Discrete Mathematics 95 (1991) 331–340
North-Holland

End-faithful forests and spanning trees in infinite graphs

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Received 12 October 1989
Revised 23 April 1990

Abstract

Širáň, J., End-faithful forests and spanning trees in infinite graphs, Discrete Mathematics 95 (1991) 331–340.

We present several sufficient conditions for an end-faithful forest to be extendable to an end-faithful spanning tree in an infinite graph. Results related to rayless spanning trees are also included. Finally, a common generalization of end-faithfulness and raylessness is discussed.

1. Introduction

In 1964, Halin [3] introduced the concept of an *end* of a graph. In the same paper, he proved that any connected countable graph contains an *end-faithful spanning tree* (i.e., a spanning tree which represents the structure of ends in the graph). One of the central questions about end-faithfulness—the problem of whether the cardinality restriction in Halin's result may be dropped—has very recently been answered in the negative by Seymour and Thomas [8]. On the other hand, numerous positive results on end-faithful spanning trees have been obtained by several authors (e.g. Halin [4], Jung [5], Polat [7], Diestel [2], Seymour and Thomas [8], etc.).

As pointed out by Polat [6–7], there is an intimate connection between end-faithful spanning trees and *rayless* spanning trees (those without infinite paths; see also [8]). We shall be interested here in another kind of relation between these two types of spanning trees, which arises from contracting end-faithful forests in graphs.

After introducing the necessary concepts, in Section 3 we characterize the countable graphs containing a rayless spanning tree. Although this result is implicitly contained in [6], we provide it with a new short proof using a method which is, in a way, dual to that used in [1]. Section 4 is devoted to the study of graphs that arise by contracting the components of end-faithful forests. Such

0012-365X/91/$03.50

graphs are shown to be closely related to the rayless spanning tree problem as well as to the problem of extending an end-faithful forest to an end-faithful spanning tree. Finally, we propose a generalization of the approach used in Section 4, this time considering forests which are 'faithful' with respect to a chosen subset of ends, and 'rayless' with respect to the remaining ends of a graph.

2. Terminology

As usual, for a finite or infinite graph H we denote by $V(H)$ and $E(H)$ the set of vertices and edges of H, respectively. A finite path with end-vertices u and v in H will be called a *u–v path*. A u–v path P will be said to be *disjoint* from a set $W \subset V(H)$ if $V(P) \cap W = \emptyset$, and *internally disjoint* from W if $(V(P) - \{u, v\}) \cap W = \emptyset$. Note that, although only undirected graphs will be considered, it will sometimes be convenient to speak about u and v as the initial and terminal vertex of P, respectively (or to say that P emanates from u and terminates at v).

Let J and K be subgraphs of H. A u–v path Q will be called a J–K path if $u \in V(J)$, $v \in V(K)$, and Q is internally disjoint from both $V(J)$ and $V(K)$. In the case when $J = \{u\}$ we simply speak of a u–K path.

A one-way infinite path in H will be referred to as a *ray* in H. (If a graph contains no rays, it will be called *rayless*.) If $P = v_0v_1 \cdots v_nv_{n+1} \cdots$ is a ray with initial vertex v_0, then any sub-ray $v_{n+1}v_{n+2} \cdots$ of P will be called a *tail* of P. The ray P will be said to be *dominated* by a vertex $u \in V(H)$ if there are infinitely many u–P paths in H no two of which have a common vertex except u. (Such a vertex u was called 'P-kritisch' in [5], and a 'neighbour of P' in [7].) Equivalently, u dominates P if there is no finite subset $W \subset V(H)$ such that u and a tail of P would lie in different components of the graph H-W; we say briefly that u and P *cannot be separated by a finite set of vertices.*

Two rays P and Q in H will be said to be *equivalent* (denoted by $P \sim_H Q$) if there is a third ray R in H which intersects both P and Q infinitely often. As was shown in [3], $P \sim_H Q$ if and only if there is an infinite set of mutually disjoint P-Q paths (some of which may be trivial one-vertex paths) in H. This is further equivalent to saying that the tails of P and Q cannot be separated by a finite set of vertices of H. It is easy to see that $\sim_H$ is an equivalence relation on the set of all rays in H [3]. An equivalence class under $\sim_H$ is called an *end* of H.

Note that if a ray R in H is dominated by a vertex u, then every ray equivalent to R is dominated by u as well. In this sense we often speak about an end being dominated by a vertex.

Ends of graphs have been studied from many points of view. We shall mainly be interested in forests (and sometimes in spanning trees) which reflect the structure of the ends of a graph. To be more precise, a forest F in a graph H will be said to be *end-faithful* ('coterminal' in [7]) if, for each end ε_H in H, there exists exactly one end ε_F in F such that $\varepsilon_F \subset \varepsilon_H$. Roughly speaking, a forest F is end-faithful in H if each end of H is represented by a unique (up to an initial segment) ray in F.

Finally, if K is a subgraph of H, then the so-called *contracted graph* H/K is obtained by collapsing every component of K to a single vertex and removing all the loops and parallel edges that arose in the process of contraction. More formally, if H-K denotes the subgraph of H induced by the set $V(H) - V(K)$ and $C(K)$ is the set of components of K, then

$$V(H/K) = V(H-K) \cup C(K) \quad \text{and} \quad E(H/K) = E(H-K) \cup \{uB; u \in V(H-K)$$

and $B \in C(K)$ with a vertex $v \in B$ such that $uv \in E(H)\} \cup \{BD; B, D \in C(K)$ such that there are vertices $u \in B$ and $v \in D$ for which $uv \in E(H)\}$. In other words, the contracted graph H/K is the image of a *contraction map* $c: H \to H/K$, which is a graph homomorphism mapping each vertex of a component of K to that component, and is the identity on $H-K$. Observe that the contraction map $c: H \to H/K$ induces a similar map $T \to T/K$ whenever $K \subset T \subset H$, so we may refer to the contracted graph T/K as $c(T)$.

3. Ray domination and rayless spanning trees

As we shall see in Section 4, there is an interesting connection between rayless spanning trees and the extendability of end-faithful forests to end-faithful spanning trees in infinite graphs. In this section we shall concentrate on the implications of ray domination for the existence of rayless spanning trees, and vice versa. The main result, although implicitly contained in [6], is presented here with a new and short proof. Moreover, our method is in a natural way dual to that used in [1] for proving the existence of special end-faithful trees in connected countable graphs. We thus have one more reason to consider raylessness and end-faithfulness as dual properties (see also [7]).

Lemma 1. *Let G be a graph containing a rayless spanning tree. Then every ray in G is dominated.*

Proof. Let T be a rayless spanning tree in G and let P be a ray in G. Pick a vertex v in G and consider the subtree T_v of T consisting of all the paths in T emanating from v and terminating at a vertex of $V(P) - \{v\}$; note that we do not require these paths to be internally disjoint from P. Since for every vertex $u \in V(P)$ there is a unique u–v path in T (possibly the trivial one), every vertex of P belongs to T_v. So, T_v is an infinite subtree of T. The fact that T has been assumed to be rayless implies that T_v must be rayless as well. By König's *Unendlichkeitslemma* (see e.g. [10]) there exists a vertex $w \in V(T_v)$ which has infinite degree in T_v. According to the definition of the tree T_v, every edge $e \in E(T_v)$ incident with w (except, possibly, the one contained in the single w-v

path in T_v) belongs to a $w-P$ path $Q_e \subset T_v$. Clearly, any two Q_e's have only the vertex w in common. Therefore, the infinite set of w–P paths Q_e shows that P is dominated in G by the vertex w. □

Note that, in Lemma 1, no cardinality restriction has been made on the number of vertices of G. The following theorem of [6–7] asserts that the converse of Lemma 1 holds true for countable graphs.

Theorem 2. *Let G be a connected countable graph in which every ray is dominated. Then G contains a rayless spanning tree.*

Proof. Let $V(G) = \{v_0, v_1, v_2, \ldots\}$. Using induction, we shall first construct a nested sequence of trees $T_0 \subset T_1 \subset T_2 \cdots$ in G such that $v_n \in V(T_n)$ for every n. As the initial step, put $T_0 = v_0$. Now assume that the tree T_{n-1} has already been constructed. If $v_n \in V(T_{n-1})$, we simply set $T_n = T_{n-1}$. If $v_n \notin V(T_{n-1})$, let m be the smallest index i for which there exists a v_n-T_{n-1} path $v_n \cdots v_i$, $v_i \in V(T_{n-1})$; its existence is guaranteed by the connectedness of G. Adjoin one of such paths $Q_n = v_n \cdots v_m$ to T_{n-1} to obtain T_n, i.e., let $T_n = T_{n-1} \cup Q_n$.

Having thus finished our inductive construction, let us consider the subgraph

$$T = \bigcup_{n=0}^{\infty} T_n$$

of G. It is clear that T is a spanning tree of G. In what follows we shall prove that T is rayless.

Suppose the contrary, i.e. that T contains a ray P. According to the properties of G, P is dominated in G by a vertex v_k, say. Since only the tails of P will be of interest to us, we shall assume (without loss of generality) that P contains no vertex v_i such that $i \leqslant k$.

We begin with an important observation: For every n, the graph $T_n \cap P$ (if non-empty) is connected. To see this, suppose for a contradiction that $T_n \cap P$ has at least two components C and D. Then there is a C-D path $J \subset T_n$ as well as another C-D path $K \subset P$. Obviously $K \neq J$, since otherwise $K \cup C \cup D \subset T_n \cap P$ and C, D would not be distinct components of the graph $T_n \cap P$. From the facts that $T_n \subset T$ and $P \subset T$ we deduce that the union $C \cup D \cup K \cup J$ is a connected subgraph of T. However, it can readily be seen that the graph $C \cup D \cup K \cup J$ contains a cycle. Therefore, $T_n \cap P$ cannot have more than one component, as claimed.

Now let j be the smallest index for which $T_j \cap P \neq T_{j-1}P \neq \emptyset$; observe that necessarily $v_k \in V(T_{j-1})$. It follows that T_j must have been obtained from T_{j-1} by adding a v_j-T_{j-1} path $Q_j = v_j \cdots v_m$, $v_m \in V(T_{j-1})$. By the connectedness and inclusion of the paths $T_{j-1} \cap P \subset T_j \cap P$, the terminal vertex v_m of Q_j is easily seen to belong to P. Thus, according to the rules of our inductive construction,

$$m = \min M \quad \text{where } M = \{i;\ \text{there is a } v_j\text{-}T_{j-1} \text{ path } v_j \cdots v_i,\ v_i \in V(T_{j-1})\}.$$

Moreover, since v_k dominates P, one can readily find a v_j-v_k path Q_j' avoiding the (finite) set $V(T_{j-1}) - \{v_k\}$. Thus, $m = \min M \leq k$, which contradicts the fact that $m > k$ by our choice of P. This shows that our spanning tree T is rayless, as desired. □

Combining the result just proved with Lemma 1, we immediately obtain the following characterization of countable graphs containing a rayless spanning tree.

Corollary 3. *A connected countable graph G has a rayless spanning tree if and only if every ray in G is dominated.*

During the Infinite Graph Theory conference in Cambridge, 1989, I asked whether the above characterization extends to graphs with arbitrary cardinality. As was shown recently by Seymour and Thomas [8], the answer in general is no, so the problem has to be restated in broader terms: Which graphs among those in which every ray is dominated have rayless spanning trees?

4. Extending and contracting end-faithful forests

In this section we shall investigate the relationship between an end-faithful forest F in a graph G and the corresponding contracted graph G/F, obtained by shrinking every component of F into a single vertex (see Section 2). Our point of departure will be the following characterization theorem of those end-faithful forests that are extendable to end-faithful spanning trees.

Theorem 4. *Let F be an end-faithful forest in a connected graph G. Then F is contained in an end-faithful spanning tree of G if and only if the graph G/F contains a rayless spanning tree.*

Proof. Assume first that T is an end-faithful spanning tree of G such that $F \subset T$. Denote by $c: G \to G/F$ the contraction map and consider the subgraph $c(T)$ in G/F. Obviously, $c(T)$ is a spanning tree in G/F. Suppose that there is a ray Q in $c(T)$. Since T is a tree and F a subforest of T, there is a ray P in T such that $c(P) = Q$. From the fact that both F and T are end-faithful in G it follows that a tail of P is contained in a single component of F. However, this implies that $c(P)$ cannot be a ray in G/F, a contradiction. Thus, $c(T)$ is a rayless spanning tree in G/F.

Conversely, let T_0 be a rayless spanning tree in G/F. As before, let $c: G \to G/F$ be the contraction homomorphism. Let $U = \{c(D); D \text{ a component of } F\}$. For each edge $uv \in E(T_0)$ choose an edge $e_{uv} \in E(G)$ from the set $c^{-1}(uv)$. Obviously, if neither u nor v belongs to U then there is a single edge $e_{uv} = uv$ in $c^{-1}(uv)$. If $u = c(B) \in U$ and $v \neq U$, the edge e_{uv} joins the vertex v to a vertex in the

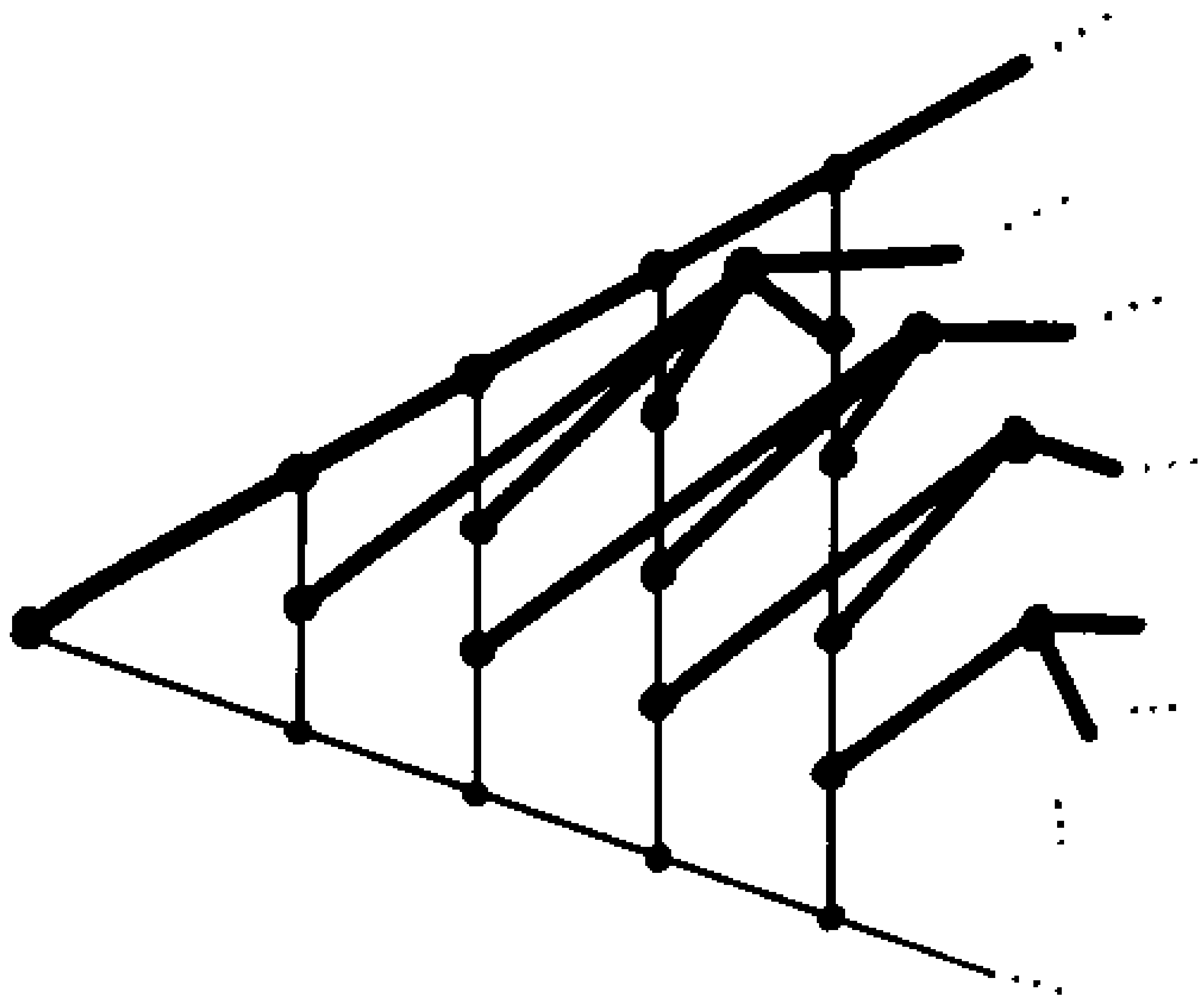

Fig. 1.

component B of F: if also $v = c(D) \in U$ then e_{uv} is one of the edges joining a vertex in B to a vertex in D.

Put $E' = \{e_{uv};\, uv \in E(T_0)\}$, and consider the subgraph T of G generated by the edge set $E(F) \cup E'$. By our construction, T comprises every vertex of G. Moreover, since $c(T) = T_0$ is a tree and the contraction involves connected components of F only, the graph T is connected and cannot contain cycles. Thus, T is a spanning tree of G. (Roughly speaking, we have obtained T from T_0 by 'blowing up' every vertex $c(D) \in U$ to the original component D of F.) It remains to prove that T is end-faithful in G. If there were a ray $R \subset T$ which was not equivalent in T to any ray of F then, clearly, $c(R)$ would be a ray in $c(T) = T_0$ (note that $c(R)$ cannot be finite). This, however, would contradict the choice of T_0. As F $(\subset T)$ represents every end of G, this completes the proof. □

It should be pointed out that, in general, an end-faithful forest need not extend to an end-faithful spanning tree. As an example, consider the graph G depicted in Fig. 1; the forest F is indicated by heavy lines. Note that, as asserted by Theorem 4, the graph G/F (Fig. 2) does not contain a rayless spanning tree.

We already saw in Section 3 that there is a relation between ray domination and the existence of rayless spanning trees. In the light of Theorems 2 and 4, it would be of interest to determine those end-faithful forests $F \subset G$ for which every ray in G/F is dominated. Towards this end we can offer only a sufficient condition. In order to state it, we shall introduce the concept of a ray-sensitive forest.

From now on, it will sometimes be useful to view any forest as a spanning forest. More explicitly, with each forest F in G we associated a new forest $\bar{F}$ by

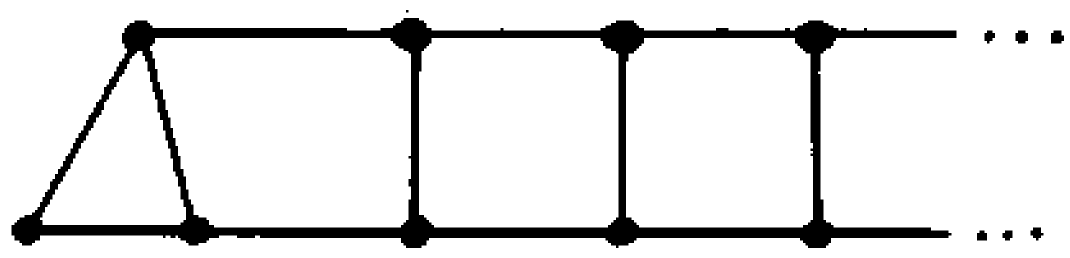

Fig. 2.

declaring every vertex not in F to be an isolated vertex of $\bar{F}$, i.e., setting $V(\bar{F}) = V(G)$ and $E(\bar{F}) = E(F)$.

Let F be an end-faithful forest in G. Then, for every ray P in G there exists a unique component C_P of F containing a ray P_F equivalent in G to P. The forest F will be called *ray-sensitive* if, for every ray P of G, either

(a) P intersects a component of F in an infinite number of vertices, or

(b) no finite set of components of $\bar{F}$ separates P from P_F (i.e., if S is any finite set of components of $\bar{F}$ other than C_P then there is a component of $G - \bigcup S$ which contains tails of both P and P_F).

Let us illustrate this concept by a simple observation.

Lemma 5. *Let F be a locally finite end-faithful forest in a graph G. Then F is ray-sensitive.*

Proof. Suppose the contrary, and let P be a ray in G which intersects no component of F in an infinite number of vertices. Let S be a finite set of components of $\bar{F}$ which separates P from P_F. Since $P \sim_G P_F$, the tails of P and P_F cannot be separated by a finite number of vertices of G. Consequently, if X is an infinite set of mutually disjoint P-P_F paths in G, then at least one component T of F, $T \in S$, intersects an infinite number of paths in X. Pick a vertex $v \in V(T)$ and let T_v be the subtree of T generated by all v-X paths. It follows that T_v is an infinite subtree of T; moreover, T_v is locally finite because of the assumed properties of F. Applying König's *Unendlichkeitslemma* we see that there is a ray R in $T_v \subset T$. But then, considering the paths in X which intersect T_v, it is easy to see that $R \sim_G P$. This, however, contradicts the fact that F is end-faithful in G, since obviously $T \neq C_P$. $\square$

We are now ready to state and prove our sufficient condition for ray domination in contracted graphs.

Theorem 6. *Let F be a ray-sensitive end-faithful forest in a graph G. Then every ray in G/F is dominated.*

Proof. Let $c: G \to G/F$ be the contraction map and let P_0 be a ray in G/F. It is easy to see that there is a ray P in G such that $c(P) = P_0$. Obviously, P cannot intersect a component of F in an infinite number of vertices; otherwise $c(P)$ would not be a ray in G/F. As before, let C_P be the unique component of F which contains a ray equivalent (in G) to P. We claim that the vertex $u = c(C_P)$ dominates the ray $P_0 = c(P)$ in G/F. For suppose this is not the case. Then u can be separated from a tail of P_0 by a finite set S_0 of vertices of G/F. Considering the pre-images of these vertices under the map c, it follows that a tail of P can be separated from the component C_P by a finite number of components of $\bar{F}$, namely, by the elements of the set $S = c^{-1}(S_0)$. But this contradicts the fact that F is ray-sensitive. □

The graph in Fig. 1 shows that if an end-faithful forest F in G is not ray-sensitive then G/F may be such that not every ray is dominated (and so, by Lemma 1, G/F has no rayless spanning tree, and therefore (Theorem 4) F cannot be extended to an end-faithful spanning tree of G). On the other hand, it is clear that the condition 'F is ray-sensitive' is not necessary for G/F to have every ray dominated.

Combining Lemma 5 with Theorem 6 we immediately obtain the following.

Corollary 7. *Let F be a locally finite end-faithful forest in a graph G. Then every ray in G/F is dominated.*

Applying now Theorem 2, the last result yields the following.

Corollary 8. *If F is a locally finite end-faithful forest in a connected countable graph G, then G/F contains a rayless spanning tree.*

By virtue of Theorem 4 we then obtain the following generalization of [9, Theorem 3].

Corollary 9. *Any locally finite end-faithful forest in a connected countable graph can be extended to an end-faithful spanning tree.*

Note that the assumption of local finiteness cannot be dropped from the last statement (see Figs. 1 and 2).

The following example (due to Diestel) shows that, in Theorem 6, the assertion that 'every ray in G/F is dominated' cannot be replaced with 'G/F has a rayless spanning tree' (so that the current version, although weaker than the potential alternative, is best possible, and it is indeed necessary for the proof of Corollaries 8–9 to invoke Theorem 2). Let H be the graph from [8] which shows that Theorem 2 does not extend to uncountable graphs; this graph H is infinitely connected (and thus every ray in it is dominated), but any spanning tree of H has uncountably many ends. Now let F be a ray disjoint from H, and let G be obtained from $H \cup F$ by inserting all edges between F and some fixed ray P in H. Clearly G and G/F are again infinitely connected, so F is an end-faithful and ray-sensitive forest in G (and every ray in G/F is dominated). However, G/F has no rayless spanning tree. For if T is such a tree and v is the vertex of G/F into which F was contracted, then $(T - v) \cup E(P)$ contains a spanning tree of H with at most one end, a contradiction.

We conclude our list of consequences with a result on extending end-faithful trees to end-faithful spanning trees [9, Theorem 2].

Corollary 10. *Let T be an arbitrary end-faithful tree in a connected countable graph G. Then T can be extended to an end-faithful spanning tree of G.*

Proof. By Theorems 6, 2 and 4, it is sufficient to notice that an end-faithful tree in G is necessarily ray-sensitive. Indeed, if P is a ray in G which intersects T in a finite number of vertices only, then a tail of P cannot be separated from a ray $R \subset T$ with $R \sim_G P$ by any finite number of vertices of G (i.e., single-vertex components of $\bar{T}$). □

5. Remarks on a common generalization of end-faithfulness and raylessness

The purpose of our last section is to introduce spanning trees which generalize both rayless and end-faithful ones. The proofs of the results can be obtained by appropriate modifications of the methods used in the previous section and are therefore left to the reader.

Let Ω be an arbitrary subset of the set of ends of a graph G. A forest $F \subset G$ will be said to be *Ω-faithful* if every ray of F belongs to an end in Ω, and for every end $\varepsilon \in \Omega$ there is precisely one end of F contained in ε. Thus an Ω-faithful forest in G is rayless if $\Omega = \emptyset$, and end-faithful in G if Ω comprises all ends of G.

Our result characterizing the end-faithful forests extendable to end-faithful spanning trees (Theorem 4) can now be stated in the following more general form.

Lemma 11. *Let Ω be an arbitrary collection of ends of a connected graph, G, and let F be an Ω-faithful forest in G. Then F can be extended to an Ω-faithful spanning tree of G if and only if the graph G/F contains a rayless spanning tree.*

In Section 4 we introduced the concept of a ray-sensitive forest. No doubt an analogous (but, unfortunately, more complicated) concept could be defined in order to generalize Theorem 6 to Ω-faithful forests. Instead of pursuing this idea, we present here a direct extension of Corollaries 7 and 9. Recall that an end is *dominated* if it contains a ray which is dominated.

Theorem 12. *Let Ω be a set of ends of a graph G, and let F be a locally finite Ω-faithful forest in G. Assume that every end of G which does not belong to Ω is dominated in G. Then every end of the graph G/F is dominated (in G/F).*

Combining the last result with Lemma 11 and Theorem 2 we immediately obtain the following.

Corollary 13. *Let Ω be a set of ends in a connected countable graph G such that every end not in Ω is dominated in G. Let F be an arbitrary locally finite Ω-faithful forest in G. Then F is contained in an Ω-faithful spanning tree of G.*

So far we have been dealing mostly with extending end-faithful or Ω-faithful forests to corresponding spanning trees. Let us conclude with a result refering directly to the existence of an Ω-faithful spanning tree.

Corollary 14. *Let Ω be a countable set of ends in a connected countable graph G. Assume that every end of G which does not belong to Ω is dominated in G. Then G contains an Ω-faithful spanning tree.*

Proof. Using induction it is easy to show that, since Ω is countable, G contains an Ω-faithful forest all of whose components are rays. Apply Corollary 13. □

In this context, it is natural to ask whether the last result extends to uncountable Ω. If so, it would be an interesting generalization of Halin's original result on end-faithful spanning trees in countable graphs [3] on the one hand, and of Theorem 2 on the other.

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