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Topological groups and infinite graphs

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Abstract

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We show how results concerning infinite, locally finite, vertex-symmetric graphs can be related with the structure theory of topological groups, when the latter is applied to automorphism groups of the graphs. In particular, we discuss polynomial growth, bounded automorphisms and infinite expanders. In an appendix, we present three problems on infinite graphs, not necessarily linked with topological considerations.

1. Introduction

The automorphism group of an infinite, locally finite graph is a topological group with the topology of pointwise convergence. In this note we show how the structure theory of topological groups can be applied to prove results concerning vertex-symmetric graphs. In particular, we discuss a series of results due to Trofimov [19–21].

After presenting preliminaries and basic facts (Section 2), we show in Section 3 how Trofimov's theorem on graphs with polynomial growth [20] can be easily proved by combining Gromov's famous theorem on discrete groups with polynomial growth [6] with a result of Losert [11] concerning topological groups with this property. In Section 4, we discuss the connection of another theorem of Trofimov [19], concerning lattice-type graphs, with the theory of FC^--groups of Grosser and Moskowitz [7]. However, deriving Trofimov's characterization of lattices directly from the results of [7] includes a difficulty which seems hard to overcome. On the other hand, one obtains from [19] that the group of bounded automorphisms of a vertex-symmetric graph is closed [21] and hence an FC^--group. Applying [7] we get that the bounded automorphisms of finite order constitute a closed group in the topology of pointwise convergence: this completes partial results by Godsil et al. [5].

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Finally in Section 5, we report a result of Soardi and Woess [16] which characterizes infinite vertex-symmetric expanders in terms of properties of the automorphism group, viewed as a topological group. The properties in question are amenability and unimodularity, and the result can be applied to show that every vertex-symmetric graph with infinitely many ends is an expander.

It should be pointed out that the spirit of this note is mainly that of an introductory survey with the aim to illustrate the use of topological groups in the study of infinite graphs. As this paper is principally addressed to readers working in the field of discrete mathematics, some space is given to the explanation of the relevant topological prerequisites.

2. Basic facts

In the sequel, $\Gamma(X, E)$ will always denote an infinite, connected, locally finite graph with vertex set X and (unoriented) edge set E. The *automorphism group* $\mathrm{AUT}(\Gamma)$ of Γ is the group of isometries of the vertex set X with respect to the discrete metric d: $d(x, y)$ is the smallest number of edges on a path in Γ connecting x and y. We shall always assume that Γ is *vertex-symmetric,* i.e., $\mathrm{AUT}(\Gamma)$ acts transitively on the vertex set.

We introduce *pointwise convergence* of a sequence (g_n) in $\mathrm{AUT}(\Gamma)$:

$$g_n \to g \in \mathrm{AUT}(\Gamma), \text{ if for every } x \in X,\ g_n x = gx \text{ for all } n \geqslant n_x.$$

Recall that a *topological group* is a group G equipped with a topology such that the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous on $G \times G$ and on G, respectively. A good introduction to the fundamentals of topological groups is given, for example, by Hewitt and Ross [10]. The topology of a topological group is completely determined by a neighbourhood base at the identity, see [10, (4.5)]. For the topology of pointwise convergence in $G = \mathrm{AUT}(\Gamma)$, a neighbourhood base at the identity is given by the family of pointwise stabilizers of finite subsets of X. We write G_x for the stabilizer of x. In other words, if we fix a reference vertex o and, for every $x \in X$, an automorphism g_x such that $g_x o = x$ (here we use transitivity!), then the family of subgroups

$$g_x \,\mathrm{AUT}(\Gamma)_o\, g_y^{-1}, \quad x, y \in X$$

is a subbasis of the topology. Thus, $\mathrm{AUT}(\Gamma)$ is a Hausdorff group with countable base, and it is locally compact, as the following well-known lemma shows.

Lemma 1. *The stabilizer* $\mathrm{AUT}(\Gamma)_x$ *is compact.*

Proof. Let (g_n) be a sequence in $\mathrm{AUT}(\Gamma)_x$, and let $\{x_0 = x, x_1, x_2, \ldots\}$ be an enumeration of X. As $g_n x = x$ for every n, and as Γ is locally finite and connected, the set $\{g_n x_k \mid n \geqslant 0\}$ is finite for every k. Hence there is a subsequence $(\tau_1(n))$ of (n) such that all $g_{\tau_1(n)}x_1$ coincide; write gx_1 for this common image. Repeating this argument inductively, we get a sub-subsequence $(\tau_k(n))$ of the preceding subsequence $(\tau_{k-1}(n))$, such that all $g_{\tau_k(n)}$, $n \geqslant 0$, send x_k to the same element of X, denoted gx_k. Thus, $g_{\tau_n(n)} \to g \in \mathrm{AUT}(\Gamma)$ pointwise. $\square$

We now see that the identity has a neighbourhood base consisting of compact-open subgroups, so that $\mathrm{AUT}(\Gamma)$ is totally disconnected. Next, we describe compactness in $\mathrm{AUT}(\Gamma)$.

Lemma 2. *A subset U of $\mathrm{AUT}(\Gamma)$ has compact closure if and only if the orbit Ux is finite for every $x \in X$.*

Proof. If U is relatively compact then it is contained in a finite union of sets $g_i \,\mathrm{AUT}(\Gamma)_x$, $i = 1, \ldots, r$. Hence, $Ux \subset \{g_i x \mid i = 1, \ldots, r\}$.

Conversely, if $Ux = \{y_1, \ldots, y_r\}$, then there are $g_i \in U$ such that $g_i x = y_i$, $i = 1, \ldots, r$. But then

$$U \subset \bigcup_{i=1}^{r} g_i \,\mathrm{AUT}(\Gamma)_x.$$

The latter set is compact by Lemma 1, so that A has compact closure. $\square$

Indeed, by local finiteness, in Lemma 2 it is enough to check finiteness of Ux for one $x \in X$. Now let G be a closed subgroup of $\mathrm{AUT}(\Gamma)$ acting transitively on X. In the relative topology, G inherits all properties of $\mathrm{AUT}(\Gamma)$ discussed so far. Choose a reference vertex $o \in X$, and define

$$V = \{g \in G \mid d(go, o) \leqslant 1\}.$$

For the following lemma, recall that $V^n = \{g_1 g_2 \cdots g_n \mid g_i \in V\}$.

Lemma 3. *V is a compact, symmetric neighbourhood of the identity in G. If $g \in G$ and $n \geqslant 0$, then $g \in V^n$ if and only if $d(go, o) \leqslant n$.*

Proof. Clearly, $V = V^{-1}$. For every $x \in X$ with $d(x, o) \leqslant 1$ choose $g_x \in G$ with $g_x o = x$. Then

$$V = \bigcup_{d(x,o) \leqslant 1} g_x G_o,$$

and V is compact by local finiteness and Lemma 2. By definition of the topology, V is also open.

The last statement is true for $n = 1$ by definition of V. Suppose it is true for n. Observe that $V^n \subset V^{n+1}$. Let $g \in G$, $go = y$ with $d(y, o) = n + 1$. Then there is a neighbour w of y such that $d(w, o) = n$. By transitivity, $ho = w$ for some $h \in G$, and by the induction hypothesis, $h \in V^n$. Now, $d(h^{-1}go, o) = d(w, y) = 1$, so that $h^{-1}g \in V$ and $g \in hV \subset V^{n+1}$. Conversely, if $g \in V_{n+1}$, then obviously $d(go, o) \leqslant n+1$. □

3. Graphs with polynomial growth

A famous theorem of Gromov [6] (in combination with a result of Wolf [27]) states that a finitely generated discrete group has polynomial growth if and only if it has a nilpotent subgroup of finite index. If one views a finitely generated group in terms of its Cayley graph with respect to some finite symmetric set of generators, then the question naturally arises if and how Gromov's Theorem generalizes to an arbitrary locally finite, vertex-symmetric graph Γ with polynomial growth. Recall that Γ is said to have *polynomial growth* if

$$\beta(n) = |\{x \in X \mid d(x, o) \leqslant n\}| \leqslant Cn^d$$

for some finite C, $d > 0$. The answer to the above question is as follows.

Theorem 1 [20, Theorem 2]. *Let $\Gamma(X, E)$ have polynomial growth, and let G be a group of automorphisms of Γ which acts transitively on X. Then there is an imprimitivity system σ of G on X with finite blocks, such that G^σ is a finitely generated nilpotent-by-finite group with finite vertex stabilizers on Γ^σ.*

For the understanding of this remarkable theorem, recall that an *imprimitivity system* of G on X is a partition of X into subsets called *blocks*, which is preserved by the action of G. By x^σ we denote the block of $x \in X$. The factor graph Γ^σ has vertex set $X^\sigma = \{x^\sigma \mid x \in X\}$ and edge set $E^\sigma = \{[x^\sigma, y^\sigma] \mid [x, y] \in E\}$. In other words, two blocks are adjacent in the factor graph if and only if they have some pair of representatives which are adjacent in Γ. The induced action of G gives rise to a homomorphic image G^σ in AUT(Γ^σ), which acts transitively on X^σ.

For a detailed survey on graphs with polynomial growth, see the article by Imrich and Seifter in this volume.

The purpose of this section is to provide a new, short proof of Theorem 1, using a result of Losert [11] concerning *topological* groups with polynomial growth. To do so, we need more preliminaries.

If G is any locally compact, Hausdorff topological group, then it carries a *left Haar measure.* This is a sigma-additive (Radon) measure λ defined on the Borel sigma-algebra of G (*i.e.*, the sigma-algebra generated by the open sets), whose important features are the following:

(1) $\lambda(K) < \infty$ if $K \subset G$ is compact,
(2) $\lambda(U) > 0$ if $U \subset G$ is open,
(3) $\lambda(gU) = \lambda(U)$ for every Borel set $U \subset G$ and every $g \in G$, and
(4) $\lambda(G) < \infty$ if and only if G is compact.

Up to multiplication with a positive constant, λ is unique. Once more, a good introduction and all relevant properties of Haar measure can be found in [10].

A *compactly generated* group G is said to have polynomial growth, if for some (equivalently, every) compact symmetric neighbourhood V of the identity,

$$\lambda(V^n) \leq Cn^d.$$

In the special case when G is a finitely generated discrete group, λ is the counting measure, and if V is chosen to be a finite symmetric set of generators containing the identity, then $\lambda(V^n)$ is the size of the n-ball in the corresponding Cayley graph. Thus, the above definition is the direct extension to topological groups of the notion of polynomial growth. If G is the abelian group $\mathbf{R}^d$, then λ is the Lebesgue measure in the corresponding dimension, and the growth of G is polynomial with degree d.

In a significant extension of Gromov's Theorem, Losert [11] describes completely the structure of topological groups with polynomial growth. As an initial step, the following important proposition is proved.

Proposition 1 [11]. *If G is a locally compact Hausdorff group with polynomial growth, then there is a compact normal subgroup K of G such that G/K is a Lie group.*

The topology on G/K is of course the one induced by the natural projection. As K is compact, G/K has the same polynomial growth as G. The *Lie groups* are a very important class of topological groups. In this paper, all we need to know is that every Lie group is locally homeomorphic with d-dimensional Euclidean space, $d \geq 0$. We emphasize that in Proposition 1 the dimension d may be zero; the corresponding 0-dimensional space has one point, so that G is discrete in this case. For the fundamentals concerning Lie groups, see e.g. Varadarajan [22].

We can use Proposition 1 to give a short and transparent proof of Theorem 1.

Proof of Theorem 1. If the Theorem is true for the closure of G in $\mathrm{AUT}(\Gamma)$ then it is true for G. Hence, we may assume that G is closed and thus locally compact. As the stabilizer G_o of our reference vertex o is open-compact, we may normalize the Haar measure of G such that $\lambda(G_o) = 1$.

Let V be as in Lemma 3. Once more, for every $x \in X$ choose $g_x \in G$ with $g_x o = x$. Then, by Lemma 3,

$$V^n = \bigcup_{d(x,o) \leq n} g_x G_o,$$

a disjoint union, and left invariance of λ yields $\lambda(V^n) = \beta(n)$ for the growth coefficients $\beta(n)$ of Γ. Thus, the group G has polynomial growth. By Proposition 1, G/K is a Lie group for some compact normal subgroup K of G. By Lemma 2, the imprimitivity system

$$\sigma = \{Kx \mid x \in X\}$$

has finite blocks, and $G^\sigma = G/K$ is a closed vertex-transitive subgroup of $\mathrm{AUT}(\Gamma^\sigma)$. Hence, in view of the properties of the automorphism group described in Section 2, G^σ is a compactly generated, totally disconnected Lie group: it must be zero-dimensional. In other words, G^σ is a finitely generated, discrete group with polynomial growth. By [6], it is nilpotent-by-finite.

Furthermore, by Lemma 2, $H = \{g \in G \mid go \in Ko\}$ is a compact subgroup of G. The stabilizer of o^σ is $H^\sigma = H/K$. This is a compact subgroup of G^σ and as such must be finite. $\square$

At this point we remark that Trofimov did not have the result of [11] (which was published later) at his disposal. We also remark that Trofimov uses the above Theorem 1 to deduce a slightly stronger result [20, Theorem 1]: working with $G = \mathrm{AUT}(\Gamma)$, the imprimitivity system σ can be constructed such that not only $\mathrm{AUT}(\Gamma)^\sigma$, but even the—possibly larger—group $\mathrm{AUT}(\Gamma^\sigma)$ is discrete, nilpotent-by-finite and has finite vertex stabilizers on Γ^σ. An important step in this deduction is [20, Proposition 2.3], see below.

4. Bounded automorphisms and FC$^-$-groups

An automorphism g of Γ is called *bounded*, if there is a constant $M = M(g) < \infty$ such that

$$d(gx, x) \leqslant M \quad \text{for every } x \in X.$$

It is obvious that the bounded automorphisms constitute a normal subgroup of $\mathrm{AUT}(\Gamma)$, denoted by $B(\Gamma)$.

An element g in a topological group G is called an FC$^-$-element, if its conjugacy class has compact closure: $\{g^h \mid h \in G\}^-$ is compact. (For a set A, A^- denotes its closure.) Recall that $g^h = h^{-1}gh$. An FC$^-$-*group* is a group consisting of FC$^-$-elements only.

Lemma 4. *$g \in B(\Gamma)$ if and only if g is an* FC$^-$*-element of* $\mathrm{AUT}(\Gamma)$.

Proof. Let $g \in B(\Gamma)$, and let $h \in \mathrm{AUT}(\Gamma)$. Then $d(g^h x, x) = d(ghx, hx) \leqslant M$ for every $x \in X$. Thus,

$$\{g^h x \mid h \in \mathrm{AUT}(\Gamma)\} \subset \{y \in X \mid d(y, x) \leqslant M\}.$$

By local finiteness, the latter set is finite, so that the conjugate class of g in $\mathrm{AUT}(\Gamma)$ is compact by Lemma 2.

Conversely, suppose that $\{g^h \mid h \in \mathrm{AUT}(\Gamma)\}$ has compact closure. By Lemma 2, $\{g^h o \mid h \in \mathrm{AUT}(\Gamma)\}$ is finite. Hence, there is $M \leqslant \infty$ such that $d(g^h o, o) \leqslant M$

for every $h \in \mathrm{AUT}(\Gamma)$. If $x \in X$, choose $h \in \mathrm{AUT}(\Gamma)$ with $ho = x$. Then

$$d(gx, x) = d(gho, ho) \leqslant M. \qquad \square$$

In his remarkable series of papers, Trofimov obtains the following two results.

Theorem 2 [19]. *$B(\Gamma)$ acts transitively on X if and only if there is an imprimitivity system σ of $B(\Gamma)$ on X, with finite blocks, such that $B(\Gamma)^\sigma$ is a free finitely generated abelian group.*

Theorem 3 [21]. *If Γ is any vertex-symmetric graph, then $B(\Gamma)$ is a closed subgroup of* $\mathrm{AUT}(\Gamma)$.

We remark that in Theorem 2, the factor graph Γ^σ must be a Cayley graph of $B(\Gamma)^\sigma$, which is isomorphic with $\boldsymbol{Z}^d$ for some $d \geqslant 1$.

One would hope that Theorem 2 could be deduced from Theorem 3 in a similar way as Theorem 1 was deduced from Proposition 1: if $B(\Gamma)$ acts transitively on X and is closed, then by Lemmas 3 and 4 it is a locally compact FC^--group; the structure theory of Grosser and Moskowitz [7, (3.13) and (3.17)] yields existence of a compact normal subgroup K of $B(\Gamma)$ such that $B(\Gamma)/K$ is discrete, torsionfree and abelian; as $B(\Gamma)$ is compactly generated, $B(\Gamma)/K$ is finitely generated, and the imprimitivity system σ can be chosen to be the one induced by K as in the proof of Theorem 1.

However, we cannot apply this simple argument, because Trofimov *uses* Theorem 2 in order to prove Theorem 3. (Indeed, I have spent—in vain—quite some time trying to find a more simple and direct proof of Theorem 3 which does not make use of Theorem 2.) In general, it is not true that the normal subgroup of FC^--elements in a locally compact group is closed.

Theorem 2 (whose proof is long and complicated, we refer to [19]) is used to deduce the following proposition, which is also of interest in itself and a key tool in deriving the stronger version of Theorem 1 mentioned at the end of Section 3.

Proposition 2 [20, Proposition 2.3]. *Let Γ be any vertex-symmetric graph. Then there is an imprimitivity system σ of* $\mathrm{AUT}(\Gamma)$ *with finite blocks, such that the stabilizer of a vertex of Γ^σ in* $\mathrm{AUT}(\Gamma^\sigma)$ *contains no bounded automorphism different from the identity.*

Thus, if $g \in B(\Gamma)$ stabilizes some $x \in X$, then it permutes each block of σ. In particular, g has finite order.

We now give an application of Theorem 3. Motivated by a problem raised by Watkins for strips (vertex-symmetric graphs with linear growth), Godsil et al. [5] show that the set $B_0(\Gamma)$ of bounded automorphisms with finite order is a *subgroup* of $B(\Gamma)$ if Γ is a vertex-symmetric graph with polynomial growth, while

$B_0(\Gamma) = B(\Gamma)$ if Γ has infinitely many ends. As a byproduct of Theorem 3, one obtains the following completion of this result.

Corollary 1. *Let Γ be a vertex-symmetric graph. Then $B_0(\Gamma)$ is a closed normal subgroup of* AUT(Γ), *and $B(\Gamma)/B_0(\Gamma)$ is a torsion free discrete abelian group.*

Proof. By Theorem 3 and Lemma 3, $B(\Gamma)$ is a locally compact FC$^-$-group. Thus, [7, 3.13] applies and yields the result. $\square$

We remark that $B(\Gamma)$ need not be compactly generated (if it does not act vertex-transitively), so that a priori $B(\Gamma)/B_0(\Gamma)$ is not necessarily finitely generated. It would be interesting to know whether this may really happen.

5. Expanders, amenability and unimodularity

In this final section, we present without proofs another result which links a structural property of vertex-symmetric graphs with their automorphism groups, seen as topological groups.

If F is a finite subset of the vertex set X of Γ, then ∂F denotes the set of vertices in F which have a neighbour in $X \setminus F$. The graph Γ is called an *infinite expander* if there is a number $\kappa > 0$ such that

$$|\partial F| \geqslant \kappa \cdot |F| \quad \text{for every finite } F \subset X.$$

See, for example, Bien [1], Dodzuik [2], and Gerl [4] for various reasons why expanders are of interest. A theorem of Soardi and Woess [16] characterizes infinite vertex-symmetric (non-)expanders in terms of their automorphism groups.

A topological group G is called *amenable*, if there is a nonnegative measure μ, defined on the Borel sets of G, with the following properties:

(1) $\mu(G) = 1$,
(2) μ is finitely additive, and
(3) $\mu(gU) = \mu(U)$ for every Borel set $U \subset G$ and every $g \in G$.

If G is compact, then we may take the Haar measure, but in general, not every group is amenable. Examples of amenable groups are abelian and solvable groups. Examples of (discrete) non-amenable groups are free groups with at least two free generators. The class of amenable groups is of interest in many respects, see the books by Pier [13] and Wagon [23].

Next, we turn to the Haar measure: the modular function Δ on a locally compact group G is defined by

$$\Delta(g) = \frac{\lambda(Ug)}{\lambda(U)}, \quad g \in G,$$

where $U \subset G$ is open with compact closure (Δ is independent of such U). A group is called *unimodular* if $\Delta \equiv 1$. Discrete and abelian groups are unimodular, but there are non-unimodular solvable groups. For details, see e.g. [10].

Now consider a graph Γ, and let G be a closed subgroup of $\mathrm{AUT}(\Gamma)$ which acts transitively on X. Then the modular function of G can be easily determined.

Lemma 5 ([14, 21]). *If $g \in G$ and $go = x$ then $\Delta(g) = |G_o x| / |G_x o|$.*

Non-expanders can be characterized as follows.

Theorem 4 [16]. *Let Γ be a vertex-symmetric graph. Then the following statements are equivalent*:

(a) *Γ is a non-expander.*

(b) *Some closed, vertex-transitive subgroup of* $\mathrm{AUT}(\Gamma)$ *is amenable and unimodular.*

(c) *Every closed, vertex-transitive subgroup of* $\mathrm{AUT}(\Gamma)$ *is amenable and unimodular.*

To conclude, we point out that this theorem is applied in [16] to prove that every vertex-symmetric graph with infinitely many ends is an infinite expander. For the definition and basic features of the space of ends, see Freudenthal [3], Halin [8] or Woess [25]. In [25], the relation between amenability of a closed group G of automorphisms of an arbitrary locally finite graph Γ and the action of G on the space of ends of Γ is described in full detail. This completes results by Tits [17] and Nebbia [12] for trees and by Seifter [15] for arbitrary graphs and nilpotent groups. Based on [25], the following is proved in [16].

Proposition 3 [16]. *Assume that Γ has infinitely many ends and that G is a closed, vertex-transitive group of automorphisms of Γ. Then G is amenable if and only if it fixes an end of Γ. In this case, G is non-unimodular.*

Appendix: Three problems on infinite graphs

The three problems presented here concern locally finite, infinite, connected, vertex-symmetric graphs.

First problem

"Are there any vertex-symmetric graphs which do not look like Cayley graphs?"

First of all, let us recall the definition of a *Cayley graph*. If G is a finitely generated group and A is a finite symmetric set of generators, then the Cayley graph of G with respect to A has vertex set G; $[g, h]$ is an (unoriented) edge if and only if $h = ga$ for some $a \in A$. By left multiplication, G acts on its Cayley graph(s) as a vertex-transitive group of graph automorphisms.

The crucial point in the above question is the definition of 'look like'. Let $\Gamma_1(X_1, E_1)$ and $\Gamma_2(X_2, E_2)$ be connected, locally finite graphs, and let d_i be the discrete metric on X_i induced by the respective graph structure, $i = 1, 2$. We say that Γ_1 and Γ_2 are *quasi-isometric* if the following holds: there are maps $\varphi: X_1 \to X_2$ and $\psi: X_2 \to X_1$ and constants $C, D > 0$ such that for every $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$ one has:

(a) $d_2(\varphi x_1, \varphi y_1) \leq C d_1(x_1, y_1) + D$,

(b) $d_1(\psi x_2, \psi y_2) \leq C d_2(x_2, y_2) + D$,

(c) $d_1(\psi\varphi x_1, x_1) \leq C$, and

(d) $d_2(\varphi\psi x_2, x_2) \leq C$.

Conditions (a) and (b) say that φ and ψ are 'quasi Lipschitz', while (c) and (d) say that they are 'quasi inverse' to each other. Thus, the two graphs are metrically equivalent up to bounded deviation. Now we can formulate our problem more precisely.

Problem 1. Is there a vertex-symmetric graph which is not quasi-isometric with some Cayley graph?

We remark that by [20], every vertex-symmetric graph with polynomial growth is quasi-isometric with the Cayley graph of a nilpotent-by-finite group in a rather strong sense, see Theorem 1 above. Furthermore, the non-Cayley graphs exhibited in [18] and [16] are quasi-isometric with homogeneous trees.

Second problem

This and the third problem regard vertex-symmetric graphs with infinitely many ends. We recall the definition of the *space of ends* of an infinite graph Γ: an infinite path in Γ is a sequence $\pi = [x_0, x_1, x_2, \ldots]$ of successively adjacent vertices in Γ without repetitions. Two infinite paths are equivalent if there is a third one which meets each of the two infinitely often. An end ω is an equivalence class of infinite paths under this relation. For more details, see [3, 8]. Let Ω denote the space of ends. Every automorphism (isometry) of Γ acts on Ω in an obvious way. If Γ is vertex-symmetric, then it is easy to see that Γ has one, two or infinitely many ends [3]. From the structural viewpoint, the first case is the most difficult and the second the simplest one. We are interested in the third.

Problem 2. Classify all vertex-symmetric graphs Γ with infinitely many ends, for which the automorphism group $\mathrm{AUT}(\Gamma)$ acts transitively on the space of ends Ω.

Typical examples are homogeneous trees and more generally, infinite distance-regular graphs. If Γ is a graph where $\mathrm{AUT}(\Gamma)$ acts transitively on Ω, then this is also true for Γ^k (the graph with the same vertex set, where two vertices are connected by an edge if their distance in Γ is bounded by k) and for $\Gamma \times \Gamma_1$ (Cartesian product), where Γ_1 is a finite, vertex-symmetric graph. In particular, all ends of Γ must have the same finite size (maximal number of disjoint equivalent paths, see [9]) and diameter (see [25] for the definition), so that Γ is quasi-isometric with a tree by [24]. Of course there are many examples of vertex-symmetric graphs whose automorphism group does not act transitively on Ω (for example, take the Cayley graph of the free product of two one-ended groups).

Third problem

The homogeneous tree T with degree at least three has another interesting property. Let ω_0 be an end of T and let G be the group of automorphisms of T which fix ω_0. Then G acts transitively on the vertex set.

In general, if Γ has infinitely many ends and $G \leqslant \mathrm{AUT}(\Gamma)$ acts transitively on the vertex set, then it is not hard to see that either $|G\omega| = \infty$ for every $\omega \in \Omega$ or G fixes an end ω_0, and $|G\omega| = \infty$ for every other end ω. The second case is quite particular and of interest in various contexts, see e.g. [12, 24, 25].

Problem 3. Classify all vertex-symmetric graphs Γ with infinitely many ends with the following property: there is a vertex-transitive group of automorphisms of Γ (not necessarily the whole automorphism group) which fixes an end of Γ.

Once more, every infinite distance-regular graph has this property. If Γ admits a vertex-transitive group of automorphisms which fixes an end, then so do Γ^k ($k \geqslant 1$) and $\Gamma \times \Gamma_1$, where Γ_1 is finite and vertex-symmetric. My impression is that one cannot get far beyond these possibilities. In particular, I conjecture that every such graph is quasi-isometric with a tree.

Notes added in proof

(1) G. Schlichting (München) has told me that he has found a direct 'topological' proof of Theorem 3, as discussed in Section 4.

(2) V.I. Trofimov (Sverdlovsk) has informed me that group theorists have well-known examples of discrete groups (Cayley graphs) with $B(\Gamma)/B_0(\Gamma)$ not finitely generated.

(3) Problem 2 has been solved recently and independently by R. Möller (Oxford) and A. Nevo (Jerusalem).

(4) More recently, R. Möller has also given a solution of Problem 3.

(5) Problem 1 remains open and seems to be difficult.

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