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Edge-transitive strips

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Abstract

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A *strip* may be characterized as an infinite locally finite connected graph admitting an automorphism with finitely many orbits. First we show that all edge-transitive strips have even valence(s). We then characterize the edge-transitive planar strips as follows. For each integer $k \geqslant 2$, there exists a unique flag-transitive example with connectivity k; it is 4-valent and 4-covalent. By introducing multiple edges to these graphs or to the 2-way infinite path and then subdividing each edge, we construct all examples that are not vertex-transitive.

1. Introduction

Highly symmetric infinite graphs of bounded valence, that is to say, locally finite graphs with a small number of orbits, such as those that are vertex- or edge-transitive, fall into classes according to their having one, two, or uncountably many ends. A now classical result due to Erdős [1] states that a countably infinite graph Γ is planar if and only if every finite subgraph is planar. Thus Γ is planar if and only if no finite subgraph is a subdivision of one of the two forbidden Kuratowski graphs K_5 and $K_{3,3}$.

Grünbaum and Shephard have studied extensively those edge-transitive infinite graphs that admit an imbedding in the plane with a unique limit point at infinity (see, for example, [4]). Their examples generally are 1-ended. J.E. Graver and the author are currently investigating infinitely-ended, vertex-transitive planar graphs (of finite valence). Some of their examples are edge-transitive, too. The latter results will be published elsewhere.

The present paper is concerned with graphs possessing exactly two ends. H.A. Jung and the author have given the name 'strip' to such a graph when it possesses finitely many orbits and have given various characterizations of strips [9]. The attention in this paper will be upon edge-transitive strips. In Section 3 it will be shown that edge-transitive strips have only even valence(s). In the final section we turn to edge-transitive planar strips and give a detailed description of all of them.

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In particular, for every integer $b \geqslant 3$, there is a unique triconnected, edge-transitive planar strip Σ with $\kappa_\infty(\Sigma) = b$.

2. Preliminaries

Capital Greek letters will be reserved for graphs and their subgraphs. The graphs we consider may be finite or infinite, but all infinite graphs will be presumed to be *locally finite*, that is, they are infinite and every vertex has finite valence. The symbols $V(\Gamma)$, $E(\Gamma)$, $G(\Gamma)$, $\rho(\Gamma)$, $\rho_{\min}(\Gamma)$ and $\kappa(\Gamma)$ will denote, respectively, the vertex set, the edge set, the (full) automorphism group, the valence (when constant on $V(\Gamma)$), the minimum valence, and the connectivity of Γ. The graph-argument will be suppressed when no ambiguity is likely.

We say that Γ is a *flag-transitive* (also called '1-transitive' in the literature) if given $\{x_i, y_i\} \in E(\Gamma)$ $(i = 1, 2)$, there exists an automorphism $\alpha \in G(\Gamma)$ such that $\alpha(x_1) = x_2$ and $\alpha(y_1) = y_2$. If $S \subseteq V(\Gamma)$, then $\partial(S)$ denotes the set of vertices in $V(\Gamma)\backslash S$ which are adjacent to at least one vertex in S. We abbreviate $\partial(\{x\})$ to $\partial(x)$ for $x \in V(\Gamma)$. Thus $\rho(x) = |\partial(x)|$ for $x \in V(\Gamma)$.

Following convention, K_n denotes the complete graph on n vertices; its complement is the empty graph Ω_n. The complete bipartite graph with $m+n$ vertices and mn edges is denoted by $K_{m,n}$. The symbol Δ_∞ will denote the graph consisting of a 2-way-infinite path.

If R is a region of a planar imbedding of Γ, then $\rho^*(R)$, called the *covalence* of R, denotes the number of edges in the boundary of R. If Γ has a uniquely determined set of regions for all planar imbeddings and if ρ^* is a constant k on that set, then we may write $\rho^*(\Gamma) = k$ and say that Γ is *k-covalent.*

By a *lobe* of Γ (also called a 'block' in the literature) we mean either a maximal biconnected subgraph of Γ or an isthmus together with its incident vertices.

The set of integers (positive integers) is denoted by $\mathbb{Z}$ ($\mathbb{Z}^+$), while $\mathbb{Z}_n$ denotes the ring of integers modulo n.

Given graphs Γ_1 and Γ_2, the *lexicographic product* of Γ_1 by Γ_2 is the graph $\Gamma_1[\Gamma_2]$ having vertex set $V(\Gamma_1) \times V(\Gamma_2)$ while $\{(x_1, y_1), (x_2, y_2)\} \in E(\Gamma_1[\Gamma_2])$ whenever either $x_1 = x_2$ and $\{y_1, y_2\} \in E(\Gamma_2)$ or $\{x_1, x_2\} \in E(\Gamma_1)$ (cf. [11]).

We use the notion of an 'end' in a graph as formulated by Halin [5]. In the special case of a locally finite graph Γ, the number of ends $\varepsilon(\Gamma)$ of Γ may be defined as the supremum of the number of infinite components of $\Gamma - S$, where S ranges over all finite subsets of $V(\Gamma)$. By combining [6, Corollary 15] and [7, Theorem 1], one has that $\varepsilon(\Gamma)$ equals 1 or 2 or is uncountable whenever Γ is infinite, locally finite, connected, and vertex-transitive. The group $G(\Gamma)$ induces a permutation group on the set of ends of Γ.

Suppose that Γ is connected. The cardinality of a smallest separating set $T \subseteq V(\Gamma)$ such that $\Gamma - T$ has at least one finite component is denoted by $\kappa_f(\Gamma)$. It is obvious that $\kappa_f(\Gamma) \leqslant \rho_{\min}(\Gamma)$. The cardinality of a smallest set $T \subseteq V(\Gamma)$ such that $\Gamma - T$ has at least two infinite components is denoted by $\kappa_\infty(\Gamma)$. Unless the appropriate 'smallest set' exists, the corresponding parameter is not defined. Otherwise, clearly $\kappa = \min\{\kappa_f, \kappa_\infty\}$. It follows from the definitions that for Γ infinite, $\varepsilon(\Gamma) > 1$ if and only if $\kappa_\infty(\Gamma)$ is defined and finite.

Proposition 2.1. *If Γ is edge-transitive, then $\kappa_f(\Gamma) = \rho_{\min}(\Gamma)$.*

Although Proposition 2.1 was originally proved for finite graphs [14, Corollary 1A], the same proof is valid for infinite graphs if one invokes [8, Corollary 1B].

The proof of the following useful proposition is often given to students as an exercise.

Proposition 2.2. *If Γ is edge-transitive but not vertex-transitive, then Γ is bipartite, admitting the bipartition $V(\Gamma) = V_0 \cup V_1$, where V_0 and V_1 are the two orbits of $G(\Gamma)$ in $V(\Gamma)$.*

If Ψ is a subgraph of Γ, we say that an end $\mathscr{E}$ 'is contained in Ψ' if for every 1-way-infinite path $\Pi \in \mathscr{E}$, a 1-way-infinite subpath of Π is contained in Ψ. We say that a set $S \subseteq V(\Gamma)$ 'separates (distinct) ends $\mathscr{E}_1$ and $\mathscr{E}_2$' if there exist distinct infinite components Θ_1 and Θ_2 of $\Gamma - S$ such that $\mathscr{E}_i$ is contained in Θ_i $(i = 1, 2)$. If $|S| = \kappa_\infty(\Gamma) < \infty$ and S separates some two ends, then S will be called a *κ_∞-separator.*

Proposition 2.3 [10]. *Let S be a κ_∞-separator of a locally finite, edge-transitive graph Γ. Then $\Gamma - S$ has no finite component.*

In [9, p. 153], H.A. Jung and the author defined a connected graph Γ to be a *strip* if there exist a set $S \subsetneq V(\Gamma)$ such that S induces a connected subgraph with $0 < |\partial(S)| < \infty$ and an automorphism $\alpha \in G(\Gamma)$ such that $\alpha[S \cup \partial(S)] \subseteq S$ and $|S \setminus \alpha[S]| < \infty$. We proceeded in [9, Theorem 5.13] to characterise a strip as a locally finite graph admitting an automorphism with finitely many orbits (all of them necessarily infinite). By [9, Lemma 5.3], all strips have exactly two ends.

If d denotes the distance function in Γ and $\beta \in G(\Gamma)$, let us call β *bounded* if there exists an integer d_β such that $d(x, \beta(x)) \leq d_\beta$ for all $x \in V(\Gamma)$. The set $B(\Gamma)$ of bounded automorphisms is a normal subgroup of $G(\Gamma)$ by [9, Lemma 5.9]. If Γ is a strip, then $B = B(\Gamma)$ is the set of automorphisms fixing the two ends of Γ, and so $[G:B] \leq 2$. (Cf. [9, Theorem 5.10].) The stabilizer in G of a vertex x is denoted by G_x. We let $B_x = B(\Gamma) \cap G_x$. By the Second Isomorphism Theorem for groups, we thus have

$$1 \leq [G_x : B_x] = [G_x B : B] \leq [G : B] \leq 2. \tag{2.4}$$

We require the following result in the last section of this paper.

Lemma 2.5. *Let Γ be a planar graph, let T be a finite set of vertices of Γ, and let Θ_1 and Θ_2 be distinct components of $\Gamma - T$. Let $x \in T$ and let u_i, $v_i \in V(\Theta_i)$ $(i = 1, 2)$ be neighbors of x. Then for no planar imbedding of Γ can the neighbors of x, in the counterclockwise cyclic ordering, form the cyclic subsequence u_1, u_2, v_1, v_2.*

Proof. For each $i = 1, 2$, there exists a u_iv_i-path Π_i contained in Θ_i. Form the circuit $\Xi_i = \Pi_i + \{u_i, x\} + \{x, v_i\}$. Since Ξ_1 and Ξ_2 have only the vertex x in common, these two circuits cannot cross at x, by the so-called 'Jordan Curve Theorem for Multigraphs' (cf. [3, p. 83, Theorem E22]). □

Let $d(k)$ denote the number of vertices at distance k from some fixed vertex of a connected, locally finite graph Γ. We say that Γ *has exponential growth* if there exist numbers $a > 1$ and $c > 0$ such that $d(k) \geqslant ca^k$ holds for all but finitely many k. The following result gives a sufficient condition for exponential growth for a directed graph (where $d(k)$ is determined using undirected distance).

Proposition 2.6 [13, Lemma 2]. *Let Δ be a directed graph with constant in-valence ρ^- and constant out-valence ρ^+, where $1 \leqslant \rho^- < \rho^+$ and $\rho = \rho^+ + \rho^-$, and assume that Δ has no 2-cycle. Then, measuring $d(k)$ from any vertex whatever, one has*

$$d(k) \geqslant 2[(\rho + 1)/(\rho - 1)]^{k-1}$$

for all $k \geqslant 1$.

One immediately obtains the following.

Theorem 2.7. *Let Γ be a connected infinite graph. If the edges of Γ can be so oriented that the in-valence ρ^- and the out-valence ρ^+ are constant on $V(\Gamma)$ and $1 \leqslant \rho^- < \rho^+$, then Γ has exponential growth.*

Corollary 2.8. *Let Γ be a locally finite, vertex-transitive, edge-transitive, nonflag-transitive graph, and let $\{x, y\} \in E(\Gamma)$. Then for every vertex $u \in V(\Gamma)$, there exists a partition $\{\partial^+(u), \partial^-(u)\}$ of $\partial(u)$ such that*

$$\partial^+(u) = \{v \in \partial(u) \colon \gamma(x) = u \text{ and } \gamma(y) = v \text{ for some } \gamma \in G(\Gamma)\}$$

and

$$\partial^-(u) = \{v \in \partial(u) \colon \gamma(x) = v \text{ and } \gamma(y) = u \text{ for some } \gamma \in G(\Gamma)\}.$$

If $|\partial^+(u)| \neq |\partial^-(u)|$ for some $u \in V(\Gamma)$, then Γ has exponential growth.

Proof. Since Γ is edge-transitive but not flag-transitive, it is elementary that for each $\{u, v\} \in E(\Gamma)$, either $\gamma(x) = u$ and $\gamma(y) = v$ for some $\gamma \in G(\Gamma)$ or $\gamma(x) = v$ and $\gamma(y) = u$ for some $\gamma \in G(\Gamma)$, but not both.

If $|\partial^+(u)| \neq |\partial^-(u)|$, we may suppose $|\partial^-(u)| < |\partial^+(u)|$; otherwise one may interchange x and y at the outset. The images of $\{x, y\}$ induce an orientation of Γ whereby $\rho^+ = |\partial^+(u)|$ and $\rho^- = |\partial^-(u)|$. The result now follows from Theorem 2.7. □

If Σ is a strip, then the sequence $\{d(k)\}$ is bounded [9, Lemma 5.4], and so the growth of Σ is certainly not exponential. Hence, in the language of the preceding corollary, for any vertex x of a vertex-transitive, edge-transitive, nonflag-transitive strip, one has $|\partial^+(x)| = |\partial^-(x)|$, and so x has even valence.

3. Applications of Halin's theorem to strips

In order that the present paper be more self-contained, we include a seminal result from Halin's substantial paper [6]. The statement of this result requires some preliminary definitions. We then apply Halin's result for locally finite graphs to the special case of strips.

Let Γ be any graph. An automorphism $\gamma \in G(\Gamma)$ is *of type* 1 if there exists a non-empty finite set $S \subset V(\Gamma)$ such that $\gamma[S] = S$. Otherwise γ is *of type* 2. If Γ is locally finite, then γ is of type 2 if and only if $\gamma[U] \subset U$ (n.b. proper containment) for some subset $U \subset V(\Gamma)$, (cf. [9, Lemma 2.4]).

If γ is of type 2, then the number of ends fixed by γ is either 1 or 2 (cf. [6, Theorem 8]). Thus if Σ is a strip and $\gamma \in G(\Sigma)$ is of type 2, then γ fixes both ends of Σ. It follows that all the type 2 automorphisms of a strip are contained in $B(\Sigma)$. Moreover, by their very definition, all strips admit type 2 automorphisms. Type 1 automorphisms may or may not belong to $B(\Sigma)$.

For an end $\mathscr{E}$, Halin denotes by $m_1(\mathscr{E})$ the maximum number of disjoint 1-way infinite paths belonging to $\mathscr{E}$. If γ fixes ends $\mathscr{E}_1$ and $\mathscr{E}_2$, then $m_1(\mathscr{E}_1) = m_1(\mathscr{E}_2)$. Moreover, $\mathscr{E}_1 \neq \mathscr{E}_2$ if and only if $m_1(\mathscr{E}_1) < \infty$. If $\mathscr{E}$ is an end of a strip Σ then clearly

$$m_1(\mathscr{E}) \leqslant \kappa_\infty(\Sigma) < \infty \tag{3.1}$$

Proposition 3.2 [6, Theorem 9e]. *Let γ be a type 2 automorphism of a locally finite connected graph Γ such that $m_1(\mathscr{E}) < \infty$ for the ends $\mathscr{E}$ fixed by γ. Then there exist γ-invariant 2-way infinite paths $\Pi_1, \ldots, \Pi_m$, a positive integer n, and a finite subset T of $V(\Gamma)$ such that the following hold*:

(a) $\gamma^n[\Pi_i] = \Pi_i$ $(i = 1, \ldots, m)$.

(b) *T separates the two ends fixed by γ.*

(c) *$|T| = m$, and for each $i = 1, \ldots, m$, there is a unique vertex $t_i \in T \cap V(\Pi_i)$.*

(d) *The sets in the family $\{\gamma^{ni}[T]: i \in \mathbb{Z}\}$ are pairwise disjoint.*

(e) *If $v \in V(\Pi_i)$, $w \in V(\Pi_j)$, and $v < t_i$, $t_j < w$, where $<$ denotes the natural order of Π_i (resp., Π_j) given by γ, then T separates v and w.*

(f) *If $p, q, r \in \mathbb{Z}$ and $p<q<r$, then $\gamma^{nq}[T]$ separates $\gamma^{np}[T]$ from $\gamma^{nr}[T]$, and no two vertices in $\gamma^{nq}[T]$ are separated by a set of the form $\gamma^{nk}[T]$, $k \in \mathbb{Z}$.*

We now apply Halin's theorem to strips.

Lemma 3.3. *Let Σ be a strip. Let $\gamma \in G(\Sigma)$ be of type 2. Then Σ contains a maximal set of 2-way-infinite, γ-invariant paths $\Pi_1, \ldots, \Pi_k$ such that the following statements hold*:

(a) $k = \kappa_\infty(\Sigma)$.

(b) *For each $i = 1, \ldots, k$, Π_i is the union of two 1-way infinite paths, each from one of the two ends of Σ.*

(c) *If S is any κ_∞-separator and $\alpha \in G(\Sigma)$, then each vertex in $\alpha[S]$ belongs to a different path Π_i.*

(d) *If Σ is edge-transitive, then for any κ_∞-separator S of Σ,*

(i) *S is an independent set*;

(ii) *If Σ is not vertex-transitive, then Σ admits a bipartition $\{V_0, V_1\}$ such that $V_0 \subseteq \bigcup_{i=1}^{k} V(\Pi_i)$.*

Proof. Let $\Pi_1, \ldots, \Pi_k$ be the paths and T the k-set postulated by Proposition 3.2. Since T separates the two ends of Σ, we have $\kappa_\infty(\Sigma) \leqslant |T| = k$. From this and (3.1), we now infer (a).

By 3.2(b,e), any two 1-way-infinite paths making up a path Π_i belong to different ends. Part (b) is now immediate.

It follows from (b) that since every κ_∞-separator separates the two ends of Σ, every κ_∞-separator of Σ meets every path Π_i. If S is a κ_∞-separator and $\alpha \in G(\Sigma)$, then $\alpha[S]$ is also a κ_∞-separator. By (a), $|\alpha[S]| = k$, and (c) follows.

Now suppose that Σ is edge-transitive and suppose e is an edge with both of its incident vertices in S. Let $e' \in E(\Pi_1)$ and pick $\alpha \in G(\Sigma)$ such that $\alpha[e] = e'$. But then Π_1 would contain two vertices of $\alpha[S]$, contrary to (c). Hence S is an independent set.

Finally, suppose that Σ is edge-transitive but not vertex-transitive. Let S be a κ_∞-separator and let $x_0 \in S$. Now Σ admits a bipartition $\{V_0, V_1\}$ where $x_0 \in V_0$. If $y_0 \in V_0$, then $\alpha(x_0) = y_0$ for some $\alpha \in G(\Sigma)$, and so $y_0 \in \alpha[S]$. By (c), $y_0 \in V(\Pi_i)$ for some $i = 1, \ldots, k$, as required. □

Definition. A 2-way-infinite path Π in a strip Σ will be called *translatable* if it is a member of a $\kappa_\infty(\Sigma)$-set of γ-invariant paths for some type 2 automorphism γ satisfying the conditions of Lemma 3.3.

If the strip Σ is edge-transitive, then clearly for any given edge there exists a translatable path of Σ containing it.

Lemma 3.4. *Let Σ be an edge-transitive or vertex-transitive strip, and let the vertex x lie between vertices w and y on some translatable path Π in Σ. Let $\alpha \in G_x$. If $\alpha(y) = w$, then $\alpha \notin B$.*

Proof. Let Π^+ denote the one-way infinite subpath of Π with initial vertex x that passes through y. Suppose that $\alpha(y) = w$ for some $\alpha \in B_x$. If T is a κ_∞-separator, then T does not separate Π^+ from $\alpha[\Pi^+]$. If $x \in T$, then $\alpha[\Pi^+]$ must meet T at some vertex other than x, contrary to Lemma 3.3(c).

Now suppose x belongs to no κ_∞-separator, and let $\Pi^- = \Pi - \Pi^+$. The neighbor x' of x in Π^- belongs to some κ_∞-separator T. Since T must not separate Π^- from $\alpha[\Pi^-]$, we have that $\alpha[\Pi^-]$ meets T at some vertex other than w, providing a contradiction. □

The author has proved [15, Theorem 5.4] that if a strip is both vertex- and edge-transitive, then it has even valence. The following theorem generalizes that result by showing *inter alia* that the hypothesis of vertex-transitivity may be dropped.

Theorem 3.5. *Let Σ be an edge-transitive strip. Then all vertices of Σ have even valence. More specifically, let T be a κ_∞-separator of Σ, and let F^+ and F^- be the components of $\Sigma - T$. Let $x \in T$.*

(A) *If Σ is not vertex-transitive or if Σ is flag-transitive, then:*
   (1) $[G:B] = 2$;
   (2) *$\partial(x)$ is a G_x-orbit;*
   (3) *$\partial(x) \cap F^+$ and $\partial(x) \cap F^-$ are B_x-orbits of equal size.*

(B) *If Σ is vertex-transitive but not flag-transitive, then either:*
   (1) *$G = B$, in which case $\partial(x) \cap F^+$ and $\partial(x) \cap F^-$ are orbits of G_x of equal size, or*
   (2) *$[G:B] = 2$, in which case:*
      (a) *Each of $\partial(x) \cap F^+$ and $\partial(x) \cap F^-$ comprises exactly two B_x-orbits, all of the same size, implying that $\rho(\Sigma)$ is divisible by 4, and*
      (b) *$\partial(x)$ comprises exactly two G_x-orbits, each being the union of one B_x-orbit in $\partial(x) \cap F^+$ and one B_x-orbit in $\partial(x) \cap F^-$.*

Proof. By Proposition 2.3, $\Sigma - T$ has no finite component. Since $\partial(x) \cap T = \emptyset$ by Lemma 3.3(d), $\{\partial(x) \cap F^+, \partial(x) \cap F^-\}$ is a partition of $\partial(x)$. There exist vertices $w \in \partial(x) \cap F^-$ and $y \in \partial(x) \cap F^+$ such that the edges $\{w, x\}$ and $\{x, y\}$ lie on some translatable path Π. Thus $\{x\} = T \cap V(\Pi)$. By Lemma 3.4, B_x admits at least two orbits in $\partial(x)$, and no orbit of B_x meets both F^+ and F^-. If r denotes the number of G_x-orbits in $\partial(x)$, then the number of B_x-orbits in $\partial(x)$ is r if $[G:B] = 1$ and is r or $2r$ if $[G:B] = 2$ by (2.4).

If Σ is not vertex-transitive, then by Proposition 2.2 G_x acts transitively on $\partial(x)$, even when x does not belong to a κ_∞-separator. Similarly $\partial(x)$ is an orbit of

G_x when Γ is flag-transitive. Hence $\partial(x)$ comprises $[G_x:B_x]=2$ orbits of B_x, namely $\partial(x)\cup F^+$ and $\partial(x)\cap F^-$. Any automorphism in the coset $G_x\setminus B_x$ interchanges these two orbits.

We now assume that Σ is vertex-transitive but not flag-transitive. Thus Σ is bipartite, and by the final paragraph of Section 2, $\rho(\Sigma)$ is even.

Suppose there exists a vertex $z\in\partial(x)\cap F^+$ that belongs to a B_x-orbit different from that of y. Since Σ is edge-transitive, there exists $\sigma\in G$ such that $\sigma(z)=x$ and $\sigma(x)=y$. At least one of y and z does not share a G_x-orbit with w; for definiteness, let us assume that w and y are in different G_x-orbits. Again by edge-transitivity, one may fix $\tau\in G$ such that $\tau(w)=x$ and $\tau(x)=y$. But then $\sigma^{-1}\tau$ fixes x while $\sigma^{-1}\tau(w)=z$, implying that w and z belong to the same G_x-orbit. This orbit must thus contain at least two — and hence exactly two — B_x-orbits, one from $\partial(x)\cap F^+$ and one from $\partial(x)\cap F^-$. By (2.4), $[G:B]=[G_x:B_x]=2$ in this case.

What we have proved is that given any two B_x-orbits in $\partial(x)\cap F^+$, one of them is contained in the G_x-orbit that includes w. Hence $\partial(x)\cap F^+$ comprises only two B_x-orbits O_1^+ and O_2^+. By symmetry, $\partial(x)\cap F^-$ comprises two B_x-orbits O_1^- and O_2^-. If $\gamma\in G_x\setminus B_x$, then γ interchanges O_i^+ with O_i^- $(i=1,2)$. It follows from the final paragraph of Section 2 that $|O_1^+\cup O_2^+|=|O_1^-\cup O_2^-|$. We conclude that the four B_x-orbits in $\partial(x)$ are all of equal size.

The remaining situation is the one in which $\partial(x)\cap F^+$ and $\partial(x)\cap F^-$ are G_x-orbits. It follows that $[G_x:B_x]=1$. That the orbits are of equal size follows from Corollary 2.8.

It remains only to prove that $G=B$. Were this not the case, (2.4) would imply that $[G:G_xB]=2$. For arbitrary $z\in V(\Sigma)$, suppose $\eta_1(x)=\eta_2(x)=z$ for some $\eta_1,\eta_2\in G$. Then $\eta_2^{-1}\eta_1\in G_x\leqslant G_xB=B$, and so η_1 and η_2 belong to the same B-coset. This yields a partition $\{V_0,V_1\}$ of $V(\Sigma)$ whereby a vertex $s\in V_1$ if and only if $\eta(x)=s$ for some $\eta\in G\setminus B$. Thus $x\in V_0$. It is immediate that V_0 and V_1 are each fixed under B and are interchanged by each automorphism in $G\setminus B$. In fact, $\{V_0,V_1\}$ is the bipartition of Σ.

Pick $w\in\partial(x)$ such that $\{w,x\}$ lies on a translatable path Π. Since $w\in V_1$ and Σ is vertex-transitive, there exists $\alpha\in G\setminus B$ such that $\alpha(w)=x$. Let $y=\alpha(x)$. Suppose there existed $\beta\in G_x$ such that $\beta(w)=y$. Then $\beta^{-1}\alpha(x)=w$ and $\beta^{-1}\alpha(w)=x$, although Σ is not flag-transitive. Hence w and y belong to different G_x-orbits: say, $w\in\partial(x)\cap F^+$ and $y\in\partial(x)\cap F^-$. In the notation of the proof of Lemma 3.4, Π^- and $\alpha[\Pi^+]$, belonging to the same end, could not be separated by T. Hence $\alpha[\Pi^+]$ would meet $T\setminus\{x\}$, contrary to Lemma 3.3c. □

In the final paragraph of [13], Thomassen and the author describe a vertex- and edge-transitive strip that is not flag-transitive. It satisfies $\rho=\kappa_f=4$ and $\kappa_\infty=9$ and exemplilfies subcase B1 of Theorem 3.5. This strip has girth 6, which implies by Theorem 4.2 below that it is nonplanar.

A strip Σ exemplifying the conditions of Subcase B2 may be constructed as follows. In [2], Folkman presented a family of finite edge-transitive nonvertex-transitive graphs of constant valence. The smallest member of this family is a 4-valent graph having 20 vertices; let Γ denote this graph. Following Proposition 2.2, let $\{A, B\}$ denote the bipartition of $V(\Gamma)$. Let $A = \{a_1, \ldots, a_{10}\}$ and $B = \{b_1, \ldots, b_{10}\}$. Note that A and B are the two vertex-orbits of $G(\Gamma)$. Now let S be the set $\{x_1, \ldots, x_{10}, y_1, \ldots, y_{10}\}$ and let

$$V(\Sigma) = S \times \mathbb{Z},$$

and for each edge $\{a_i, b_j\} \in E(\Gamma)$ we have for each $n \in \mathbb{Z}$ the following four edges in Σ:

$$\{(x_i, n), (x_j, n+1)\},$$
$$\{(y_i, n), (y_j, n+1)\},$$
$$\{(x_i, n), (y_j, n-1)\},$$
$$\{(y_i, n), (x_j, n-1)\}.$$

Thus $\rho(\Sigma) = 4\rho(\Gamma) = 16$. One verifies straightforwardly that Σ has the desired properties.

4. Planar edge-transitive strips

Let Σ be a planar strip. It is not hard to see that Σ admits an imbedding on an infinite cylinder in such a way that its two ends are separated by every circumference of the cylinder. Consider an automorphism $\alpha \in G(\Sigma)$ like the one in the definition of 'strip' in Section 2. Some properties of α are that $\alpha \in B(\Sigma)$ and that α has infinite order and a finite number m of orbits, all of them infinite. For any given $d_0 \in \mathbb{Z}^+$, we can pick $n \in \mathbb{Z}^+$ such that $d(x, \alpha^n(x)) \geqslant d_0$ for all $x \in V(\Sigma)$. For $x, y \in V(\Sigma)$ write $x \cong y$ if $y = \alpha^{kn}(x)$ for some $k \in \mathbb{Z}$. Obviously $\cong$ is an equivalence relation. The quotient graph $\Sigma/\cong$ is finite, having mn vertices, and is imbeddable on the surface of a torus. Hence $\rho_{\min}(\Sigma/\cong) \leqslant 6$. By choosing d_0 sufficiently large, we can be assured that for all $x \in V(\Sigma)$, $\rho(x)$ in Σ equals $\rho([x])$ in $\Sigma/\cong$, where $[x]$ denotes the equivalence class containing x. We proved the following.

Lemma 4.1. *Let Σ be a planar strip. Then $\rho_{\min}(\Sigma) \leqslant 6$.*

Theorem 4.2. *Let Σ be a planar, edge-transitive strip. Then Σ has the following characterization:*

(A) *If $\kappa(\Sigma) \geqslant 3$, let $b \geqslant 3$, and let*

$$W = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 0 \leqslant x + y \leqslant 2b\}$$

where $(-x, x)$ has been identified with $(-x+b, x+b)$ for all $x \in \mathbb{Z}$. Let $V((\Sigma) = W$, and when $0 \leqslant x+y < 2b$, let (x, y) be adjacent to $(x+1, y)$ and $(x, y+1)$. In this case, Σ is flag-transitive, $\rho = \rho^* = \kappa_f = 4$, $\kappa_\infty = b$, and $\kappa = \min\{4, b\}$.

(**B**) *If $\kappa(\Sigma) = 2$, then*:

(1) *$\Sigma = \Delta_\infty[\Omega_2]$, or*

(2) *Σ is obtained from one of the graphs in* (**A**) *or from $\Delta_\infty[\Omega_2]$ by constructing the multigraph wherein each edge is replicated $r \geqslant 1$ times and subdividing each edge once.*

(**C**) *If $\kappa(\Sigma) = 1$, then*:

(1) *Σ is a 2-way-infinite path, or*

(2) *for some given $m \geqslant 2$, all the lobes of Σ are isomorphic to $K_{2,m}$ and all the separating vertices are incident with exactly two lobes and belong to the 2-vertex side of each copy of $K_{2,m}$.*

Proof. Let Σ be a planar, edge-transitive strip.

(**A**) Suppose that $\kappa(\Sigma) \geqslant 3$. Thus the imbedding of Σ in the plane is essentially unique; we will be able to speak of the 'regions' of Σ as well as of its dual. (See [12] for an extension to infinite graphs of Whitney's classical result [16] concerning the uniqueness of planar imbeddings of finite graphs.)

We first show that Σ is 4-valent. Let $x \in V(\Sigma)$ be given.

Subcase 1: *$x \in T$ for some κ_∞-separator T.*

By Theorem 3.5, $\rho(x)$ is even. If $\rho(x) \neq 4$, then $\rho(x) \geqslant 6$ and we may write $\partial(x) \cap F^+ = \{u_1, \ldots, u_m\}$ and $\partial(x) \cap F^- = \{u_{m+1}, \ldots, u_{2m}\}$, where $m \geqslant 3$ and F^+ and F^- are the vertex sets of the two infinite components of $\Sigma - T$. By Lemma 2.5 we may assume that the elements of $\partial(x)$ have been indexed in such a way that as one proceeds in counterclockwise fashion around x, one encounters in cyclic order the incident edges $\{x, u_1\}$, $\{x, u_2\}, \ldots, \{x, u_{2m}\}$.

Note that the restriction of G_x (and hence of B_x) to $\partial(x)$ is a subpermutation group of the full dihedral group on the cyclic list of symbols $u_1, \ldots, u_{2m}$. Suppose for some $\beta \in B_x$ that $\beta(u_1) = u_2$. By Theorem 3.5, $\beta(u_{2m}) \neq u_1$. This forces $\beta(u_2) = u_1$ and, in turn, $\beta(u_3) = u_{2m}$, contrary to Theorem 3.5. Hence u_1 and u_2 belong to different B_x-orbits, and so Σ is subject to the conditions of Case B2 of Theorem 3.5, whence Σ is vertex-transitive and so $\rho(x) = \rho_{\min}(\Gamma)$, a number divisible by 4. By Lemma 4.1 together with our initial assumption in this subcase, $\rho_{\min}(\Gamma) \leqslant 6 \leqslant \rho(x)$, giving a contradiction.

Subcase 2: *x belongs to no κ_∞-separator.*

In this case Σ is not vertex-transitive; Σ is bipartite, and every neighbor of x belongs to a κ_∞-separator. By edge-transitivity, some translatable path Π passes through x. By Lemma 3.3, x has two neighbors $t_0, t_1 \in V(\Pi)$ and no κ_∞-separator contains both t_0 and t_1. Let T_0 be a κ_∞-separator that contains t_0. Since $t_1 = \gamma_1(t_0)$ for some $\gamma_1 \in G_x \setminus B_x$, we let $T_1 = \gamma_1[T_0]$. For $i = 0, 1$, let F_i be the vertex set of the component of $\Sigma - T_i$ that contains x. Since Π cannot 'recross' T_0 or T_1, x is the only vertex of Π in $F_0 \cap F_1$. Since Σ has only two ends, it follows that $F_0 \cap F_1$ is finite. We may assume that T_0 has been chosen so as to minimize $|F_0 \cap F_1|$.

We claim that $\partial(x) \subseteq T_0 \cup T_1$. Otherwise let $t_2 \in \partial(x) \setminus (T_0 \cup T_1)$. For some $i = 0, 1$, $\gamma_2(t_i) = t_2$ for some $\gamma_2 \in B_x$. Let $T_2 = \gamma_2[T_i]$ and let F_2 be the component of $\Sigma - T_2$ that contains x. Thus $F^* = F_i \cap F_2$ is infinite. By the proof of [8, Theorem 2], $\partial(F^*)$ is a κ_∞-separator and $\partial(F^*) = (F_i \cap T_2) \cup (T_i \cap T_2) \cup (T_i \cap F_2)$. For $j \in \{0, 1\} \setminus \{i\}$, $|F_j \cap F^*| < |F_0 \cap F_1|$ since $t_2 \notin V(F^*)$. But $x \in F_j \cap F^*$, contrary to the minimality assumption.

Were x to have a neighbor $t \in T_0 \cap T_1$, then a translatable path through $\{x, t\}$ would meet T_0 or T_1 twice. Hence $T_0 \cap T_1 = \emptyset$. For $\alpha \in G_x$, $\alpha(t_0) \in T_0$ if and only if $\alpha \in B_x$. It follows that $\partial(x) \cap T_0$ and $\partial(x) \cap T_1$ are the two B_x-orbits in $\partial(x)$ and hence are of equal size.

[We remark that thus far in the present subcase, no assumption about planarity, connectivity, or valence has yet been used. We note this because the foregoing argument will be cited below in the proof of Part (B) of this theorem.]

Letting $m = |\partial(x) \cap T_0|$, we have that $2m = \rho(x) \geqslant \kappa(\Sigma) \geqslant 3$, and so $m \geqslant 2$. We observe that $\Sigma - (\partial(T_0) \cap F_0)$ has two infinite components and that $\partial(x) \cap T_1 \subseteq \partial(\partial(T_0) \cap F_0) \cap F_0$. Thus we may repeat the argument of Subcase 1 by replacing T, $\partial(x) \cap F^-$, and $\partial(x) \cap F^+$ by $F_0 \cap F_1$, T_0, and T_1, respectively. The argument this time is abbreviated by the fact that $\partial(x)$ is a G_x-orbit. We readily deduce that $m \leqslant 2$. Thus Σ is a 4-valent graph.

Now consider the dual graph Σ^*, which must similarly be 3-connected and edge-transitive. Also, Σ and Σ^* have the same number of ends, i.e., Σ^* is a strip. Applying the foregoing argument to Σ^*, we obtain that Σ^* is 4-valent. Thus Σ is 4-covalent. In fact, Σ must be bipartite, whether it is vertex-transitive or not.

Let us now investigate the structure of Σ. Let $x_1 \in V(\Sigma)$ and let T be a κ_∞-separator such that $x_1 \in T$. Let F^+ and F^- denote the vertex sets of the two components of $\Sigma - T$. Let $T' = \partial(T) \cap F^+$. In the event that Σ is not vertex-transitive, an enumeration of the $2\kappa_\infty$ TT'-edges proves that $|T| = |T'|$, and so every vertex belongs to a κ_∞-separator. Proceeding counterclockwise about x_1, let its neighbors encountered in cyclic order be u_1, v_1, v_2, u_2, where $u_1, u_2 \in F^+$ and $v_1, v_2 \in F^-$. The edges $\{x_1, u_2\}$ and $\{x_1, v_2\}$ are two of the edges on the boundary of a region R_1. There exists a vertex x_2 such that $\{x_1, u_2, x_2, v_2\}$ determines R_1. Since T separates u_2 from v_2, it follows that $x_2 \in T$. There are vertices $u_3 \in V(F^+)$, $v_3 \in V(F^-)$ such that the neighbors of x_2 in counterclockwise cyclic order are u_2, v_2, v_3, u_3. If $u_3 = u_1$, then $\{u_1, x_1, v_2, x_2, u_1\}$ determines a circuit that separates u_2 from other vertices such as v_1, and so $\{x_1, x_2\}$ is a separating set although $\kappa(\Sigma) > 2$. Hence u_1, u_2, u_3 are distinct, and similarly so are v_1, v_2, v_3. Now the edges $\{x_2, u_3\}$ and $\{x_2, v_3\}$ lie on the boundary of a region R_2, whose fourth boundary vertex is a vertex $x_3 \in T$. If $x_3 = x_1$, then $\{x_1, x_2\}$ would be a κ_∞-separator. Hence x_1, x_2, x_3 are distinct.

Continuing in this way, the process terminates with $T = \{x_1, x_2, \ldots, x_b\}$, where $b = \kappa_\infty$ and the neighbors of x_i in counterclockwise cyclic order are $u_i, v_i, v_{i+1}, u_{i+1}$, where subscripts are understood modulo b.

Passing to Σ^*, we see that $T^* = \{R_1, \ldots, R_b\}$ is a κ_∞-separator. One constructs the regions of Σ^* adjacent to those in T^*, then passes back to $\Sigma^{**} = \Sigma$. In this way, all of Σ is forced. (Note that Σ is isomorphic to Σ^*. See Fig. 1 for a cylindrical representation of the unique edge-transitive planar strip with connectivity 3.)

(**B**) Suppose that $\kappa(\Sigma) = 2$.

Case 1: $\kappa_f(\Sigma) > 2$.

Thus $\kappa_\infty = 2$. Since all valences are even, all vertices have valence at least 4. Let x be any vertex adjacent to a κ_∞-separator of Σ. As noted in the proof of Subcase 2 above, there exist κ_∞-separators T_0 and T_1 of Σ such that $\partial(x) \cap T_0$ and $\partial(x) \cap T_1$ are the two orbits of B_x in $\partial(x)$. Since $|T_0| = |T_1| = 2$ and $|\partial(x)| \geq 4$, we have that $\rho(x) = 4$ and $\partial(x) = T_0 \cup T_1$.

Let $T_0 = \{w_1, w_2\}$ and let F_i be the vertex set of the component of $\Sigma - T_i$ that contains x, ($i = 0, 1$).

We assert that if $z \in F_0$, then $\{w_1, z\}$ is not a κ_∞-separator. By Lemma 3.3(d), $z \notin \partial(T_0)$, and clearly $z \notin V(\Sigma) \setminus (T_1 \cup F_1)$. But then some translatable path passes through w_2, x, and one vertex in T_1, thus avoiding $\{w_1, z\}$. Similarly, $\{w_2, z\}$ is not a κ_∞-separator.

Suppose there were a vertex $x' \in \partial(w_1) \cap F_0$ but $x' \notin \partial(w_2)$. There exists $\beta \in B_{w_1}$ such that $\beta(x) = x'$. But $\beta(w_2) \neq w_2$ and so $\beta[T_0]$ is not a κ_∞-separator. We infer that every vertex in $T' = \partial(T_0) \cap F_0$ is adjacent to both w_1 and w_2.

If $|T'| > 2$, then one can construct a homeomorph of $K_{3,3}$ in Σ, the nodes of which are w_1, w_2, three of the vertices in T' and one vertex in $F_0 \setminus T'$, contrary to the planarity of Σ. Hence $|T'| = 2$, and it is now routine to show that the structure of Σ is forced; $\Sigma = \Delta_\infty[\Omega_2]$, the copies of Ω_2 being the κ_∞-separators.

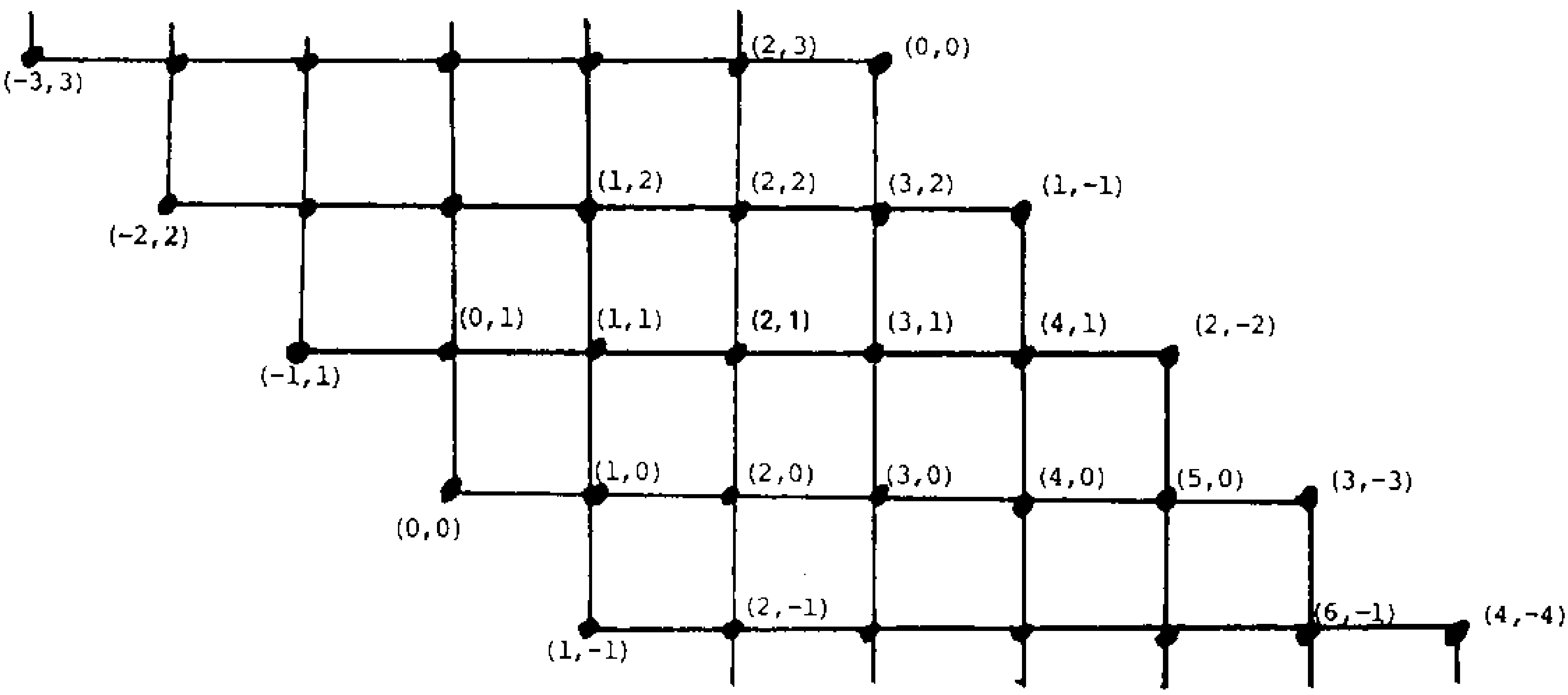

Fig. 1. Edge-transitive strip with $\kappa = 3$.

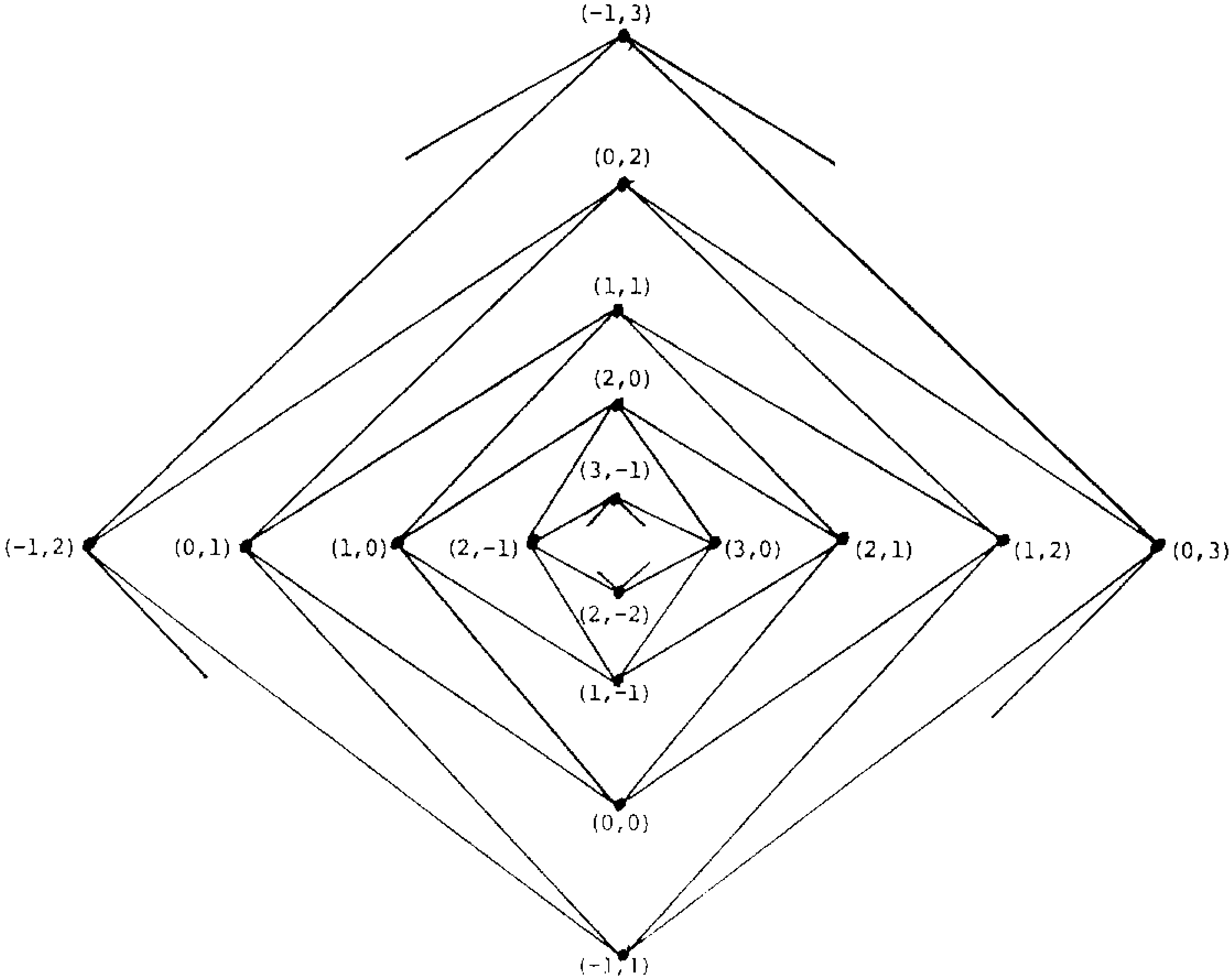

Fig. 2. $\Delta_\infty[\Omega_2]$.

[We note that were we to let $b = 2$ in part (A), the same graph $\Delta_\infty[\Omega_2]$ would be produced. See Fig. 2 for a planar representation of this graph.]

Case 2: $\kappa_f(\Sigma) = 2$.

Thus $\kappa_\infty \geq 2$. By Proposition 2.1, Σ admits 2-valent vertices, although Σ is not itself 2-valent. The situation is that Σ is not vertex-transitive; it admits a bipartition $\{V_0, V_1\}$ where V_1 is the set of all 2-valent vertices. Consider the strip Ψ with $V(\Psi) = V_0$ and $E(\Psi)$ consisting of pairs of vertices of V_0 having a common neighbor in Σ. Since $\kappa_f(\Psi) = \rho_{\min}(\Psi) > 2$ and Ψ is edge-transitive and planar, either $\Psi = \Delta_\infty[\Omega_2]$ or Ψ has been characterized in part (A). By replicating each edge of Ψ some fixed number of times and then subdividing all the edges once, one recovers the original strip Σ.

(C) Suppose $\kappa(\Sigma) = 1$. Since Σ admits a separating vertex, every edge is incident with a separating vertex. Since Σ is locally finite, it admits infinitely many separating vertices and infinitely many lobes.

Every lobe Λ of Σ is itself an edge-transitive graph, for if $e, e' \in E(\Lambda)$, we have $\alpha(e) = e'$ for some $\alpha \in G(\Sigma)$, whence $\alpha[\Lambda] = \Lambda$ and so $\alpha_{|\Lambda} \in G(\Lambda)$. By extension, all the lobes of Σ are pairwise isomorphic.

If a lobe Λ were incident with three or more separating vertices of Σ, then the deletion of any three of these vertices would leave at least three infinite components. On the other hand, edge-transitivity precludes that Λ contain a single separating vertex of Σ. Hence each lobe of Σ is incident with exactly two separating vertices. If some edge of Λ is incident with both of the separating vertices in Λ, then $\Lambda = K_2$ and Σ is a 2-way-infinite path.

The situation remains in which each edge of Λ is incident with one separating vertex of Σ. The two separating vertices in Λ belong to the same orbit of G and hence of $G_{|\Lambda}$. It follows that Λ is bipartite and hence is a subgraph of $K_{2,m}$ for some m. By definition of a lobe, Λ is biconnected, and so $\Lambda = K_{2,m}$ for some $m \geqslant 2$. [The graph defined in part (A) with $b = 1$ is precisely this graph when $m = 2$.] The proof is now complete. □

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