## 8.6 Graphs with ends: the complete picture

In this section we shall develop a deeper understanding of the global structure of infinite graphs, especially locally finite ones, that can be attained only by studying their ends. This structure is intrinsically topological, because topology best captures our intuition about convergence.[9]

Our first goal will be to make precise our intuitive idea that the ends of a graph are the 'points at infinity' to which its rays converge. To do so, we shall define a topological space $|G|$ associated with a graph $G = (V, E, \Omega)$ and its ends.[10] By considering topological versions of paths, cycles and spanning trees in this space, we shall then be able to extend to infinite graphs some parts of finite graph theory that would not otherwise have infinite counterparts; see the notes for more examples. Thus, the ends of an infinite graph turn out to be more than a curious phenomenon: they form an integral part of the picture, without which it cannot be properly understood. $V, E, \Omega$

To build the space $|G|$ formally, we start with the set $V \cup \Omega$. For every edge $e = uv$ we add a set $\mathring{e} = (u, v)$ of continuum many points, making these sets $\mathring{e}$ disjoint from each other and from $V \cup \Omega$. We then choose for each $e$ some fixed bijection between $\mathring{e}$ and the real interval $(0, 1)$, and extend this bijection to one between $[u, v] := \{u\} \cup \mathring{e} \cup \{v\}$ and $[0, 1]$. This bijection defines a metric on $[u, v]$; we call $[u, v]$ a *topological edge* with *inner points* $x \in \mathring{e}$. Given any $F \subseteq E$ we write $\mathring{F} := \bigcup\{\, \mathring{e} \mid e \in F \,\}$. When we speak of a 'graph' $H \subseteq G$, we shall often also mean its corresponding point set $V(H) \cup \mathring{E}(H)$. $(u, v)$ $[u, v]$ $\mathring{F}$

Having thus defined the point set of $|G|$, let us choose a basis of open sets to define its topology. For every edge $uv$, declare as open all subsets of $(u, v)$ that correspond, by our fixed bijection between $(u, v)$ and $(0, 1)$, to an open set in $(0, 1)$. For every vertex $u$ and $\epsilon > 0$, declare as open the 'open star around $u$ of radius $\epsilon$', that is, the set of all points on edges $[u, v]$ at distance less than $\epsilon$ from $u$, measured individually for each edge in its metric inherited from $[0, 1]$. Finally, for every end $\omega$ and every finite set $S \subseteq V$, there is a unique component $C(S, \omega)$ of $G - S$ that contains rays from $\omega$. Let $\Omega(S, \omega) := \{\, \omega' \in \Omega \mid C(S, \omega') = C(S, \omega) \,\}$. For every $\epsilon > 0$, write $\mathring{E}_\epsilon(S, \omega)$ for the set of all inner points of $S$–$C(S, \omega)$ edges at distance less than $\epsilon$ from their endpoint in $C(S, \omega)$. Then declare as open all sets of the form $C(S, \omega)$

$$\hat{C}_\epsilon(S, \omega) := C(S, \omega) \cup \Omega(S, \omega) \cup \mathring{E}_\epsilon(S, \omega)\,.$$

$\hat{C}_\epsilon(S, \omega)$

This completes the definition of $|G|$, whose open sets are the unions of the sets we explicitly chose as open above. $|G|$

[9] Only point-set topology is needed for the text. See the exercises for more.

[10] The notation of $|G|$ comes from topology and clashes with our notation for the order of $G$. But there is little danger of confusion, so we keep both.

*closure* $\overline{X}$

The *closure* of a set $X \subseteq |G|$ will be denoted by $\overline{X}$. For example, $\overline{V} = V \cup \Omega$ (because every neighbourhood of an end contains a vertex), and the closure of a ray is obtained by adding its end. More generally, the closure of the set of teeth of a comb contains a unique end, the end of its spine. Conversely, if $U \subseteq V$ and $R \in \omega \in \Omega \cap \overline{U}$, there is a comb with spine $R$ and teeth in $U$ (Exercise 77). In particular, the closure of the subgraph $C(S,\omega)$ considered above is the set $C(S,\omega) \cup \Omega(S,\omega)$.

*standard subspace*

$V(X), E(X)$

The subspaces $X$ of $|G|$ we shall be interested in are usually the closure of a subgraph $H$ of $G$, i.e., of the form $X = \overline{U} \cup \mathring{D}$ for $H = (U,D)$. We write $V(X)$ for $U$ and $E(X)$ for $D$, and call such subspaces *standard*. We also refer to such $X$ as $\overline{H}$, or even as $\overline{D}$ if $H$ has no isolated vertices, and then say that $X$ is *spanned by* $D$. Note that the ends in $X$ are always ends of $G$, not of $H$; in particular, they need not have a ray in $H$.

By definition, $|G|$ is always Hausdorff; indeed one can show that it is normal. When $G$ is connected and locally finite, then $|G|$ is compact:[11]

**Proposition 8.6.1.** *If $G$ is connected and locally finite, then $|G|$ is a compact Hausdorff space.*

(8.1.2)

*Proof.* Let $\mathcal{O}$ be an open cover of $|G|$; we show that $\mathcal{O}$ has a finite subcover. Pick a vertex $v_0 \in G$, write $D_n$ for the (finite) set of vertices at distance $n$ from $v_0$, and put $S_n := D_0 \cup \ldots \cup D_{n-1}$. For every $v \in D_n$, let $C(v)$ denote the component of $G - S_n$ containing $v$, and let $\hat{C}(v)$ be its closure together with all inner points of $C(v)$–$S_n$ edges. Then $G[S_n]$ and these $\hat{C}(v)$ together partition $|G|$.

$\hat{C}(v)$

We wish to prove that, for some $n$, each of the sets $\hat{C}(v)$ with $v \in D_n$ is contained in some $O(v) \in \mathcal{O}$. For then we can take a finite subcover of $\mathcal{O}$ for $G[S_n]$ (which is compact, being a finite union of edges and vertices), and add to it these finitely many sets $O(v)$ to obtain the desired finite subcover for $|G|$.

Suppose there is no such $n$. Then for each $n$ the set $V_n$ of vertices $v \in D_n$ such that no set from $\mathcal{O}$ contains $\hat{C}(v)$ is non-empty. Moreover, for every neighbour $u \in D_{n-1}$ of $v \in V_n$ we have $C(v) \subseteq C(u)$ because $S_{n-1} \subseteq S_n$, and hence $u \in V_{n-1}$; let $f(v)$ be such a vertex $u$. By the infinity lemma (8.1.2) there is a ray $R = v_0 v_1 \ldots$ with $v_n \in V_n$ for all $n$. Let $\omega$ be its end, and let $O \in \mathcal{O}$ contain $\omega$. Since $O$ is open, it contains a basic open neighbourhood of $\omega$: there exist a finite set $S \subseteq V$ and $\epsilon > 0$ such that $\hat{C}_\epsilon(S,\omega) \subseteq O$. Now choose $n$ large enough that $S_n$ contains $S$ and all its neighbours. Then $C(v_n)$ lies inside a component of $G - S$. As $C(v_n)$ contains the ray $v_n R \in \omega$, this component must be $C(S,\omega)$. Thus

$$\hat{C}(v_n) \subseteq \hat{C}_\epsilon(S,\omega) \subseteq O \in \mathcal{O},$$

contradicting the fact that $v_n \in V_n$. □

[11] Topologists call $|G|$ the *Freudenthal compactification* of $G$.

If $G$ has a vertex of infinite degree then $|G|$ cannot be compact. (Why not?) But $\Omega \subseteq |G|$ can be compact; see Exercise 85 for when it is.

What else can we say about the space $|G|$ in general? For example, is it metrizable? Using a normal spanning tree $T$ of $G$, it is indeed not difficult to define a metric on $|G|$ that induces its topology. But not every connected graph has a normal spanning tree, and it is not easy to determine in graph-theoretical terms which graphs do. Surprisingly, though, it is possible to deduce the existence of a normal spanning tree from that of a defining metric on $|G|$. Thus whenever $|G|$ is metrizable, a metric can be made visible in a natural and structural way.

**Theorem 8.6.2.** *For a connected graph $G$, the following assertions are equivalent:*

(i) *The space $|G|$ is metrizable.*

(ii) *$G$ has a normal spanning tree.*

(iii) *All minors of $G$ have countable colouring number.*

The proof of the equivalence of (i) and (ii) in Theorem 8.6.2 is indicated in Exercises 41 and 86. More on (iii) can be found in the notes.

Our next aim is to review, or newly define, some topological notions of paths and connectedness, of cycles, and of spanning trees. By substituting these topological notions with respect to $|G|$ for the corresponding graph-theoretical notions with respect to $G$ one can extend to locally finite infinite graphs a number of theorems about paths, cycles and spanning trees in finite graphs whose ordinary infinite versions are false. We shall do this, as a case in point, for the tree packing theorem of Nash-Williams and Tutte, Theorem 2.4.1; see the notes for more.

Let $X$ be an arbitrary Hausdorff space. (Later, this will be a subspace of $|G|$.) $X$ is *(topologically) connected* if it is not a union of two disjoint non-empty open subsets.[12] Note that continuous images of connected spaces are connected. For example, since the real interval $[0, 1]$ is connected,[13] so are its continuous images in $X$. *$X$ connected*

A *homeomorphic* image of $[0, 1]$ in $X$ is an *arc* in $X$; it *links* the images of 0 and 1, which are its *endpoints*. Every finite path in $G$ defines an arc in $|G|$ in an obvious way. Similarly, every ray defines an arc linking its starting vertex to its end, and a double ray in $G$ forms an arc with the two ends of its tails if these ends are distinct. *arc*

The *(topological) degree* of an end $\omega$ of $G$ *in* a standard subspace $X$ of $|G|$ is the supremum, in fact maximum, of all integers $k$ such that $X$ contains $k$ arcs that end in $\omega$ and are otherwise disjoint. *end degrees in subspaces*

[12] These subsets would be complements of each other, and hence also be closed. Note that 'open' and 'closed' means open and closed *in* $X$: when $X$ is a subspace of $|G|$ with the subspace topology, the two sets need not be open or closed in $|G|$.

[13] This takes a few lines to prove—can you prove it?

$G = (V, E, \Omega)$ For the remainder of this section let, unless otherwise mentioned, $G = (V, E, \Omega)$ be a fixed connected locally finite graph.

Unlike ordinary paths, arcs in $|G|$ can jump across a cut without containing an edge from it—but only if the cut is infinite:

[8.7.1] **Lemma 8.6.3.** (Jumping Arc Lemma)
*Let $F \subseteq E$ be a cut of $G$ with sides $V_1, V_2$.*

(i) *If $F$ is finite, then $\overline{V_1} \cap \overline{V_2} = \emptyset$, and there is no arc in $|G| \setminus \mathring{F}$ with one endpoint in $V_1$ and the other in $V_2$.*

(ii) *If $F$ is infinite, then $\overline{V_1} \cap \overline{V_2} \neq \emptyset$, and there will be such an arc if both $V_1$ and $V_2$ are connected in $G$.*

(8.2.2) *Proof.* (i) Suppose that $F$ is finite. Let $S$ be the set of vertices incident with edges in $F$. Then $S$ is finite and separates $V_1$ from $V_2$, so for every $\omega \in \Omega$ the connected graph $C(S, \omega)$ misses either $V_1$ or $V_2$. But then so does every basic open set of the form $\hat{C}_\epsilon(S, \omega)$. Therefore no end $\omega$ lies in the closure of both $V_1$ and $V_2$.

As $|G| \setminus \mathring{F} = \overline{G[V_1]} \cup \overline{G[V_2]}$ and this union is disjoint, no connected subset of $|G| \setminus \mathring{F}$ can meet both $V_1$ and $V_2$. Since arcs are continuous images of $[0, 1]$ and hence connected, there is no $V_1$–$V_2$ arc in $|G| \setminus \mathring{F}$.

(ii) Suppose now that $F$ is infinite. Since $G$ is locally finite, the set $U$ of endvertices of $F$ in $V_1$ is also infinite. By the star-comb lemma (8.2.2), there is a comb in $G$ with teeth in $U$; let $\omega$ be the end of its spine. Then every basic open neighbourhood $\hat{C}_\epsilon(S, \omega)$ of $\omega$ meets $U \subseteq V_1$ infinitely and hence also meets $V_2$, giving $\omega \in \overline{V_1} \cap \overline{V_2}$.

To obtain a $V_1$–$V_2$ arc in $|G| \setminus \mathring{F}$, all we need now is an arc in $\overline{G[V_1]}$ and another in $\overline{G[V_2]}$, both ending in $\omega$. Such arcs exist if the graphs $G[V_i]$ are connected: we can then pick a sequence of vertices in $V_i$ converging to $\omega$, and apply the star-comb lemma in $G[V_i]$ to obtain a comb whose spine is a ray in $G[V_i]$ converging to $\omega$. Concatenating these two rays yields the desired jumping arc. $\square$

To some extent, arcs in $|G|$ assume the role that paths play in finite graphs. So arcs are important—but how do we find them? It is not always possible to construct arcs as explicitly as in the proof of Lemma 8.6.3 (ii). Figure 8.6.1, for example, shows an arc that goes through continuum many ends; such arcs cannot be constructed greedily by following a ray into its end and emerging from that end on another ray, etc.

There are two basic methods to obtain an arc between two given points, say two vertices $x$ and $y$. One is to use compactness to obtain, as a limit of finite $x$–$y$ paths, a *topologial $x$–$y$* path, a continuous map $\pi \colon [0, 1] \to |G|$ sending 0 to $x$ and 1 to $y$. A lemma from general topology then tells us that this path can be made injective:

[8.7.3] **Lemma 8.6.4.** *The image of a topological $x$–$y$ path in a Hausdorff space contains an $x$–$y$ arc.*

To illustrate this method, we will use it in the proof of Theorem 8.7.3.

Another method is to prove that the subspace in which we wish to find our $x$–$y$ arc is topologically connected, and use this to deduce that it contains the desired arc. Our next three lemmas provide the tools needed to implement this approach in practice; we shall then illustrate its use in the proof of Theorem 8.6.9.

Being linked by an arc is an equivalence relation on the points of our Hausdorff space $X$: every $x$–$y$ arc $A$ has a first point $p$ on any $y$–$z$ arc $A'$ (because $A'$ is closed), and the obvious segments $Ap$ and $pA'$ together form an $x$–$z$ arc in $X$. The corresponding equivalence classes are the *arc-components* of $X$. If $X$ has only one arc-component, then $X$ is *arc-connected*. *arc-component* *arc-connected*

Since $[0,1]$ is connected, arc-connectedness implies connectedness. The converse implication is false in general, even for spaces $X \subseteq |G|$ with $G$ locally finite. But it holds in all the cases that matter:

**Lemma 8.6.5.** *Connected standard subspaces of $|G|$ are arc-connected.* [8.7]

Our proof of Theorem 8.7.3 will show how one can prove Lemma 8.6.5. Two further proofs are indicated in Exercises 88 and 129.

**Lemma 8.6.6.** *Arc-components of standard subspaces of $|G|$ are closed.* [8.7]

*Proof.* Let $A$ be an arc-component of a standard subspace of $|G|$. Since $A$ is connected, so is its closure $\overline{A}$. If $\overline{A} \setminus A \neq \emptyset$ then its points are limits of vertices in $A$ (why?), so $\overline{A}$ is again standard. Hence $\overline{A}$ is arc-connected, either because $\overline{A} = A$ or by Lemma 8.6.5. But then $\overline{A} = A$, by definition of $\overline{A}$. Hence $A$ is closed, as claimed. □

Connected standard subspaces of $|G|$ containing two given points are much easier to construct than an arc between two points. This has to do with the fact that they can be described in purely graph-theoretical terms, with reference only to finite subgraphs of $G$ rather than to $|G|$. The description can be viewed as a topological analogue of the fact that a subgraph $H$ of $G$ is connected if and only if it contains an edge from every cut of $G$ that separates two of its vertices:

**Lemma 8.6.7.** *A standard subspace of $|G|$ is connected if and only if it contains an edge from every finite cut of $G$ of which it meets both sides.* [8.7.1]

*Proof.* Let $X \subseteq |G|$ be a standard subspace. For the forward implication, suppose that $G$ has a finite cut $F = E(V_1, V_2)$ such that $X$ meets both $V_1$ and $V_2$ but has no edge in $F$. Then

$$X \subseteq |G| \setminus \mathring{F} = \overline{G[V_1]} \cup \overline{G[V_2]}\,,$$

and this union is disjoint by Lemma 8.6.3 (i). The induced partition of $X$ into non-empty closed subsets of $X$ shows that $X$ is not connected.

The backward implication holds vacuously if $X$ meets more than one component of $G$; we may therefore assume that $G$ is connected. If $X$ is not connected, we can partition it into disjoint non-empty open subsets $O_1$ and $O_2$. As $X$ is standard, $U_i := O_i \cap V(X) \neq \emptyset$ for both $i$. Let $\mathcal{P}$ be a maximal set of edge-disjoint $U_1$–$U_2$ paths in $G$, and put

$$F := \bigcup \{ E(P) \mid P \in \mathcal{P} \} .$$

Then $E(X) \cap F = \emptyset$, and no component of $G - F$ meets both $U_1$ and $U_2$. Extending $\{U_1, U_2\}$ to a partition of $V$ in such a way that each component of $G - F$ has all its vertices in one class, we obtain a cut $F' \subseteq F$ of $G$ of which $X$ meets both sides. As $E(X) \cap F = \emptyset$, it thus suffices to show that $F$ is finite.

If $F$ is infinite, then so is $\mathcal{P}$. As $G$ is locally finite, the vertices of each $P \in \mathcal{P}$ are incident with only finitely many edges of $G$. We can thus inductively find an infinite subset of $\mathcal{P}$ consisting of paths that are not only edge-disjoint but disjoint. As $G$ is connected, the endvertices in $U_1$ of these paths have a limit point $\omega$ in $|G|$ (Proposition 8.6.1), which is also a limit point of their endvertices in $U_2$. Since both $O_1$ and $O_2$ are closed in $|G|$, we thus have $\omega \in O_1 \cap O_2$, contradicting the choice of the $O_i$. □

*circle*

A *circle* in a topological space is a homeomorphic image of the unit circle $S^1 \subseteq \mathbb{R}^2$. For example, if $G$ is the 2-way infinite ladder shown in Figure 8.1.3, and we delete all its rungs (the vertical edges), what remains is a disjoint union of two double rays; its closure in $|G|$, obtained by adding the two ends of $G$, is a circle. Similarly, the double ray 'round the outside' of the 1-way ladder forms a circle together with the unique end of that ladder.

*circuit*

It is not hard to show that no arc in $|G|$ can consist entirely of ends. This implies that every circle in $|G|$ is a standard subspace; the set of edges spanning it will be called its *circuit*.

A more adventurous example of a circle is shown in Figure 8.6.1. Suppose $G$ is the graph obtained from the binary tree $T_2$ by joining for every finite 0–1 sequence $\ell$ the vertices $\ell 01$ and $\ell 10$ by a new edge $e_\ell$. Together with all the (uncountably many) ends of $G$, the double rays $D_\ell \ni e_\ell$ shown in the figure form an arc $A$ in $|G|$, whose union with the bottom double ray $D$ is a circle in $|G|$ (Exercise 94). Note that no two of the double rays in $A$ are consecutive: between any two there lies a third (cf. Exercise 95).

*topological spanning tree*

A *topological spanning tree* of $G$ is a connected standard subspace $T$ of $|G|$ that contains every vertex but contains no circle. Since standard subspaces are closed, $T$ also contains every end, and by Lemma 8.6.5 it is even arc-connected. With respect to the deletion or addition of edges, it is both minimally connected and maximally 'acirclic' (Exercise 99).

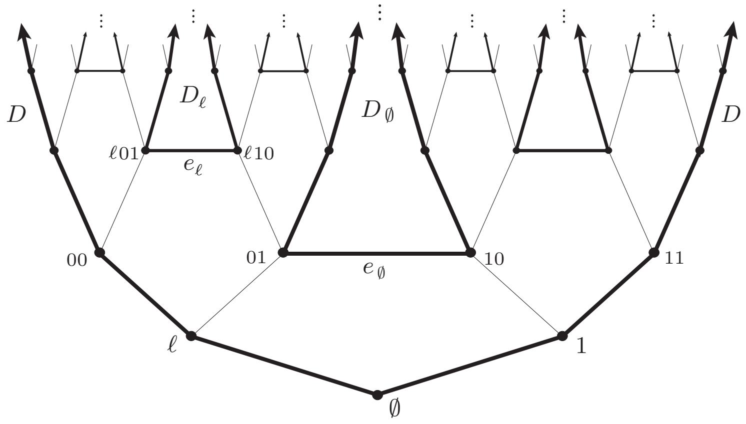


*Fig. 8.6.1.* The Wild Circle

One might expect that the closure $\overline{T}$ of an ordinary spanning tree $T$ of $G$ is always a topological spanning tree of $|G|$, but this is not the case: $\overline{T}$ may well contain a circle (Figure 8.6.2). Conversely, a subgraph whose closure is a topological spanning tree may well be disconnected: the 'vertical' rays in the $\mathbb{N} \times \mathbb{N}$ grid, for example, form a topological spanning tree together with the unique end.

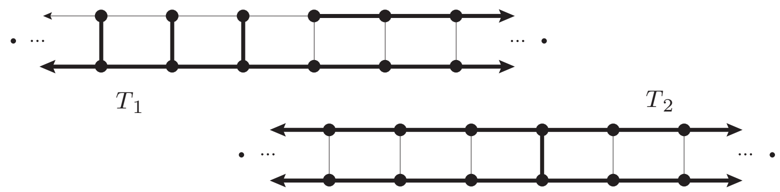


*Fig. 8.6.2.* $T_1$ is a topological spanning tree, but $T_2$ contains three circles

Topological spanning trees can be constructed much as spanning trees of finite graphs: Lemma 8.6.11 will find one by iteratively deleting edges from $|G|$, but they can also be built up 'from below' (Exercise 102). Their mere existence even comes as a corollary of Theorem 8.2.4: (8.2.4)

**Lemma 8.6.8.** *The closure in $|G|$ of any normal spanning tree of $G$ is a topological spanning tree of $G$.* [8.7]

*Proof.* Let $T$ be a normal spanning tree of $G$. By Lemma 8.2.3, every end $\omega$ of $G$ contains a normal ray $R$ of $T$. Then $R \cup \{\omega\}$ is an arc linking $\omega$ to the root of $T$, so $\overline{T}$ is arc-connected. (1.5.5) (8.2.3)

It remains to check that $\overline{T}$ contains no circle. Suppose it does, and let $A$ be the $u$–$v$ arc obtained from that circle by deleting the inner

$f$ points of an edge $f = uv$ it contains. Clearly, $f \in T$. Assume that $u < v$ in the tree-order of $T$, let $T_u$ and $T_v$ denote the components of $T - f$ containing $u$ and $v$, and notice that $V(T_v)$ is the up-closure $\lfloor v \rfloor$ of $v$ in $T$.

Now let $S := \lceil u \rceil$. By Lemma 1.5.5 (ii), $\lfloor v \rfloor$ is the vertex set of a component $C$ of $G - S$. Thus, $V(C) = V(T_v)$ and $V(G - C) = V(T_u)$, so the set $E(C, S)$ of edges between these sets meets $E(T)$ precisely in $f$. Thus, $\overline{C}$ and $\overline{G - C}$ partition $|G| \setminus \mathring{E}(C, S) \supseteq A$ into two open sets both meeting $A$. This contradicts the fact that $A$ is topologically connected. □

Note that the proof of Lemma 8.6.8 did not use our assumption that $G$ is locally finite: whenever a graph $G$ has a normal spanning tree $T$, the closure of $T$ in $|G|$ is an arc-connected subspace that contains no circle.

As a first application of our new concepts, let us now extend the tree packing theorem (2.4.1) of Nash-Williams and Tutte to locally finite graphs. Its naive extension, with ordinary spanning trees, fails. Indeed, for every $k \in \mathbb{N}$ one can construct a $2k$-edge-connected locally finite graph that is left disconnected by the deletion of the edges in any one finite circuit (Exercise 19). Such a graph will have at least $k(\ell - 1)$ edges across any vertex partition into $\ell$ sets, but it cannot have more than two edge-disjoint spanning trees: adding an edge of one of these to another creates a (finite) fundamental circuit there, whose deletion would disconnect any third spanning tree.

As soon as we replace ordinary spanning trees with topological ones, however, Theorem 2.4.1 does extend:

$G = (V, E)$ **Theorem 8.6.9.** *The following statements are equivalent for all $k \in \mathbb{N}$ and connected locally finite multigraphs $G = (V, E)$:*

(i) *$G$ has $k$ edge-disjoint topological spanning trees.*

(ii) *For every finite partition of $V$, into $\ell$ sets say, $G$ has at least $k(\ell - 1)$ cross-edges.*

We begin our proof of Theorem 8.6.9 with a compactness extension of the finite theorem. This yields a weaker, 'finitary', statement at the limit (cf. Lemma 8.6.7):

**Lemma 8.6.10.** *If for every finite partition of $V$, into $\ell$ sets say, $G$ has at least $k(\ell - 1)$ cross-edges, then $G$ has $k$ edge-disjoint spanning submultigraphs whose closures in $|G|$ are topologically connected.*

(2.4.1)
(8.1.2) *Proof.* Pick an enumeration $v_0, v_1, \ldots$ of $V$. For every $n \in \mathbb{N}$ let $G_n$ be the finite multigraph obtained from $G$ by contracting every component of $G - \{v_0, \ldots, v_n\}$ to a vertex, deleting any loops but no parallel edges that arise in the contraction. Then $G[v_0, \ldots, v_n]$ is an induced submultigraph of $G_n$. Let $\mathcal{V}_n$ denote the set of all $k$-tuples $(H_n^1, \ldots, H_n^k)$ of edge-disjoint connected spanning submultigraphs of $G_n$.

Since every partition $P$ of $V(G_n)$ induces a partition of $V$, since $G$ has enough cross-edges for that partition, and since all these cross-edges are also cross-edges of $P$, Theorem 2.4.1 implies that $\mathcal{V}_n \neq \emptyset$. As every $(H_n^1, \dots, H_n^k) \in \mathcal{V}_n$ induces an element $(H_{n-1}^1, \dots, H_{n-1}^k)$ of $\mathcal{V}_{n-1}$, the infinity lemma (8.1.2), yields a sequence $(H_n^1, \dots, H_n^k)_{n \in \mathbb{N}}$ of $k$-tuples, one from each $\mathcal{V}_n$, with a limit $(H^1, \dots, H^k)$ defined by the nested unions

$$H^i := \bigcup_{n \in \mathbb{N}} H_n^i[v_0, \dots, v_n]\,.$$

These $H^i$ are edge-disjoint for distinct $i$ (because the $H_n^i$ are), but they need not be connected. To show that they have connected closures, it suffices by Lemma 8.6.7 to show that each of them has an edge in every finite cut $F$ of $G$. Given $F$, choose $n$ large enough that all the edges of $F$ lie in $G[v_0, \dots, v_n]$. Then $F$ is also a cut of $G_n$. Now consider the $k$-tuple $(H_n^1, \dots, H_n^k)$ which the infinity lemma picked from $\mathcal{V}_n$. Each of these $H_n^i$ is a connected spanning submultigraph of $G_n$, so it contains an edge from $F$. But $H_n^i$ agrees with $H^i$ on $\{v_0, \dots, v_n\}$, so $H^i$ too contains this edge from $F$. $\square$

**Lemma 8.6.11.** *Every connected standard subspace of $|G|$ that contains $V$ also contains a topological spanning tree of $G$.*

*Proof.* Let $X$ be a connected standard subspace of $|G|$ containing $V$. Then $G$ too must be connected, so it is countable. Let $e_0, e_1, \dots$ be an enumeration of $E(X)$, and consider these edges in turn. Starting with $X_0 := X$, define $X_{n+1} := X_n \setminus \mathring{e}_n$ if this keeps $X_{n+1}$ connected; if not, put $X_{n+1} := X_n$. Finally, let $T := \bigcap_{n \in \mathbb{N}} X_n$.

Since $T$ is closed and contains $V$, it is still a standard subspace. And $T$ has an edge in every finite cut of $G$, because $X$ does and its last edge in that cut will never be deleted. So $T$ is connected, by Lemma 8.6.7. But $T$ contains no circle: that would contain an edge, which should have got deleted since deleting an edge from a circle cannot destroy connectedness. $\square$

**Proof of Theorem 8.6.9.** The implication (ii)→(i) follows from our two lemmas. For (i)→(ii), let $G$ have edge-disjoint topological spanning trees $T_1, \dots, T_k$, and consider a partition $P$ of $V$ into $\ell$ sets. If there are infinitely many cross-edges, there is nothing to show; so we assume there are only finitely many. For each $i \in \{1, \dots, k\}$, let $T_i'$ be the multigraph of order $\ell$ which the edges of $T_i$ induce on $P$.

To establish that $G$ has at least $k(\ell - 1)$ cross-edges, we show that the multigraphs $T_i'$ are connected. If not, then some $T_i'$ has a vertex partition crossed by no edge of $T_i$. This partition induces a cut of $G$ that contains no edge of $T_i$. By our assumption that $G$ has only finitely many cross-edges, this cut is finite. By Lemma 8.6.7, this contradicts the connectedness of $T_i$. $\square$

## (1.9) 8.7 The topological cycle space

As a more comprehensive application of our new theory, let us now look at how the cycle space theory of finite graphs extends to locally finite graphs $G = (V, E)$ with infinite circuits and topological spanning trees.

*$G = (V, E)$*

Every two points of a topological spanning tree $T$ are joined by a unique arc in $T$: existence follows from Lemma 8.6.5, while uniqueness is proved as for finite graphs. Adding a new edge $e$ to $T$ therefore creates a unique circle in $T \cup e$; its edges form the *fundamental circuit* $C_e$ of $e$ with respect to $T$. Note that $C_e$ can be infinite.

*fundamental circuit $C_e$*

Similarly, for every edge $f \in E(T)$ the space $T \setminus \mathring{f}$ has exactly two arc-components; the set of edges between these is the *fundamental cut* $D_f$ of $T$. Since the two arc-components of $T \setminus \mathring{f}$ are closed (Lemma 8.6.6) but disjoint, Lemma 8.6.3 (ii) implies that $D_f$ is finite.

*fundamental cut $D_f$*

As in finite graphs, we have $e \in D_f$ if and only if $f \in C_e$, for all $f \in E(T)$ and $e \in E \setminus E(T)$. Topological spanning trees that are the closure of a normal spanning tree, as in Lemma 8.6.8, are particularly useful in this context: their fundamental circuits and cuts are both finite.

(8.6.8)

For locally finite graphs there will be two cycle spaces: the usual 'finitary' one from Chapter 1.9, and a new 'topological' one based on topological circuits. The former will be a subspace of the latter, much as the space of all finite cuts is a subspace of the space of all cuts. These four spaces are cross-related by matroid duality in a surprising way; see the notes and Exercise 118.

Call a family $(D_i)_{i \in I}$ of subsets of $E$ *thin* if no edge lies in $D_i$ for infinitely many $i$. Let the *thin sum* $\sum_{i \in I} D_i$ of this family be the set of all edges that lie in $D_i$ for an odd number of indices $i$. The *topological cycle space* $\mathcal{C}(G)$ of $G$ is the subspace of its edge space $\mathcal{E}(G)$ consisting of all thin sums of circuits.

*thin sum*

*topological cycle space $\mathcal{C}(G)$*

We say that a given set $\mathcal{Z}$ of circuits *generates* $\mathcal{C}(G)$ if every element of $\mathcal{C}(G)$ is a thin sum of elements of $\mathcal{Z}$. For example, the topological cycle space of the ladder in Figure 8.1.3 can be generated by all its squares (the 4-element circuits), or by the infinite circuit consisting of all horizontal edges and all squares but one. Similarly, the 'wild circuit' of Figure 8.6.1 is the thin sum of all the finite face boundaries of that graph, which thus generate it.

*generates*

Let us use $\mathcal{C}_{\text{fin}}(G)$ to denote the *finitary cycle space* of $G$ as defined in Chapter 1.9 : the (finite) sums of its finite circuits. Clearly $\mathcal{C}_{\text{fin}}(G) \subseteq \mathcal{C}(G)$. We shall see later that $\mathcal{C}_{\text{fin}}(G)$ contains all the finite elements of $\mathcal{C}(G)$, but this is not obvious from the definitions; see Exercise 115. When $G$ is finite, however, clearly $\mathcal{C}_{\text{fin}}(G) = \mathcal{C}(G)$.

*finitary cycle space $\mathcal{C}_{\text{fin}}(G)$*

As shown in Chapter 1.9, a finite set of edges of $G$ lies in $\mathcal{C}_{\text{fin}}(G)$ if and only if it meets every cut of $G$ evenly, and the fundamental circuits of any ordinary spanning tree generate $\mathcal{C}_{\text{fin}}(G)$ by finite sums: just copy the proofs given there. For $\mathcal{C}(G)$ we have the following topological analogue:

**Theorem 8.7.1.** *The following statements are equivalent for every set $D$ of edges of a locally finite connected graph $G$:*

(i) *$D \in \mathcal{C}(G)$;*

(ii) *$D$ meets every finite cut $F$ of $G$ in an even number of edges;*

(iii) *$D$ is a thin sum of fundamental circuits of any topological spanning tree of $G$.*

*Proof.* The implication (iii)→(i) holds by definition of $\mathcal{C}(G)$ and the fact that $G$ has a topological spanning tree (Lemma 8.6.11). (8.2.4) (8.6.3) (8.6.7)

Let us prove (i)→(ii). By assumption, $D$ is a thin sum of circuits. Only finitely many of these can meet $F$, so it suffices to show that every circuit meets $F$ evenly. This follows from Lemma 8.6.3 (i): given a circle $C$ in $|G|$, the segments of $C$ between its edges in $F$ (if any) are arcs whose vertices all lie on the same side of the cut $F$. These sides alternate as we follow $C$ round. Therefore, there is an even number of such arcs, and hence also of edges that $C$ has in $F$.

It remains to prove (ii)→(iii). Write $C_e$ for the fundamental circuit of an edge $e \notin E(T)$, and $D_f$ for the fundamental cut of an edge $f \in E(T)$. Recall that, by Lemma 8.6.3 (ii), these $D_f$ are finite cuts. We show that

$$D = \sum_{e \in D \setminus E(T)} C_e \,. \qquad (*)$$

This sum is well defined: since $f \in C_e \Leftrightarrow e \in D_f$ and fundamental cuts are finite, the $C_e$ in this sum form a thin family. To prove $(*)$ we show that $D' := D + \sum_{e \in D \setminus E(T)} C_e = \emptyset$.

Note first that $D' \subseteq E(T)$: any chord of $T$ that lies in $D$ also lies in exactly one of the $C_e$ in the sum. Hence any $f \in D'$ is the unique edge of $T$, and hence of $D'$, in the finite cut $D_f$, giving $|D' \cap D_f| = 1$. This is a contradiction, since $D$ meets $D_f$ evenly by (ii), and every $C_e$ does by Lemma 8.6.3. □

**Corollary 8.7.2.** *$\mathcal{C}(G)$ is generated by finite circuits.*

*Proof.* Apply Theorem 8.7.1 with the closure of a normal spanning tree, which is a topological spanning tree by Lemma 8.6.8. □ (8.2.4)

Our second aim in this section is to prove the analogue of Proposition 1.9.1 (ii) for the topological cycle space: that its elements $D$ are not only thin sums but even disjoint unions of circuits. For finite graphs, it was easy to find these circuits greedily: we would 'follow the edges of $D$ round' until a circuit was found, delete it, and repeat.

This will still be our overall strategy when $G$ is infinite. But it is no longer straightforward now to isolate a single circuit from $D$. For example, without using our knowledge that the edge set $D$ of the wild circle in the graph $G$ of Figure 8.6.1 is a circuit, we can see at once that it must lie in $\mathcal{C}(G)$: it is the thin sum of all the finite circuits bounding a face. Our proof must therefore be able to 'decompose' $D$ into disjoint circuits. Since $D$ itself is the only circuit contained in $D$, the proof thus has to reconstruct the complicated wild circle just from the information that $D \in \mathcal{C}(G)$. And it has to do so generically, without appealing to the special structure of this particular graph.

**Theorem 8.7.3.** *For every locally finite graph $G$, every element of $\mathcal{C}(G)$ is a disjoint union of circuits.*

(1.9.1)
(8.1.2)
(8.6.4)

*Proof.* We may assume that $G$ is connected, and hence countable. Let $D \in \mathcal{C}(G)$ be given, and enumerate its edges. We inductively construct a sequence of disjoint circuits $C \subseteq D$ each containing the smallest edge in our enumeration of $D$ that is not yet contained in the circuits constructed before. Then all these circuits will form the desired partition of $D$.

$D'$, $e$

Suppose we have already constructed finitely many disjoint circuits all contained in $D$. Deleting these edges from $D$ leaves a set $D'$ of edges that is again in $\mathcal{C}(G)$; let $e$ be its smallest edge in our enumeration of $D$. We shall find a topological path $\pi$ between the endvertices of $e$ in the standard subspace that $D' \smallsetminus \{e\}$ spans in $|G|$. By Lemma 8.6.4, the image of $\pi$ will contain an arc $A$ between these vertices, and $A \cup e$ will be the circle defining our next circuit.

$e = v_0v_1$

$G_n$

$v'_n$, $V_n$

Enumerate the vertices of $G$ as $v_0, v_1, \ldots$, with $e = v_0v_1$. Let $S_n := \{v_0, \ldots, v_n\}$. For each $n \geqslant 1$, let $G_n$ be the finite multigraph obtained from $G$ by contracting every component of $G - S_n$ to a vertex, deleting any loops but keeping parallel edges that arise in the contraction. Note that both $V(G_n)$ and $E(G_n)$ are finite, and that $G[S_n] \subseteq G_n$. Let $v'_n$ denote the vertex of $G_{n-1} - S_{n-1}$ whose branch set $V_n$ contains $v_n$.

We may think of $E(G_n)$ as a subset of $E(G)$. Then the cuts of $G_n$ are also cuts of $G$. By Theorem 8.7.1, $D'$ meets these evenly; in particular, every vertex of $G_n$ is incident with an even number of edges in $D'$. Hence $D' \cap E(G_n) \in \mathcal{C}(G_n)$, by Proposition 1.9.1, so $G_n$ contains a cycle through $e$ that has all its edges in $D'$. Let $P_n$ be the unique $v_0$–$v_1$ walk in this cycle that does not contain $e$ and does not repeat any vertices.

Let $\mathcal{V}_n$ be the set of all $v_0$–$v_1$ walks in $G_n - e$ in which none of the vertices $v_0, \ldots, v_n$, and hence no edge, occurs more than once. Then $P_n \in \mathcal{V}_n \neq \emptyset$, and $\mathcal{V}_n$ is finite. Every walk $W \in \mathcal{V}_n$ with $n \geqslant 2$ induces a walk $W' \in \mathcal{V}_{n-1}$ consisting of the edges that $W$ has in $G_{n-1}$, traversed in the same order and direction.[14] Thus, $W'$ arises from $W$ by replacing any subwalk of vertices and edges not in $G_{n-1}$ with $v'_n$. The vertices of any such subwalk of $W$ will be $v_n$ or vertices of $G_n - S_n$ whose branch set is contained in $V_n$. By the infinity lemma, there exists a choice of

[14] These are well defined: every edge $e \in W$ that is an edge of $G_{n-1}$ has at least one endvertex in $S_{n-1}$, which either precedes it in $W$ or follows it. In $W'$, this vertex will likewise precede or follow $e$, respectively.

walks $W_n \in \mathcal{V}_n$ such that $W'_n = W_{n-1}$ for all $n \geqslant 2$.

Our next aim is to turn these walks $W_n$ into topological paths $\pi_n\colon [0,1] \to |G_n|$ that traverse them from $v_0$ to $v_1$ and reflect their compatibility. We shall define these $\pi_n$ for $n = 1, 2, \ldots$ in turn, as follows.

For $n = 1$, note that $W_1$ has exactly two edges: at least two, because $e$ has no parallel edge, and at most two, because every edge of $G_1$ is adjacent to either $v_0$ or $v_1$. Let $\pi_1$ map $[0, \frac{1}{3}]$ onto the first edge, $[\frac{1}{3}, \frac{2}{3}]$ to the unique inner vertex of $W_1$, and $[\frac{2}{3}, 1]$ onto the second edge.

For $n \geqslant 2$, assume inductively that $\pi_{n-1}$ traverses the edges of $W_{n-1}$ in their given order and direction, and that $\pi_{n-1}$ 'pauses' at each vertex $v \in G_{n-1} - S_{n-1}$ on $W_{n-1}$ for a non-singleton closed interval $I \subseteq [0,1]$, mapping $I$ constantly to that vertex. (Thus, if $W_{n-1}$ visits $v$ five times, then $\pi_{n-1}^{-1}(v)$ is a disjoint union of five such intervals.) We start our definition of $\pi_n$ by letting $\pi_n(\lambda) := \pi_{n-1}(\lambda)$ for all $\lambda$ with $\pi_{n-1}(\lambda) \in |G_n|$.

Every other $\lambda \in [0,1]$ satisfies $\pi_{n-1}(\lambda) = v'_n$. These $\lambda$ form a disjoint union of closed intervals, one for every occurrence of $v'_n$ on $W_{n-1}$. Recall that $W_n$ arises from $W_{n-1}$ by replacing each occurrence of $v'_n$ by a subwalk of $W_n$ whose vertices are either $v_n$ or vertices of $G_n - S_n$ whose branch set is contained in $V_n$. For every occurrence of $v'_n$ on $W_{n-1}$, let $\pi_n$ on the corresponding interval $I$ with $\pi_{n-1}(I) = \{v'_n\}$ traverse this subwalk of $W_n$, once more pausing for a non-singleton interval at any vertex that this subwalk has in $G_n - S_n$.

These maps $\pi_n$ tend to a limit $\pi\colon [0,1] \to |G|$, defined as follows. Let $\lambda \in [0,1]$ be given. If $\pi_n(\lambda) \in |G|$ for some $n$, then $\pi_m(\lambda) = \pi_n(\lambda)$ for all $m > n$, and we let $\pi(\lambda) := \pi_n(\lambda)$. Otherwise $\pi_n(\lambda) \in V(G_n) \smallsetminus S_n$ for all $n$; let $U_n$ be the branch set of this vertex $u_n := \pi_n(\lambda)$ of $G_n$ in $G$. By our inductive construction of the maps $\pi_n$, we have $U_1 \supseteq U_2 \supseteq \ldots$. Since $U_n$ spans a component $C_n = C_n(\lambda)$ of $G - S_n$, we can find a ray in $G$ that has a tail in each $C_n$; let $\pi(\lambda)$ be the end $\omega$ of this ray. Note that $\omega$, and hence $\pi(\lambda)$, is well defined: every end $\omega' \neq \omega$ is separated from $\omega$ by some $S_n$, and then fails to have a ray in $C_n$.

For a proof that $\pi$ is our desired topological $v_0$–$v_1$ path in $|G|$, we need to check continuity at every $\lambda$. If $\pi(\lambda) = \pi_n(\lambda)$ for some $n$, then $\pi$ agrees with $\pi_n$ also in a small neighbourhood of $\lambda$, so this follows from the continuity of $\pi_n$. Otherwise $\pi(\lambda)$ is an end, $\omega$ say. Then $\omega$ has a neighbourhood basis in $|G|$ consisting of open sets $\hat{C}_\epsilon(S_n, \omega)$. Here $C(S_n, \omega)$ is the component $C_n(\lambda)$ defined earlier, since $\omega$ has a ray in it.

Now $\lambda$ is an inner point of an interval $I \subseteq [0,1]$ which $\pi_n$ maps to the vertex $u_n = \pi_n(\lambda)$. By construction, $\pi(I) \subseteq \overline{C_n(\lambda)} \subseteq \hat{C}_\epsilon(S_n, \omega)$, completing our continuity proof for $\pi$. □

**Corollary 8.7.4.** *$\mathcal{C}(G)$ is closed under infinite thin sums.*

*Proof.* Consider a thin sum $\sum_{i \in I} D_i$ of elements of $\mathcal{C}(G)$. By Theorem 8.7.3, each $D_i$ is a disjoint union of circuits. Together, these form a thin family, whose sum lies in $\mathcal{C}(G)$ and equals $\sum_{i \in I} D_i$. □

## 8.8 Infinite graphs as limits of finite ones

In the last section we saw how the space $|G|$, for a locally finite graph $G$, seems to appear as a 'limit' of the finite minors $G_n$ of $G$ obtained by contracting the components left on deleting its first $n$ vertices. We now make this relationship between $|G|$ and the $G_n$ more formal. Clarifying this can help a lot with transferring theorems for finite graphs to infinite ones—which, after all, is the idea behind considering $|G|$ in the first place.

*directed*

Let $(P, \leqslant)$ be a *directed* partially ordered set, one such that for all $p, q$ there exists an $r$ such that $p \leqslant r$ and $q \leqslant r$. A subset $Q \subseteq P$ is *cofinal* in $P$ if for every $p \in P$ there exists some $q \in Q$ with $p \leqslant q$.

*cofinal*

*inverse system*

*bonding maps*

For every $p \in P$ let $X_p$ be a compact Hausdorff topological space; later, these will represent finite graphs. Assume that we have continuous maps $f_{qp}\colon X_q \to X_p$ for all $q > p$, which are *compatible* in that, whenever $r > q > p$, we have $f_{qp} \circ f_{rq} = f_{rp}$. The family $\mathcal{X} = (\, X_p \mid p \in P\,)$, together with these *bonding maps* $f_{qp}$, is called an *inverse system*.

*inverse limit*

The set $X$ of all $x = (\, x_p \mid p \in P\,)$ with $x_p \in X_p$ and $f_{qp}(x_q) = x_p$ for all $p < q$ in $P$ is the *inverse limit* $X = \varprojlim \mathcal{X}$ of $\mathcal{X}$. We give it the subspace topology from the product space $\prod_{p \in P} X_p$ which, like the $X_p$, is Hausdorff and compact by Tychonoff's theorem.

The space $X = \varprojlim \mathcal{X}$ is the intersection, over all $q \in P$, of the sets $X_{<q}$ of all $(\, x_p \mid p \in P\,) \in \prod_p X_p$ that satisfy $f_{qp}(x_q) = x_p$ for all $p < q$. Using the fact that the $X_p$ are Hausdorff and the maps $f_{qp}$ are continuous, one can show that these subsets $X_{<q}$ of $\prod_p X_p$ are closed. Thus, $X = \bigcap_{q \in P} X_{<q}$ is closed in the compact space $\prod_p X_p$, and therefore compact.

As $P$ is directed, the sets $X_{<q}$ have the finite intersection property, as long as the $X_p$ are non-empty. Then $X = \bigcap_q X_{<q}$ is also non-empty:

**Lemma 8.8.1.** *$X = \varprojlim (\, X_p \mid p \in P\,)$ is a compact Hausdorff space. It is non-empty if $X_p \neq \emptyset$ for all $p \in P$.* □

$G,\ V, E, \Omega$

$P(G),\ P$

Given a graph $G = (V, E, \Omega)$, consider as $P = P(G)$ the set of all finite partitions of $V$ with only finitely many cross-edges. Letting $p \leqslant q$ whenever $q$ refines $p$ makes $P$ into a directed partially ordered set. For each $p$, let $G/p$ be the finite multigraph on $p$ whose edges are the cross-edges of $p$.[15] The vertices of $G/p$ that are non-singleton partition classes are its *dummy vertices*. The other vertices of $G/p$, those of the form $\{v\}$, we consider to be vertices of $G$ and refer to them as $v$.

$G/p$

*dummy vertices*

$X_p,\ f_{qp}$

On the compact spaces $X_p := |G/p|$ we have compatible quotient maps $f_{qp}\colon X_q \to X_p$ for $q > p$ which send the vertices of $G/q$ to the vertices of $G/p$ that contain them as subsets; which are the identity on the edges of $G/q$ that are also edges of $G/p$; and which send any other edge of $G/q$

[15] If the partition classes $U \in p$ are connected in $G$, then $G/p$ is the minor of $G$ obtained by contracting them. But we do not require them to be connected.

to the dummy vertex of $G/p$ that contains both its endvertices in $G/q$. Let

$$\|G\| := \varprojlim \,(\, X_p \mid p \in P\,), \qquad \|G\|$$

with these $f_{qp}$ as bonding maps.

**Theorem 8.8.2.** *If $G$ is locally finite and connected, then $\|G\|$ is homeomorphic to $|G|$.*

*Proof.* As $\|G\|$ is compact and $|G|$ is Hausdorff, it suffices to construct a continuous bijection $\sigma\colon \|G\| \to |G|$. Let $x = (\, x_p \mid p \in P\,) \in \|G\|$ be given.

If there exists $p \in P$ such that $x_p$ is not a dummy vertex of $G/p$, then $x_p \in |G| \setminus \Omega$ and we let $\sigma(x) := x_p$. To see that this is well defined, consider two such points $x_p$ and $x_{p'}$ and pick $q > p, p'$. Then $x_q$ is not a dummy vertex either, and $x_p = x_q = x_{p'}$ by the definition of $f_{qp}$ and $f_{qp'}$.

Suppose now that $x_p$ is a dummy vertex for every $p$. For every $n \in \mathbb{N}$ let $S_n$ be the set of the first $n$ vertices of $G$ in some fixed enumeration, and let $p_n \in P$ consist of the vertices in $S_n$ as singleton partition classes and the vertex sets of the components of $G - S_n$ as the remaining partition classes. This sequence $p_0, p_1, \dots$ is cofinal in $P$, since every $p \in P$ is refined by every $p_n$ with $n$ large enough that all the cross-edges of $p$ have their endvertices in $S_n$.

As $f_{qp}(x_q) = x_p$ whenever $p = p_m < p_n = q$, the connected vertex sets $U_n = x_{p_n}$ form a descending sequence $U_0 \supseteq U_1 \supseteq \dots$. It is straightforward to construct a ray $R$ in $G$ that has a tail in $G[U_n]$ for every $n$. Let $\omega$ be the end of $R$.

For every $p \in P$ the set $U = x_p$ contains every $U_n$ with $p < p_n$ as a subset. As the $p_n$ are cofinal in $P$, every $G[x_p]$ thus contains a tail of $R$. Conversely, for every end $\omega' \neq \omega$ there is an $n$ such that $G[U_n]$ contains no ray from $\omega'$. Thus, $\omega$ is the unique end of $G$ that has a ray in $G[x_p]$ for every $p \in P$. Let $\sigma(x) := \omega$. This completes the definition of $\sigma$.

To see that $\sigma$ is injective, consider distinct points $x, x' \in \|G\|$, differing in their components $x_p \neq x'_p$ say. If $p$ can be chosen so that one of these is not a dummy vertex of $G/p$, then clearly $\sigma(x) \neq \sigma(x')$. Otherwise $U = x_p$ and $U' = x'_p$ are disjoint vertex sets in $G$ separated by finitely many edges, and $\sigma(x)$ is an end with a ray in $G[U]$ while $\sigma(x')$ is an end with a ray in $G[U']$. Thus again, $\sigma(x) \neq \sigma(x')$.

To see that $\sigma$ is surjective, let $x \in |G|$ be given. If $x$ is not an end, choose $p(x) \in P$ so as to contain the vertex $x$, or the endvertices of the edge containing $x$, as singleton partition classes. For every $q \geqslant p(x)$ in $P$ let $x_q := x$, and for every $p' < q$ for some such $q$ let $x_{p'} := f_{qp'}(x)$. Then $(\, x_p \mid p \in P\,)$ is a well-defined point in $\|G\|$ which $\sigma$ maps to $x$.

If $x$ is an end, it has a ray in $G[x_p]$ for exactly one dummy vertex $x_p$ of $G/p$ for every $p \in P$. These satisfy $f_{qp}(x_q) = x_p$ whenever $p < q$, so $(\, x_p \mid p \in P\,)$ is a point in $\|G\|$ which $\sigma$ maps to $x$.

Let us show that $\sigma$ is continuous at every point $x = (\, x_p \mid p \in P\,)$ of $\|G\|$. If $\sigma(x)$ is not an end, there exists some $p(x) \in P$ such that $\sigma(x) = x_{p(x)}$, which is a point in $X_{p(x)}$ but not a dummy vertex. Then every basic open neighbourhood $O$ of $\sigma(x)$ in $|G|$ is also a basic neighbourhood of this same point $x_{p(x)}$ in $X_{p(x)}$. Then the set $\prod_{p \in P} O_p$ with $O_{p(x)} = O$ and $O_p = X_p$ for all $p \neq p(x)$ is a basic open neighbourhood of $x$ in $\prod_p X_p$. Its intersection with $\|G\|$ is an open neighbourhood of $x$ in $\|G\|$ which $\sigma$ maps to $O$.

If $\sigma(x)$ is an end, $\omega$ say, consider any basic open neighbourhood $O = \hat{C}_\epsilon(S, \omega)$ of $\omega$ in $|G|$. Let $p(\omega) \in P$ be the partition of $V$ into the vertex sets of the components of $G - S$ and the singletons in $S$. Then $V(C)$ is a dummy vertex of $G/p(\omega)$; call it $x_{p(\omega)}$. Let $O_{p(\omega)} \subseteq X_{p(\omega)}$ consist of $x_{p(\omega)}$ and the inner points in $O$ of any $C$–$S$ edges; these are also points of $X_{p(\omega)}$. As earlier, $x$ has a basic open neighbourhood $\prod_p O_p$ in $\prod_p X_p$ with $O_p = X_p$ for all $p \neq p(\omega)$, whose intersection with $\|G\|$ maps to $O$ under $\sigma$. $\square$

Note that our proof did not use that $|G|$ is compact: we reobtain Proposition 8.6.1 as a corollary.

In the proof of Theorem 8.8.2 we found it convenient to work with a cofinal sequence in $P$ instead of the entire set $P$. This is justified more generally by the following easy lemma:

**Lemma 8.8.3.** *Let $(\, X_p \mid p \in P\,)$ be an inverse system of compact spaces, let $Q \subseteq P$ be cofinal in $P$, and consider $(\, X_p \mid p \in Q\,)$ with the same bonding maps. Mapping every point $(\, x_p \mid p \in P\,)$ to its restriction $(\, x_p \mid p \in Q\,)$ then defines a homeomorphism from $\varprojlim\,(\, X_p \mid p \in P\,)$ to $\varprojlim\,(\, X_p \mid p \in Q\,)$.* $\square$

By Theorem 8.8.2 and this lemma, our familiar $|G|$ for locally finite $G$ is the inverse limit of the finite contraction minors $G_n$ of $G$ defined as in Section 8.6. Indeed, for the cofinal sequence $p_0, p_1, \ldots$ in $P$ defined in the proof of the theorem, we have $G_n = G/p_n$, and by the lemma $|G|$ is the inverse limit of the corresponding compact spaces $X_{p_n}$.

Just like $|G|$ itself, every standard subspace $X'$ of $X = |G|$ can be obtained as an inverse limit of finite multigraphs. Indeed, the projections $f_p \colon X \to X_p$ defined by $(\, x_p \mid p \in P\,) \mapsto x_p$ are continuous, so their images $X'_p \subseteq X_p$ of $X'$ are compact since $X'$ is, and the $f_{qp}$ send $X'_q$ to $X'_p$. Thus, $(\, X'_p \mid p \in P\,)$ is an inverse system with bonding maps $f'_{qp} := f_{qp} \restriction X'_q$, and $X' = \varprojlim\,(\, X'_p \mid p \in P\,)$.

More typically, we would like to *find* a standard subspace $X'$ with certain desired properties—for example, a topological spanning tree. We can then try to construct some $X'_p$ whose inverse limit is $X'$. It may not be straightforward, however, to find such compatible $X'_p$ for all $p \in P$. Here, Lemma 8.8.3 can help: it is only necessary to find them for all

$p$ in some cofinal $Q \subseteq P$. For example, we can construct spanning trees inductively in all the $G_n$ by expanding a dummy vertex in the tree $T_n \subseteq G_n$ to a star in $T_{n+1} \subseteq G_{n+1}$. Then our given bonding maps $X_{p_n} \to X_{p_m}$ will map the subspace $X'_{p_n}$ induced by $T_n$ to that induced by $T_m$, and these $X'_{p_n}$ will have a topological spanning tree in $X = |G|$ as their inverse limit. This construction is possible only because the partition classes of the $p_n$ are connected in $G$; we could not perform it on all of $P(G)$.

Arcs and circles in $|G|$, or in a standard subspace, can be obtained easily by applying the following *lifting lemma* with $Y = [0,1]$ or $Y = S^1$. Let $(\, X_p \mid p \in P\,)$ be any inverse system of compact spaces, with bonding maps $f_{qp}$ say, and let $X$ be its inverse limit. Let $Y$ be a topological space with continuous *compatible* maps $g_p\colon Y \to X_p$: maps that commute with the $f_{qp}$ in that $g_p = f_{qp} \circ g_q$ whenever $p < q$. Let us call the family $(\, g_p \mid p \in P\,)$ *eventually injective* if for all distinct $y, y' \in Y$ there exists some $p \in P$ with $g_p(y) \neq g_p(y')$.

*compatible maps*

*lifting lemma*

**Lemma 8.8.4.** *There is a unique continuous map $g\colon Y \to X$ that commutes with the projections $f_p\colon X \to X_p$ in that $g_p = f_p \circ g$ for all $p \in P$. If the $g_p$ are eventually injective, then $g$ is injective.* □

For example, suppose we wish to find an arc in $X$ between some points $x$ and $y$. We can find a topological $x$–$y$ path $g\colon [0,1] \to X$ by finding topological $f_p(x)$–$f_p(y)$ paths $g_p\colon [0,1] \to X_p$ that commute with the $f_{qp}$. If we can make these $g_p$ eventually injective, then $g$ will be injective, and its image will be the desired arc.

Similarly, if we can find compatible circles $g_p\colon S^1 \to X_p$ that are eventually injective, whose images contain all the vertices of $G/p$, and which commute with the $f_{qp}$, then $g$ will define a *Hamilton circle* of $G$, a circle in $|G|$ that traverses every vertex.

*Hamilton circle*

## Exercises

1.$^-$ Show that a connected graph is countable if all its vertices have countable degrees.

2.$^-$ Given countably many sequences $\sigma^i = s_1^i, s_2^i, \dots$ $(i \in \mathbb{N})$ of natural numbers, find one sequence $\sigma = s_1, s_2, \dots$ that beats every $\sigma^i$ eventually, i.e. such that for every $i$ there exists an $n(i)$ such that $s_n > s_n^i$ for all $n \geqslant n(i)$.

3.$^-$ Can a countable set have uncountably many subsets whose intersections have finitely bounded size?

4.$^-$ Let $T$ be an infinite rooted tree. Show that every ray in $T$ has an increasing tail, that is, a tail whose sequence of vertices increases in the tree-order associated with $T$ and its root.