \circ \varrho^{-1}$ is a topological (respectively, combinatorial) isomorphism between $\varrho(G)$ and $\varrho'(G)$. If G is 2-connected, the two definitions coincide by Theorem 4.3.1, and we simply speak of *equivalent* embeddings. Clearly, this is indeed an equivalence relation on the set of planar embeddings of any given graph.

Note that two drawings of G resulting from inequivalent embeddings may well be topologically isomorphic (exercise): for the equivalence of two embeddings we ask not only that some (topological or combinatorial) isomorphism exist between the their images, but that the canonical isomorphism $\varrho' \circ \varrho^{-1}$ be a topological or combinatorial one.

Even in this strong sense, 3-connected graphs have only one embedding up to equivalence:

[12.7.4] **Theorem 4.3.2.** (Whitney 1933)
Any two planar embeddings of a 3-connected graph are equivalent.

(4.2.7) *Proof.* Let G be a 3-connected graph with planar embeddings $\varrho\colon G \to H$ and $\varrho'\colon G \to H'$. By Theorem 4.3.1 it suffices to show that $\varrho' \circ \varrho^{-1}$ is a graph-theoretical isomorphism, i.e. that $\varrho(C)$ bounds a face of H if and only if $\varrho'(C)$ bounds a face of H', for every subgraph $C \subseteq G$. This follows at once from Proposition 4.2.7. □

4.4 Planar graphs: Kuratowski's theorem

planar A graph is called *planar* if it can be embedded in the plane: if it is isomorphic to a plane graph. A planar graph is *maximal*, or *maximally planar*, if it is planar but cannot be extended to a larger planar graph by adding an edge (but no vertex).

Drawings of maximal planar graphs are clearly maximally plane. The converse, however, is not obvious: when we start to draw a planar graph, could it happen that we get stuck half-way with a proper subgraph that is already maximally plane? Our first proposition says that this can never happen, that is, a plane graph is never maximally plane just because it is badly drawn:

Proposition 4.4.1.

(i) *Every maximal plane graph is maximally planar.*

(ii) *A planar graph with $n \geqslant 3$ vertices is maximally planar if and only if it has $3n-6$ edges.*

Proof. Apply Proposition 4.2.8 and Corollary 4.2.10. □ (4.2.8) (4.2.10)

Which graphs are planar? As we saw in Corollary 4.2.11, no planar graph contains K^5 or $K_{3,3}$ as a topological minor. Our aim in this section is to prove the surprising converse, a classic theorem of Kuratowski: any graph without a topological K^5 or $K_{3,3}$ minor is planar.

Before we prove Kuratowski's theorem, let us note that it suffices to consider ordinary minors rather than topological ones:

Lemma 4.4.2. *A graph contains K^5 or $K_{3,3}$ as a minor if and only if it contains K^5 or $K_{3,3}$ as a topological minor.*

Proof. By Proposition 1.7.3 it suffices to show that every graph G with a (1.7.3) K^5 minor contains either K^5 as a topological minor or $K_{3,3}$ as a minor. So suppose that $G \succcurlyeq K^5$, and let K be a minimal model of K^5 in G. Then every branch set of K induces a tree in K, and between any two branch sets K has exactly one edge. If we take the tree induced by a branch set V_x and add to it the four edges joining it to other branch sets, we obtain another tree, T_x say. By the minimality of K, the tree T_x has exactly 4 leaves, the 4 neighbours of V_x in other branch sets (Fig. 4.4.1).

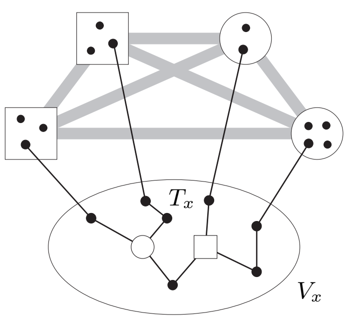

Fig. 4.4.1. Every IK^5 contains a TK^5 or $IK_{3,3}$

If each of the five trees T_x is a $TK_{1,4}$ then K is a TK^5, and we are done. If one of the T_x is not a $TK_{1,4}$ then it has exactly two vertices of degree 3. Contracting V_x onto these two vertices, and every other branch set to a single vertex, we obtain a graph on 6 vertices containing a $K_{3,3}$. Thus, $G \succcurlyeq K_{3,3}$ as desired. □

We first prove Kuratowski's theorem for 3-connected graphs. This is the heart of the proof: the general case will then follow easily.

Lemma 4.4.3. *Every 3-connected graph G without a K^5 or $K_{3,3}$ minor is planar.*

(3.2.4)
(4.2.1)
(4.2.6)

Proof. We apply induction on $|G|$. For $|G| = 4$ we have $G = K^4$, and the assertion holds. Now let $|G| > 4$, and assume the assertion is true
xy for smaller graphs. By Lemma 3.2.4, G has an edge xy such that G/xy is again 3-connected. Since the minor relation is transitive, G/xy has no K^5 or $K_{3,3}$ minor either. Thus, by the induction hypothesis, G/xy has
$\tilde{G}$ a drawing $\tilde{G}$ in the plane. Let f be the face of $\tilde{G} - v_{xy}$ containing the
f, C point v_{xy}, and let C be the boundary of f. Let $X := N_G(x) \setminus \{y\}$ and
X, Y $Y := N_G(y) \setminus \{x\}$; then $X \cup Y \subseteq V(C)$, because $v_{xy} \in f$. Clearly,

$\tilde{G}'$
$$\tilde{G}' := \tilde{G} - \{\, v_{xy}v \mid v \in Y \setminus X \,\}$$

may be viewed as a drawing of $G - y$, in which the vertex x is represented by the point v_{xy} (Fig. 4.4.2). Our aim is to add y to this drawing to obtain a drawing of G.

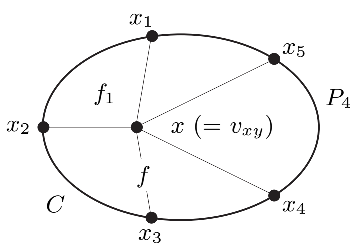

Fig. 4.4.2. $\tilde{G}'$ as a drawing of $G - y$: the vertex x is represented by the point v_{xy}

Since $\tilde{G}$ is 3-connected, $\tilde{G} - v_{xy}$ is 2-connected, so C is a cycle
$x_1, \dots, x_k$ (Proposition 4.2.6). Let $x_1, \dots, x_k$ be an enumeration along this cycle of
P_i the vertices in X, and let $P_i = x_i \dots x_{i+1}$ be the X-paths on C between them ($i = 1, \dots, k$; with $x_{k+1} := x_1$). Let us show that $Y \subseteq V(P_i)$ for some i. Suppose not. If y has a neighbour $y' \in \mathring{P}_i$ for some i, it has another neighbour $y'' \in C - P_i$, and these are separated in C by $x' := x_i$ and $x'' := x_{i+1}$. If $Y \subseteq X$ and $|Y \cap X| \leqslant 2$, then y has exactly two neighbours y', y'' on C but not in the same P_i, so again y' and y'' are separated on C by two vertices $x', x'' \in X$. In either case, x, y', y'' and y, x', x'' are the branch vertices of a $TK_{3,3}$ in G, a contradiction. The only remaining case is that y and x have three common neighbours on C. Then these form a TK^5 with x and y, again a contradiction.

Fix i so that $Y \subseteq P_i$. The point set $C \setminus P_i$ is contained in one of
C_i the two faces of the cycle $C_i := x x_i P_i x_{i+1} x$; we denote the *other* face of

C_i by f_i. Clearly, f_i lies inside a face of C. By Lemma 4.2.1 (ii), one of these is f. Since f_i meets f (close to x), we thus have $f_i \subseteq f$. Moreover, the plane edges xx_j with $j \notin \{i, i+1\}$ meet C_i only in x and end outside f_i in $C \smallsetminus P_i$, so f_i meets none of those edges. Hence $f_i \subseteq \mathbb{R}^2 \smallsetminus \tilde{G}'$, that is, f_i is contained in (and hence equal to) a face of $\tilde{G}'$. We may therefore extend $\tilde{G}'$ to a drawing of G by placing y and its incident edges in f_i. □ f_i

Compared with other proofs of Kuratowski's theorem, the above proof has the attractive feature that it can easily be adapted to produce a drawing in which every inner face is convex (exercise); in particular, every edge can be drawn straight. Note that 3-connectedness is essential here: a 2-connected planar graph need not have a drawing with all inner faces convex (example?), although it always has a straight-line drawing (Exercise 15).

It is not difficult, in principle, to reduce the general Kuratowski theorem to the 3-connected case by manipulating and combining partial drawings assumed to exist by induction. For example, if $\kappa(G) = 2$ and $G = G_1 \cup G_2$ with $V(G_1 \cap G_2) = \{x, y\}$, and if G has no TK^5 or $TK_{3,3}$ subgraph, then neither $G_1 + xy$ nor $G_2 + xy$ has such a subgraph, and we may try to combine drawings of these graphs to one of $G + xy$. (If xy is already an edge of G, the same can be done with G_1 and G_2.) For $\kappa(G) \leqslant 1$, things become even simpler. However, the geometric operations involved require some cumbersome shifting and scaling, even if all the plane edges occurring are assumed to be straight.

The following more combinatorial route offers an ingenious alternative. In order to show that a given graph $G \not\supseteq TK^5, TK_{3,3}$ is planar we start by adding edges to G until it is edge-maximal with the property of not containing a TK^5 or $TK_{3,3}$. In Lemma 4.4.5 we show that this makes our graph 3-connected, so by Lemma 4.4.3 it is planar.

For the proof of Lemma 4.4.5 we need another lemma. We state this a little more generally, so that we can use it again in another context in Chapter 7. For our application here put $\mathcal{X} := \{K^5, K_{3,3}\}$.

Lemma 4.4.4. *Let $\mathcal{X}$ be a set of 3-connected graphs. Let G be a graph with a proper separation $\{V_1, V_2\}$ of order $\kappa(G) \leqslant 2$. If G is edge-maximal without a topological minor in $\mathcal{X}$, then so are $G_1 := G[V_1]$ and $G_2 := G[V_2]$, and $G_1 \cap G_2 = K^2$.* [7.3.1]

Proof. Note first that every vertex $v \in S := V_1 \cap V_2$ has a neighbour in every component of $G_i - S$, $i = 1, 2$: otherwise $S \smallsetminus \{v\}$ would separate G, contradicting $|S| = \kappa(G)$. By the maximality of G, every edge e added to G lies in a $TX \subseteq G + e$ with $X \in \mathcal{X}$. For all the choices of e considered below, the 3-connectedness of X will imply that the branch vertices of this TX all lie in the same V_i, say in V_1. (The position of e will always be symmetrical with respect to V_1 and V_2, so this assumption entails no loss S X

P of generality.) Then the TX meets V_2 at most in a path P corresponding to an edge of X.

If $S = \emptyset$, we obtain an immediate contradiction by choosing e with one end in V_1 and the other in V_2. If $S = \{v\}$ is a singleton, let e join a neighbour v_1 of v in $V_1 \smallsetminus S$ to a neighbour v_2 of v in $V_2 \smallsetminus S$ (Fig. 4.4.3). Then P contains both v and the edge $e = v_1v_2$; replacing its segment vPv_2v_1 with the edge vv_1 we obtain a TX in $G_1 \subseteq G$, a contradiction.

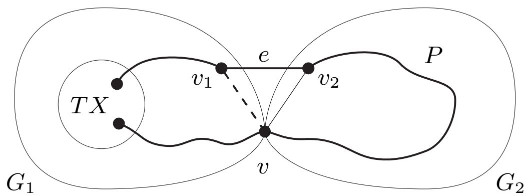

Fig. 4.4.3. If $G+e$ contains a TX, then so does G_1 or G_2

x, y So $|S| = 2$, say $S = \{x, y\}$. If $xy \notin G$, we let $e := xy$, and in the arising TX replace e by an x–y path through G_2. This yields a TX in G, a contradiction. Hence $xy \in G$, and $G[S] = K^2$ as claimed.

It remains to show that G_1 and G_2 are edge-maximal without a topological minor in $\mathcal{X}$. So let e' be an additional edge for G_1, say. Replacing xPy with the edge xy if necessary, we obtain a TX either in $G_1 + e'$ (which shows the edge-maximality of G_1, as desired) or in G_2 (which contradicts $G_2 \subseteq G$). □

Lemma 4.4.5. *If $|G| \geqslant 4$ and G is edge-maximal with $TK^5, TK_{3,3} \not\subseteq G$, then G is 3-connected.*

(4.2.11) *Proof.* We apply induction on $|G|$. For $|G| = 4$, we have $G = K^4$ and the assertion holds. Now let $|G| > 4$, and let G be edge-maximal without a TK^5 or $TK_{3,3}$. Suppose $\kappa(G) \leqslant 2$, and choose G_1 and G_2 as in Lemma 4.4.4. For $\mathcal{X} := \{K^5, K_{3,3}\}$, the lemma says that $G_1 \cap G_2$ is a K^2, with vertices x, y say, and that G_1 and G_2 too are edge-maximal without a TK^5 or $TK_{3,3}$. Hence, G_1 and G_2 are either a triangle or 3-connected by the induction hypothesis. Since they cannot contain K^5 or $K_{3,3}$ even as an ordinary minor (Lemma 4.4.2), they are thus planar by Lemma 4.4.3.

G_1, G_2

x, y

f_i For each $i = 1, 2$ separately, choose a drawing of G_i, a face f_i with
z_i the edge xy on its boundary, and a vertex $z_i \neq x, y$ on the boundary of f_i.
K Let K be a TK^5 or $TK_{3,3}$ in the abstract graph $G + z_1z_2$ (Fig. 4.4.4).

If all the branch vertices of K lie in the same G_i, then either $G_i + xz_i$ or $G_i + yz_i$ (or G_i itself, if z_i is already adjacent to x or y, respectively) contains a TK^5 or $TK_{3,3}$; this contradicts Corollary 4.2.11, since these graphs are planar by the choice of z_i. Since $G + z_1z_2$ does not contain four independent paths between $(G_1 - G_2)$ and $(G_2 - G_1)$, these subgraphs cannot both contain a branch vertex of a TK^5, and cannot both contain

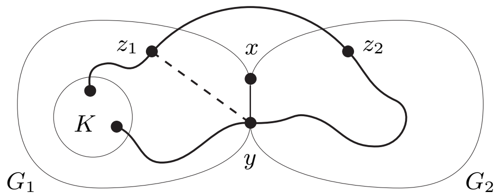

Fig. 4.4.4. A TK^5 or $TK_{3,3}$ in $G + z_1z_2$

two branch vertices of a $TK_{3,3}$. Hence K is a $TK_{3,3}$ with only one branch vertex v in, say, $G_2 - G_1$. But then also the graph $G_1 + v + \{vx, vy, vz_1\}$, which is planar by the choice of z_1, contains a $TK_{3,3}$. This contradicts Corollary 4.2.11. □

[4.5.1]
[5.1.2]
[12.6.4]

Theorem 4.4.6. (Kuratowski 1930; Wagner 1937)
The following assertions are equivalent for graphs G:

(i) *G is planar;*

(ii) *G contains neither K^5 nor $K_{3,3}$ as a minor;*

(iii) *G contains neither K^5 nor $K_{3,3}$ as a topological minor.*

Proof. Combine Corollary 4.2.11 with Lemmas 4.4.2, 4.4.3 and 4.4.5. (4.2.11)
□

Corollary 4.4.7. *Every maximal planar graph with at least four vertices is 3-connected.*

Proof. Apply Lemma 4.4.5 and Theorem 4.4.6. □

4.5 Algebraic planarity criteria

One of the most conspicuous features of a plane graph G are its *facial cycles*, the cycles that bound a face. If G is 2-connected it is covered by its facial cycles, so in a sense these form a 'large' set. In fact, the set of facial cycles is large even in the sense that they generate the entire cycle space: every cycle in G is easily seen to be the sum of the facial cycles (see below). On the other hand, the facial cycles only cover G 'thinly', as every edge lies on at most two of them. Our first aim in this section is to show that the existence of such a large yet thinly spread family of cycles is not only a conspicuous feature of planarity but lies at its very heart: it characterizes it. *facial cycles*

Let $G = (V, E)$ be any graph. We call a subset $\mathcal{F}$ of its edge space $\mathcal{E}(G)$ *sparse* if every edge of G lies in at most two sets of $\mathcal{F}$. For example, the cut space $\mathcal{B}(G)$ has a sparse basis consisting of atomic cuts (Proposition 1.9.2). *sparse*

[4.6.3] **Theorem 4.5.1.** (MacLane 1937)
A graph is planar if and only if its cycle space has a sparse basis.

(1.9.1) *Proof.* The assertion being trivial for graphs of order at most 2, we
(1.9.5) consider a graph G of order at least 3. If $\kappa(G) \leqslant 1$, then G is the union
(4.1.1) of two proper induced subgraphs G_1, G_2 with $|G_1 \cap G_2| \leqslant 1$. Then $\mathcal{C}(G)$
(4.2.2) is the direct sum of $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$, and hence has a sparse basis if
(4.2.6) and only if both $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ do (proof?). Moreover, G is planar if
(4.4.6) and only if both G_1 and G_2 are: this follows at once from Kuratowski's theorem, but also from easy geometrical considerations. The assertion for G thus follows inductively from those for G_1 and G_2. For the rest of the proof, we now assume that G is 2-connected.

We first assume that G is planar and choose a drawing. By Proposition 4.2.6, the face boundaries of G are cycles, so they are elements of $\mathcal{C}(G)$. We shall show that the face boundaries generate all the cycles in G; then $\mathcal{C}(G)$ has a sparse basis by Lemma 4.2.2. Let $C \subseteq G$ be any cycle, and let f be its inner face. By Lemma 4.2.2, every edge e with $\mathring{e} \subseteq f$ lies on exactly two face boundaries $G[f']$ with $f' \subseteq f$, and every edge of C lies on exactly one such face boundary. Hence the sum in $\mathcal{C}(G)$ of all those face boundaries is exactly C.

Conversely, let $\{C_1, \ldots, C_k\}$ be a sparse basis of $\mathcal{C}(G)$. Then, for every edge $e \in G$, also $\mathcal{C}(G-e)$ has a sparse basis. Indeed, if e lies in just one of the sets C_i, say in C_1, then $\{C_2, \ldots, C_k\}$ is a sparse basis of $\mathcal{C}(G-e)$; if e lies in two of the C_i, say in C_1 and C_2, then $\{C_1 + C_2, C_3, \ldots, C_k\}$ is such a basis. (Note that $C_1 + C_2$ is indeed an element of $\mathcal{C}(G-e)$, by Proposition 1.9.1.) Thus every subgraph of G has a cycle space with a sparse basis. For our proof that G is planar, it thus suffices to show that the cycle spaces of K^5 and $K_{3,3}$ (and hence those of their subdivisions) do *not* have a sparse basis: then G cannot contain a TK^5 or $TK_{3,3}$, and so is planar by Kuratowski's theorem.

Let us consider K^5 first. By Theorem 1.9.5, $\dim \mathcal{C}(K^5) = 6$; let $\mathcal{B} = \{C_1, \ldots, C_6\}$ be a sparse basis, and put $C_0 := C_1 + \ldots + C_6$. As $\mathcal{B}$ is linearly independent, none of the sets $C_0, \ldots, C_6$ is empty, so each of them contains at least three edges (cf. Proposition 1.9.1). Moreover, as every edge from C_0 lies in just one of $C_1, \ldots, C_6$, the set $\{C_0, \ldots, C_6\}$ is still sparse. But this implies that K^5 should have more edges than it does, i.e. we obtain the contradiction of

$$21 = 7 \cdot 3 \leqslant |C_0| + \ldots + |C_6| \leqslant 2\,\|K^5\| = 20\,.$$

For $K_{3,3}$, Theorem 1.9.5 gives $\dim \mathcal{C}(K_{3,3}) = 4$; let $\mathcal{B} = \{C_1, \ldots, C_4\}$ be a sparse basis, and put $C_0 := C_1 + \ldots + C_4$. As $K_{3,3}$ has girth 4, each C_i contains at least four edges. We then obtain the contradiction of

$$20 = 5 \cdot 4 \leqslant |C_0| + \ldots + |C_4| \leqslant 2\,\|K_{3,3}\| = 18\,.$$

□

A constructive proof of the backward implication of MacLane's theorem is indicated in Exercise 32. That proof shows that the generating set we chose in our proof of the forward implication, the set of face boundaries, is canonical in the following sense: given any set $\mathcal{D}$ of cycles in a 2-connected planar graph G that generates $\mathcal{C}(G)$ and is such that every edge of G lies on exactly two of those cycles, there is a drawing of G in which the cycles in $\mathcal{D}$ are precisely the face boundaries.

It is one of the hidden beauties of planarity theory that two such abstract and seemingly unintuitive results about generating sets in cycle spaces as MacLane's theorem and Tutte's theorem 3.2.6 conspire to produce a very tangible planarity criterion for 3-connected graphs:

Theorem 4.5.2. (Kelmans 1978)
A 3-connected graph is planar if and only if every edge lies on at most (equivalently: exactly) two non-separating induced cycles.

(3.2.6)
(4.2.2)
(4.2.6)
(4.2.7)

Proof. The forward implication follows from Proposition 4.2.7 and Lemma 4.2.2 (and Proposition 4.2.6 for the 'exactly two' version); the backward implication follows from Theorems 3.2.6 and 4.5.1. □

Let us conclude this section with another characterization of planarity, one with a very different flavour. A *linear extension* of a partial ordering $\leqslant$ of a set P is a total ordering $\leqslant'$ on P which includes $\leqslant$ as a subset of P^2. Thus, for any $p \leqslant q$ in P we still have $p \leqslant' q$, and for incomparable $p, q \in P$ we have either $p <' q$ or $p >' q$ in addition. The *dimension* of the partially ordered set $(P, \leqslant)$ is the least number of linear extensions of $\leqslant$ on P whose intersection is exactly $\leqslant$: for any incomparable $p, q \in P$ there must be a linear extension $\leqslant'$ with $p <' q$ and another linear extension $\leqslant''$ with $p >'' q$ in this collection.

linear extension

poset dimension

With every graph $G = (V, E)$ one can associate its *incidence poset*, the partially ordered set $(V \cup E, \leqslant)$ in which $v < e$ if and only if v is a vertex and e is an edge at v. (Thus, as a relation, $<$ is the same as $\in$.)

incidence poset

Theorem 4.5.3. (Schnyder 1989)
A graph is planar if and only if its incidence poset has dimension $\leqslant 3$.

4.6 Plane duality

In this section we shall use MacLane's theorem to uncover another connection between planarity and algebraic structure: a connection between the duality of plane graphs, defined below, and the duality of the cycle and cut space hinted at in Chapters 1.9 and 2.4.

plane multigraph

A *plane multigraph* is a pair $G = (V, E)$ of finite sets (of *vertices* and *edges*, respectively) satisfying the following conditions:

(i) $V \subseteq \mathbb{R}^2$;

(ii) every edge is either an arc between two vertices or a polygon containing exactly one vertex (its *endpoint*);

(iii) apart from its own endpoint(s), an edge contains no vertex and no point of any other edge.

We shall use terms defined for plane graphs freely for plane multigraphs. Note that, as in abstract multigraphs, both loops and double edges count as cycles.

Let us consider the plane multigraph G shown in Figure 4.6.1. Let us place a new vertex inside each face of G and link these new vertices up to form another plane multigraph G^*, as follows: for every edge e of G we link the two new vertices in the faces incident with e by an edge e^* crossing e; if e is incident with only one face, we attach a loop e^* to the new vertex in that face, again crossing the edge e. The plane multigraph G^* formed in this way is then dual to G in the following sense: if we apply the same procedure as above to G^*, we obtain a plane multigraph very similar to G; in fact, G itself may be reobtained from G^* in this way.

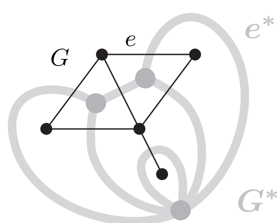

Fig. 4.6.1. A plane graph and its dual

To make this idea more precise, let $G = (V, E)$ and (V^*, E^*) be any two plane multigraphs, and put $F(G) =: F$ and $F((V^*, E^*)) =: F^*$. We call (V^*, E^*) a *plane dual* of G, and write $(V^*, E^*) =: G^*$, if there are bijections

plane dual G^*

$$\begin{array}{ccc} F \to V^* & E \to E^* & V \to F^* \\ f \mapsto v^*(f) & e \mapsto e^* & v \mapsto f^*(v) \end{array}$$

satisfying the following conditions:

(i) $v^*(f) \in f$ for all $f \in F$;

(ii) $|e^* \cap G| = |\mathring{e}^* \cap \mathring{e}| = |e \cap G^*| = 1$ for all $e \in E$, and in each of e and e^* this point is an inner point of a straight line segment;

(iii) $v \in f^*(v)$ for all $v \in V$.

Every connected plane multigraph has a plane dual. Indeed, to satisfy condition (i) we start by picking from each face f of G a point $v^*(f)$ as a vertex for G^*. We can then link these vertices up by independent arcs as required by (ii), and using the connectedness of G show that there is indeed a bijection $V \to F^*$ satisfying (iii) (Exercise 37).

If G_1^* and G_2^* are two plane duals of G, then clearly $G_1^* \cong G_2^*$; in fact, one can show that the natural bijection $v_1^*(f) \mapsto v_2^*(f)$ is a topological isomorphism between G_1^* and G_2^*. In this sense, we may speak of *the* plane dual G^* of G.

Finally, G is in turn a plane dual of G^*. Indeed, this is witnessed by the inverse maps of the bijections from the definition of G^*: setting $v^*(f^*(v)) := v$ and $f^*(v^*(f)) := f$ for $f^*(v) \in F^*$ and $v^*(f) \in V^*$, we see that conditions (i) and (iii) for G^* transform into (iii) and (i) for G, while condition (ii) is symmetrical in G and G^*. As duals are easily seen to be connected (Exercise 36), this symmetry implies that connectedness is also a necessary condition for G to have a dual.

Perhaps the most interesting aspect of plane duality is that it relates geometrically two types of edge sets – cycles and bonds – that we have previously seen to be algebraically related (Theorem 1.9.4):

Proposition 4.6.1. *For any connected plane multigraph G, an edge set $E \subseteq E(G)$ is the edge set of a cycle in G if and only if $E^* := \{ e^* \mid e \in E \}$ is a bond in G^*.* [6.5.2]

(4.1.1)
(4.2.4)

Proof. By conditions (i) and (ii) in the definition of G^*, two vertices $v^*(f_1)$ and $v^*(f_2)$ of G^* lie in the same component of $G^* - E^*$ if and only if f_1 and f_2 lie in the same region of $\mathbb{R}^2 \setminus \bigcup E$: every $v^*(f_1)$–$v^*(f_2)$ path in $G^* - E^*$ is an arc between f_1 and f_2 in $\mathbb{R}^2 \setminus \bigcup E$, and conversely every such arc P (with $P \cap V(G) = \emptyset$) defines a walk in $G^* - E^*$ between $v^*(f_1)$ and $v^*(f_2)$.

Now if $C \subseteq G$ is a cycle and $E = E(C)$ then, by the Jordan curve theorem and the above correspondence, $G^* - E^*$ has exactly two components. So E^* is a bond of G^*, a minimal non-empty cut.

Conversely, if $E \subseteq E(G)$ is such that E^* is a cut in G^* then, by Proposition 4.2.4 and the above correspondence, E contains the edges of a cycle $C \subseteq G$. If E^* is a bond, E cannot contain any further edges (by the implication shown before). Hence $E = E(C)$. $\square$

abstract dual

Proposition 4.6.1 suggests the following generalization of plane duality to abstract multigraphs.[5] Call a multigraph G^* an *abstract dual* of a multigraph G if $E(G^*) = E(G)$ and the bonds in G^* are precisely the edge sets of cycles in G. (Neither G nor G^* need be connected now.)

This correspondence between cycles and bonds extends to the spaces they generate:

Proposition 4.6.2. *If G^* is an abstract dual of G, then the cut space of G^* is the cycle space of G, i.e.,*

$$\mathcal{B}(G^*) = \mathcal{C}(G)\,.$$

(1.9.3) *Proof.* Since the cycles of G are precisely the bonds of G^*, the subspace $\mathcal{C}(G)$ they generate in $\mathcal{E}(G) = \mathcal{E}(G^*)$ is the same as the subspace generated by the bonds in G^*. By Lemma 1.9.3, this is the space $\mathcal{B}(G^*)$. □

(1.9.4) By Theorem 1.9.4, Proposition 4.6.2 implies at once that if G^* is an abstract dual of G then G is an abstract dual of G^*. One can show that if G is 3-connected, then G^* is unique (up to isomorphism and the
(3.1.3) addition of isolated vertices). By Lemma 3.1.3, a non-empty subset of
(3.1.2) $E(G) = E(G^*)$ is the edge set of a block of G if and only if it is the edge set of a block of G^*. By Lemma 3.1.2, this implies that the blocks of G^* are duals of the blocks of G.

Although the notion of abstract duality arose as a generalization of plane duality, it could have been otherwise. We knew already from Theorem 1.9.4 that the cycles and the bonds of a graph form natural and related sets of edges. It would not have been unthinkable to ask whether, for some graphs, the orthogonality between these collections of edge sets might give them sufficiently similar intersection patterns that a collection forming the cycles in one graph could form the bonds in another, and vice versa. In other words, for which graphs can we move their entire edge set to a new set of vertices, redefining incidences, so that precisely those sets of edges that used to form cycles now become bonds (and vice versa)? Put in this way, it seems surprising that this could ever be achieved, let alone for such a large and natural class of graphs as all planar graphs.

As one of the highlights of classical planarity theory we now show that the planar graphs are *precisely* those for which this can be done. Admitting an abstract dual thus appears as a new planarity criterion. Conversely, the theorem can be read as a surprising topological characterization of the equally fundamental property of admitting an abstract dual:

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[5] In what follows we shall use some lemmas from earlier chapters that were stated for graphs only. These lemmas extend to multigraphs with proofs unchanged.

Theorem 4.6.3. (Whitney 1932)
A graph is planar if and only if it has an abstract dual.

Proof. Let G be a planar graph, and consider any drawing. Every component C of this drawing has a plane dual C^*. Consider these C^* as abstract multigraphs, and let G^* be their disjoint union. Then the bonds of G^* are precisely those of the C^*, which by Proposition 4.6.1 correspond to the cycles in G. (1.9.2) (4.5.1)

Conversely, suppose that G has an abstract dual G^*. For a proof that G is planar, it suffices by Theorem 4.5.1 and Proposition 4.6.2 to show that $\mathcal{B}(G^*)$ has a sparse basis. By Proposition 1.9.2, it does. □

The duality theory for both abstract and plane graphs can be extended to infinite graphs. As these can have infinite bonds, their duals must then have 'infinite cycles'. Such things do indeed exist, and are fascinating: they arise as topological circles in a space formed by the graph and its *ends*; see Chapter 8.6.

Exercises

1. Show that every graph can be embedded in $\mathbb{R}^3$ with all edges straight.

2.$^-$ Show directly by Lemma 4.1.2 that $K_{3,3}$ is not planar.

3. Here is an inductive 'proof' that every maximal plane graph of order $\geqslant 4$ is a plane triangulation of minimum degree 3. The induction starts with K^4. For the induction step, consider an arbitrary maximal plane graph G of order $n \geqslant 4$, and consider all possible ways of extending it to a maximal plane graph G' of order $n+1$ by adding a new vertex v. No matter how this is done, v will come to sit in a face of G, which by the inductive assumption is bounded by a triangle. Since G' is maximally planar, v must be joined to all three vertices of that triangle. Clearly, G' is another plane triangulation, and $\delta(G') = d(v) = 3$.
   (i)$^-$ Find the flaw in this 'proof'.
   (ii) Find a counterexample, and explain why the 'proof' overlooks it.

4. Show that every planar graph is a union of three forests.

5. The ancient Greeks loved regular plane graphs whose faces were bounded by cycles of the same length.
   (i) Show that such graphs exist for only finitely many pairs (d, ℓ) of degree $d \geqslant 3$ and cycle length ℓ. Can you give an upper bound?
   (ii)$^+$ Show that there are only finitely many such plane graphs, up to topological isomorphism.

6. A *fullerene* is a molecule that is made up entirely of carbon atoms forming a cubic plane graph all whose faces are pentagons or hexagons. Show that, since carbon atoms can form double bonds, every such graph can be realized in principle by (4-valent) carbon atoms.

7. A football is made of pentagons and hexagons, not necessarily of regular shape. They are sewn together so that their seams form a cubic planar graph. How many pentagons does the football have?

8.$^-$ (continued from Exercises 6 and 7)
Fullerenes are less stable if they contain adjacent pentagons. Show that stable fullerenes have at least 60 carbon atoms.

9. Let G be a graph of order n that is embedded in a surface of Euler characteristic χ and cannot be embedded in a simpler surface (one of larger Euler characteristic). Show that G has at most $3n - 3\chi$ edges.

   (Hint. You may use that every face of such an embedded graph is a topological disc. Such embeddings satisfy the general Euler formula, $n - m + \ell = \chi$.)

10. Find a direct proof for planar graphs of Tutte's theorem on the cycle space of 3-connected graphs (Theorem 3.2.6).

11.$^-$ Show that the two plane graphs in Figure 4.3.1 are not combinatorially (and hence not topologically) isomorphic.

12. Show that the two graphs in Figure 4.3.2 are combinatorially but not topologically isomorphic.

13.$^-$ Show that our definition of equivalence for planar embeddings does indeed define an equivalence relation.

14. Find a 2-connected planar graph whose drawings are all topologically isomorphic but whose planar embeddings are not all equivalent.

15.$^+$ Show that every plane graph is combinatorially isomorphic to a plane graph whose edges are all straight.

   (Hint. Given a plane triangulation, construct inductively a graph-theoretically isomorphic plane graph whose edges are straight. Which additional property of the inner faces could help with the induction?)

Do not use Kuratowski's theorem in the following two exercises.

16. Show that any minor of a planar graph is planar. Deduce that a graph is planar if and only if it is the minor of a grid. (*Grids* are defined in Chapter 12.4.)

17. (i) Show that the planar graphs can in principle be characterized as in Kuratowski's theorem: that there exists a set $\mathcal{X}$ of graphs such that a graph G is planar if and only if G has no minor in $\mathcal{X}$.

   (ii) Can every graph property be characterized in this way? If not, which can?

18. Does every planar graph have a drawing with all inner faces convex?

19. Modify the proof of Lemma 4.4.3 so that all inner faces become convex.

20. Does every minimal non-planar graph G (i.e., every non-planar graph G whose proper subgraphs are all planar) contain an edge e such that $G - e$ is maximally planar? Does the answer change if we define 'minimal' with respect to minors rather than subgraphs?

21. Show that adding a new edge to a maximal planar graph of order at least 6 always produces both a TK^5 and a $TK_{3,3}$ subgraph.

22. Prove the general Kuratowski theorem from its 3-connected case by manipulating plane graphs, i.e. avoiding Lemma 4.4.5.

    (This is not intended as an exercise in elementary topology; for the topological parts of the proof, a rough sketch will do.)

23.$^-$ A graph is called *outerplanar* if it has a drawing in which every vertex lies on the boundary of the outer face. Show that a graph is outerplanar if and only if it contains neither K^4 nor $K_{2,3}$ as a minor.

24. Show that a 2-connected plane graph is bipartite if and only if every face is bounded by an even cycle.

25. Let $G = G_1 \cup G_2$, where $|G_1 \cap G_2| \leqslant 1$. Show that $\mathcal{C}(G)$ has a sparse basis if both $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ have one.

26. Find a cycle space basis among the face boundaries of a 2-connected plane graph.

27. Show that a 3-connected graph of order n has at least $n/2$ peripheral cycles. Is this lower bound sharp?

28.$^+$ Find an algebraic proof of Euler's formula for 2-connected plane graphs, along the following lines. Define the *face space* $\mathcal{F}$ (over $\mathbb{F}_2$) of such a graph in analogy to its vertex space $\mathcal{V}$ and edge space $\mathcal{E}$. Define *boundary maps* $\mathcal{F} \to \mathcal{E} \to \mathcal{V}$ in the obvious way, specifying them first on single faces or edges (i.e., on the standard bases of $\mathcal{F}$ and $\mathcal{E}$) and then extending these maps linearly to all of $\mathcal{F}$ and $\mathcal{E}$. Determine the kernels and images of these homomorphisms, and derive Euler's formula from the dimensions of those subspaces of $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{V}$.

A family of subgraphs of G is said to form a *double cover* of G if every edge of G lies in exactly two of those subgraphs. A double cover by cycles is a *cycle double cover*. A *surface map* is a graph embedded in a surface so that every face is bounded by a cycle.

29. (for topologists)

    (i) Show that a cubic graph has a cycle double cover if and only if it isomorphic to a surface map.

    (ii) Does *every* cycle double cover of a cubic graph occur as the family of face boundaries of a surface graph isomorphic to it?

    (iii) Is the assumption of being cubic relevant in (ii)?

Given a 2-connected graph $G = (V, E)$ and an integer χ, call another graph $H = (W, E)$ with the same edge set E as G an *abstract dual of G with parameter χ* if all the atomic bonds of G are cycles in H and $|V| - |E| + |W| = \chi$.

30.$^+$ (continuing Exercise 29)
    Show that a graph is isomorphic to a surface map for a surface of Euler characteristic χ if and only if it has an abstract dual with parameter χ. You may use the results from Appendix B as needed.

31. Let G be a 2-connected graph whose cycle space is generated by a sparse set $\mathcal{C}'$ of cycles. From MacLane's theorem we know that G even admits a double cover by cycles generating $\mathcal{C}(G)$: the face boundaries in any drawing of G. Show directly (without using MacLane's theorem) that $\mathcal{C}'$ extends to a cycle double cover $\mathcal{D}$ of G.

32.$^+$ (for topologists) Prove the non-trivial implication in MacLane's theorem constructively, as follows. Assume that the given graph G is 2-connected and, by the previous exercise, has a double cover $\mathcal{D}$ by cycles generating $\mathcal{C}(G)$. For each of these cycles C take a disc and identify its boundary with C.

(i) Show that the space obtained is a surface, i.e., a compact 2-manifold without boundary.

(ii) Use Theorem 1.9.5 to show that this surface has Euler characteristic at least 2. (This implies that it must be the sphere, a fact you may assume as known.)

33. Deduce from the last two exercises that, given any 2-connected planar graph and a sparse basis $\mathcal{C}'$ of $\mathcal{C}(G)$ consisting of cycles, there is a drawing of G in which the cycles in $\mathcal{C}'$ are precisely the boundaries of the inner faces.

34.$^+$ Let C be a closed curve in the plane that intersects itself at most once in any given point of the plane, and where every such self-intersection is a proper crossing. Call C *alternating* if we can turn these crossings into over- and underpasses in such a way that when we run along the curve the overpasses alternate with the underpasses.

(i) Prove that every such curve is alternating, or find a counterexample.

(ii) Does the solution to (i) change if the curves considered are not closed?

35.$^-$ What does the plane dual of a plane tree look like?

36.$^-$ Show that the plane dual of a plane multigraph is connected.

37.$^+$ Show that a connected plane multigraph has a plane dual.

38. Show that any two plane duals of a plane multigraph are combinatorially isomorphic.

39. Let G^* be an abstract dual of G, and let $e = e^*$ be an edge. Prove the following two assertions:

(i) G^*/e^* is an abstract dual of $G - e$.

(ii)$^+$ $G^* - e^*$ is an abstract dual of G/e.

40. Find a connected graph that has two non-isomorphic abstract duals. Can you find a 2-connected example?

41. Let G, G^* be dual plane graphs. Prove the following statements:

(i) If G is 2-connected, then G^* is 2-connected.

(ii) If G is 3-connected, then G^* is 3-connected.

(iii) If G is 4-connected, then G^* need not be 4-connected.

42. Give detailed proofs for the statements made after Proposition 4.6.2, except for the uniqueness of G^* (which is proved in Exercise 43 (ii)).

43. Let $G^* = (V^*, E^*)$ be a connected abstract dual of a connected multigraph $G = (V, E)$. Does G have a drawing whose plane dual is isomorphic to G^*? (Perhaps even 'canonically' isomorphic? In which sense?)
   (i) For 2-connected G, prove this using the approach of Exercise 32.
   (ii) Deduce that abstract duals of 3-connected graphs are unique. (What exactly could this mean? Suggest a definition of uniqueness that is stronger than 'up to isomorphism'.)
   (iii) Find a counterexample to the general statement.

44. Show that the following statements are equivalent for connected multigraphs $G = (V, E)$ and $G' = (V', E)$ with the same edge set:
   (i) G and G' are abstract duals of each other;
   (ii) given any set $F \subseteq E$, the multigraph (V, F) is a tree if and only if $(V', E \setminus F)$ is a tree.

Notes

There is a very thorough monograph on the embedding of graphs in surfaces, including the plane: B. Mohar & C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University Press 2001. Proofs of the results cited in Section 4.1, as well as all references for this chapter, can be found there. A good account of the Jordan curve theorem, both polygonal and general, is given also in J. Stillwell, *Classical topology and combinatorial group theory*, Springer 1980.

The short proof of Corollary 4.2.10 uses a trick that deserves special mention: the so-called *double counting* of pairs, illustrated in the text by a bipartite graph whose edges can be counted alternatively by summing its degrees on the left or on the right. Double counting is a technique widely used in combinatorics, and there will be more examples later in the book.

The material of Section 4.3 is not normally standard for an introductory graph theory course, and the rest of the chapter can be read independently of this section. However, the results of Section 4.3 are by no means unimportant. In a way, they have fallen victim to their own success: the shift from a topological to a combinatorial setting for planarity problems which they achieve has made the topological techniques developed there dispensable for most of planarity theory.

In its original version, Kuratowski's theorem was stated only for topological minors; the version for general minors was added by Wagner in 1937. Our proof of the 3-connected case (Lemma 4.4.3) is a weakening of a proof due to C. Thomassen, Planarity and duality of finite and infinite graphs, *J. Comb. Theory, Ser. B* **29** (1980), 244–271, which yields a drawing in which all the inner faces are convex (Exercise 19). The existence of such 'convex' drawings for 3-connected planar graphs follows already from the theorem of Steinitz (1922) that these graphs are precisely the 1-skeletons of 3-dimensional convex polyhedra. Compare also W.T. Tutte, How to draw a graph, *Proc. Lond. Math. Soc.* **13** (1963), 743–767.

As one readily observes, adding an edge to a maximal planar graph (of order at least 6) produces not only a topological K^5 *or* $K_{3,3}$, but both. In Chapter 7.3 we shall see that, more generally, every graph with n vertices and more than $3n-6$ edges contains a TK^5 and, with one easily described class of exceptions, also a $TK_{3,3}$ (Ex. 25, Ch. 7).

Theorem 4.5.2 is widely known as 'Tutte's planarity criterion', because it follows at once from Tutte's 1963 Theorem 3.2.6 and the even earlier planarity criterion of MacLane, Theorem 4.5.1. However, Tutte appears to have been unaware of this. Theorem 4.5.2 was first noticed in the late 1970s, and proved independently of both Theorems 3.2.6 and 4.5.1, by A.K. Kelmans, The concept of a vertex in a matroid, the non-separating cycles in a graph and a new criterion for graph planarity, in *Algebraic Methods in Graph Theory*, Vol. 1, Conf. Szeged 1978, *Colloq. Math. Soc. János Bolyai* **25** (1981) 345–388. Kelmans also reproved Theorem 3.2.6 (being unaware of Tutte's proof), and noted that it can be combined with MacLane's criterion to a proof of Theorem 4.5.2.

Theorem 4.5.3 is due to W. Schnyder, Planar graphs and poset dimension, *Order* **5** (1989), 323–343. For an alternative proof and further references see F. Barrera-Cruz and P. Haxell, A note on Schnyder's theorem, *Order* **28** (2011), 221–226, arXiv:1606.08943.

The proper setting for cycle-bond duality in abstract finite graphs (and beyond) is the theory of *matroids*; see J.G. Oxley, *Matroid Theory*, Oxford University Press 1992, and H. Bruhn & R. Diestel, Infinite matroids in graphs, *Discrete Math.* **311** (2011), 1461–1471.arXiv:1011.4749 The axioms of infinite matroids are given in H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh & P. Wollan, Axioms for infinite matroids, *Adv. Math.* **239** (2013), 18–46, arXiv:1003.3919 Duality in infinite graphs is treated without matroids in H. Bruhn & R. Diestel, Duality in infinite graphs, *Comb. Probab. Comput.* **15** (2006), 75–90, and in R. Diestel & J. Pott, Dual trees must share their ends, *J. Comb. Theory, Ser. B* **123** (2017), 32–53, arXiv:1106.1324.