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Fast growing functions based on Ramsey theorems

H.J. Prömel
Forschungsinstitut für Diskrete Mathematik, Universität Bonn, Nassestr. 2, W-5300 Bonn 1, Germany

W. Thumser
Fakultät für Mathematik, Universität Bielefeld, W-4800 Bielefeld 1, Germany

B. Voigt
Fakultät für Mathematik, Universität Bielefeld, W-4800 Bielefeld 1, Germany

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Abstract

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Introduction

Gödel's paper (1931) on formally undecidable propositions of first order Peano Arithmetic showed that any recursive axiomatic system which contains the axioms of Peano Arithmetic still admits propositions which are not decidable, i.e., which are neither provable nor refutable on the basis of the given axiomatic system. For the reader who is not used to work in Peano Arithmetic we mention that (for statements about natural numbers) Peano Arithmetic is equivalent to the result of replacing the axiom of infinity by its negation in the usual axioms of Zermelo-Fraenkel set theory (see, e.g., Jech [10] for these axioms). Gödel's original example of such a proposition was not that illuminating, as it merely formalized the well-known antinomy of the lyer. That raised the problem to find intuitively meaningful propositions which are valid in the 'real world' but which are not provable in Peano arithmetic (of course, such statements should be expressible in terms of first-order logic).

In this paper we present a general framework to define fast growing functions based on Ramsey theorems. This framework is suggested by the work of Ketonen and Solovay [13] and Kanamori and McAloon [12]. One advantage of our approach is that we are dealing just with colorings of pairs. Such colorings may be interpreted as edge colorings of graphs. To appreciate the fast growingness of our functions we need certain sample functions to compare with. These sample functions are defined by ordinal recursion and are in fact generating functions for hierarchies of recursive functions.

In Section 1 we investigate a certain function $\mathrm{KM}:\omega \to \omega$, which has been introduced by Kanamori and McAloon [12]. Using a model theoretic argument they show that this function $\mathrm{KM}(m)$ fails to be primitive recursive. We use a purely combinatorial argument to achieve the same result, thereby obtaining explicit lower bounds. The ideas behind this argument are extended in Section 2, where an auxiliary example of a fast growing function is presented. A general framework which is extracted from the prototypical functions of Section 1 and 2 is provided in Section 3. Section 4, then, contains the main result as well as some applications. One might be tempted to object that the fast growingness of our Ramsey functions is merely due to the implicit use of ordinals in its definition. But this is not quite justified. Other basic approaches to fast growing functions, viz., via well-quasi-ordered sets (cf. Simpson [25]) also implicitly rely on the notion of ordinals.

1. A nonprimitive recursive Ramsey function

In this section we consider a rapidly growing Ramsey function which proves not to be primitive recursive. However, as it turns out, its growth rate is only slightly above all primitive recursively growing functions. The ideas behind our approach, based upon the work of McAloon and Kanamori [12], are generalized in the sequel in order to come up with faster and faster growing functions giving rise to noncollapsing hierarchies (cf., Rose [22]).

If we estimate functions and want to visualize their growth rates we are used to compare with certain well-understood functions which are supposed to be prototypical. So we speak, e.g., of polynomial or exponential growth rate. The kind of functions we are dealing with here are growing much faster. To appreciate this we provide a sample of fast growing functions.

We start with the so called Grzegorczyk hierarchy of primitive recursive functions.

For mappings $F:\omega \to \omega$ we denote by $F^n:\omega \to \omega$ the n-fold iterate of F, $F^{n+1}(x) = F(F^n(x))$ where $F^0(x) = x$. Consider the family $(F_k)_{k<\omega}$ of functions $F_k:\omega \to \omega$ which is defined by

$$F_0(n) = n + 1, \qquad F_{k+1}(n) = F_k^n(n).$$

Note that the functions F_k speed up with a phantastic acceleration, e.g.,

$$F_1(n) = 2 \cdot n, \qquad F_2(n) = 2^n \cdot n, \qquad F_3(n) \geqslant \left.2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}\right\}n \text{ twos}.$$

F_2 is an exponential function, F_3 is called the stack-of-twos function, F_4 is an iteration of the stack-of-twos function, and so forth. Spencer suggests to call F_4 the *wow-function*, as its growth rate is already beyond imagination.

However, all these functions are still primitive recursive. Moreover, for every primitive recursive function $f:\omega\to\omega$ there exist k and n_0 such that $f(n)\leqslant F_k(n)$ for all $n\geqslant n_0$, i.e., the function F_k eventually dominates f. Thus we may define a function $F_\omega:\omega\to\omega$ which is recursive but not primitive recursive by $F_\omega(n)=F_n(n)$. This is an *Ackermann function* (though Ackermann's original definition is somewhat different, compare Peter [20]).

Lemma 1.1. *For every positive integer m there exists a smallest positive integer $n=\mathrm{KM}(m)$ such that for every regressive mapping $d:[{n \atop 2}]\to[n]$, meaning that $d(x,y)<x$, for all $0<x<y<n$, there exists an m-element set $M\in[{n \atop m}]$ such that $d(x,y)=d(x,z)$ for all $x<y<z$ in M.*

Proof. By the Erdős-Rado canonization theorem [4], compare also Graham, Rothschild and Spencer [9], for every mapping $d:[{\omega \atop 2}]\to\omega$ there exists an infinite set $F\in[{\omega \atop \omega}]$ such that $d\rceil[{\omega \atop 2}]$ is a canonical coloring. If d is regressive, i.e., $d(A)<\min A$ for $A\in[{\omega \atop 2}]$ it follows that $d(A)=d(B)$ for all $A,B\in[{F \atop 2}]$ with $\min A=\min B$. Here we use the infinity of F. Alternatively, this may also be established by iterated applications of the pigeon hole principle. Notice that for fixed $x<\omega$ the two element subsets $\{x,y\}$, $x<y$, are partitioned into just x classes. Hence the result follows by a compactness argument. $\square$

Kanamori and McAloon [12] give a model-theoretic proof to show that the function $\mathrm{KM}(m)$ is not primitive recursive. We use a purely combinatorial argument.

Notation. $\mathrm{Ram}(l,m,k)$ is the least integer n such that $n\to(m)^l_k$, this is the ordinary Ramsey function, in other words, for every coloring $d:[{n \atop l}]\to[k]$ there exists an m-element set $M\in[{n \atop m}]$ such that $d(X)=d(Y)$ for all $X,Y\in[{M \atop l}]$.

Lemma 1.2. $\mathrm{KM}(\mathrm{Ram}(2,m+3,k))\geqslant F_k(m)$.

Proof. Let $m^*=\mathrm{Ram}(2,m+3,k)$ and define a regressive mapping $d:[{\mathrm{KM}(m^*) \atop 2}]\to[\mathrm{KM}(m^*)]$ by $d(x,y)=0$ if $F_k(x)\leqslant y$ and $d(x,y)=l$ otherwise, where the numbers $0\leqslant k^*<k$ and $1\leqslant l<x$ are defined by

$$F^l_{k^*}(x)\leqslant y<F^{l+1}_{k^*}(x).$$

Note that this is a proper definition as $F_{k^*}(x)<F_{k^*+1}(x)=F^x_{k^*}(x)$ by definition of the functions F_k.

Let $M^*\in[{\mathrm{KM}(m^*) \atop m^*}]$ be such that $d(x,y)=d(x,z)$ for all $x<y<z$ in M^*. We define a k-coloring $d^*:[{M^* \atop 2}]\to[k]$ by $d^*(x,y)=0$ if $F_k(x)\leqslant y$ and $d(x,y)=k^*$ otherwise, where k^* is as above. Let $M\in[{M^* \atop m+3}]$ be such that $d^*\rceil[{M \atop 2}]$ is a constant coloring and let $x<y<z$ be the three largest elements of M. Then $m\leqslant x$ and

thus it suffices to show that $F_k(x) \leqslant y$. Assume to the contrary that $F_k(x) > y$. Then also $F_k(x) > z$, as $d(x, y) = d(x, z) \neq 0$. Hence also $F_k(y) > z$, as $F_k(x) \leqslant F_k(y)$ (one readily sees that the functions F_k are increasing). Say, $d(x, y) = d(x, z) = l$ and $d^*(x, y) = d^*(x, z) = d^*(y, z) = k^*$. Then

$$F^l_{k^*}(x) \leqslant y < z < F^{l+1}_{k^*}(x).$$

Apply F_{k^*} to this inequality. Then $z < F^{l+1}_{k^*}(x) \leqslant F_{k^*}(y)$. But this contradicts $F_{k^*}(y) \leqslant z$. □

Corollary 1.3. *The function* KM(m) *is not primitive recursive.*

Proof. It is well known that $\mathrm{Ram}(2, m+3, k) \leqslant k^{(m+3)\cdot k}$ (cf. Graham, Rothschild and Spencer [9]), in particular, it is primitive recursive. But $\mathrm{KM}(\mathrm{Ram}(2, m+3, m)) \geqslant F_m(m) = F_\omega(m)$ by the lemma. As primitive recursive functions are closed under composition the assertion follows. □

Let us note that the growth rate of the function KM(m) is approximately that of F_ω, viz., it may be shown that (cf., Prömel and Voigt [21]):

Theorem 1.4. $\mathrm{KM}(m) < F_{m-1}(3) < F_\omega(m)$.

2. A prototypical fast growing Ramsey function

We extend the ideas of the previous section and define a Ramsey function which is growing much faster than all primitive recursive functions. For doing so we need additional sample functions to compare with. To enlarge the family $(F_k)_{k<\omega}$ into the transfinite we first require an effective system of notations for ordinals less than ϵ_0.

We assume that the reader is acquainted with the arithmetic of ordinals, compare, e.g., Bachmann [1].

In the following greek letters denote (countable) ordinals.

Lemma 2.1. *Every ordinal $\alpha > 0$ can be represented uniquely as*

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \cdots + \omega^{\alpha_k} \cdot n_k,$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_k \geqslant 0$ are ordinals and $n_1, \ldots, n_k$ are positive integers. Additionally, for $\alpha < \epsilon_0$ it follows that $\alpha_1 < \alpha$.

Recall that

$$\epsilon_0 = \omega^{\omega^{\cdot^{\cdot^{\cdot}}}} = \lim_{n\to\infty} \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\Big\}{\scriptstyle n \text{ times}}$$

is the first fixed point of the ordinal function $\alpha \to \omega^\alpha$.

The representation of α hinted at in Lemma 2.1 is the *Cantor normal form* of α. A coding of ordinals $\alpha < \epsilon_0$ with positive integers can be defined straightforwardly relying on the Cantor normal form. The details are somewhat technical. We refer, e.g., to Schütte [24].

Next we define *fundamental sequences* which are exploited in the definition of the sample functions F_α, $\alpha < \epsilon_0$. We need these fundamental sequences to handle limit ordinals properly.

To every limit ordinal $\alpha < \epsilon_0$ we associate a strictly monotone sequence $\alpha[n]$, $n < \omega$, which approaches α from below. For convenience we introduce $\alpha[n]$ for successor ordinals as well as for 0.

Let $0[n] = 0$ and let $(\alpha + 1)[n] = \alpha$ for all $n < \omega$. In general, if $\alpha < \epsilon_0$ is given in its Cantor normal form $\alpha = \alpha' + \omega^{\alpha_k} \cdot n_k$, where α_k is the minimal exponent, let

$$\alpha[n] = \begin{cases} \alpha' + \omega^{\alpha_k} \cdot (n_k - 1) + \omega^{\alpha_k[n]} \\ \quad \text{if } \alpha_k \text{ is a limit ordinal,} \\ \alpha' + \omega^{\alpha_k} \cdot (n_k - 1) + \omega^{\alpha_k - 1} \cdot n \\ \quad \text{if } \alpha_k \text{ is a successor ordinal,} \\ \alpha' + n_k - 1 \\ \quad \text{if } \alpha_k = 0. \end{cases}$$

For example, $\omega[n] = n$, $\omega^\omega[n] = \omega^n$, $\omega^{\omega^\omega}[n] = \omega^{\omega^n}$ and $\omega^{\omega^n}[n] = \omega^{\omega^{n-1} \cdot n}$.

With the aid of these fundamental sequences we define functions F_α, $\alpha < \epsilon_0$, by extending the definitions from Section 1:

$$F_0(n) = n + 1, \qquad F_{\alpha+1}(n) = F_\alpha^n(n), \qquad F_\alpha(n) = F_{\alpha[n]}(n)$$

for limit ordinals α.

Finally we define a function F_{ϵ_0} by

$$F_{\epsilon_0}(n) = F_{\gamma_n}(n), \quad \text{where } \gamma_n = \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}\right\} n \text{ times}.$$

This is the so called *Wainer hierarchy* of provably recursive functions (actually, it is a slight modification of Wainer's original approach which is due to Ketonen and Solovay [13]).

The significance of the Wainer hierarchy in connection with unprovability results is emphasized by a result of Wainer which relates the F_α's to the class of ordinal recursive functions. Futhermore, from Kreisel [17] it is known that the provably total recursive functions (provably total with respect to Peano arithmetic) can be characterized in terms of ordinal recursion up to ϵ_0. What we actually need is the following result, compare also Bucholz and Wainer [2].

Theorem 2.2. *Let $f : \omega \to \omega$ be a provably total recursive function (with respect to Peano arithmetic). Then f is eventually dominated by some F_α for an $\alpha < \epsilon_0$. Moreover, F_{ϵ_0} eventually dominates every provably total recursive function but it is itself not provably total recursive.*

The sample functions F_α are used to estimate the growth rate of rapidly growing functions. In particular, if such a function eventually dominates every provably total recursive function then the formula expressing its totality cannot be proved in Peano arithmetic.

Suppose we want to compute $F_\beta(n)$. The computation proceeds along the recursive definition of the F_α's. The set of names of previous functions we need to know is given by

$$N(\beta, n) = \{\beta[n]\underbrace{[n]\cdots[n]}_{i \text{ times}} \mid i < \omega\}.$$

For example,

$$\begin{aligned} N(\omega, n) &= \{n, n-1, \ldots, 0\}, \\ N(\omega^\omega, n) &= \{\omega^n, \omega^{n-1}\cdot n, \omega^{n-1}\cdot(n-1) + \omega^{n-2}\cdot n, \\ &\qquad \omega^{n-1}\cdot(n-1) + \omega^{n-2}\cdot(n-1) + \omega^{n-3}\cdot n, \ldots\} \\ &\subseteq \left\{\sum_{k=0}^{n} \omega^k \cdot n_k \mid n_k \in [n+1]\right\}. \end{aligned}$$

Properties of the fundamental sequences, as defined above, which are relevant in our proof, are summarized in the following observation.

Observation 2.3. (1) *$F_\alpha(k) \leqslant F_\alpha(l)$ for all $\alpha < \epsilon_0$ and positive integers $k \leqslant l$,*

(2) *$F_\alpha(k) \leqslant F_\beta(k)$ for all $\alpha \in N(\beta, k)$,*

(3) *$N(\alpha, k) \subseteq N(\alpha, l)$ for all positive integers $k \leqslant l$.*

Proof. The assertions (1), (2) and (3) resp. are proved by straightforward inductions on α. We show how to prove (3). If $\alpha = \beta + 1$ is a limit number then

$$N(\beta + 1, k) = \{\beta\} \cup N(\beta, k) \subseteq \{\beta\} \cup N(\beta, l) = N(\beta + 1, l),$$

as $N(\beta, k) \subseteq N(\beta, l)$ by induction. Next let α be a limit ordinal, say, $\alpha = \beta + \omega^\gamma$ with $\beta > \omega^\gamma$ and $\gamma > 0$. If $\gamma = \delta + 1$ is a successor then

$$N(\alpha, k) = \{\beta + \omega^\delta \cdot k\} \cup N(\beta + \omega^\delta \cdot k, k) \subseteq \{\beta + \omega^\delta \cdot l\} \cup N(\beta + \omega^\delta \cdot l, l)$$

as $N(\beta + \omega^\delta \cdot k, k) \subseteq N(\beta + \omega^\delta \cdot k, l) \subseteq N(\beta + \omega^\delta \cdot l, l)$ by induction and since

$$\beta + \omega^\delta \cdot k \in N(\beta + \omega^\delta \cdot l, l) \quad \text{for } k < l.$$

If γ is a limit number then

$$N(\alpha, k) = \{\beta + \omega^{\gamma[k]}\} \cup N(\beta + \omega^{\gamma[k]}, k) \subseteq N(\alpha, l)$$

as $N(\gamma, k) \subseteq N(\gamma, l)$ by induction. □

Recall that we assume that a primitive recursive coding of ordinals $\alpha < \epsilon_0$ into ω is available and thus we may talk about mappings into ϵ_0, knowing that these are actually mappings into ω.

Equipped with these tools we define a fast growing Ramsey function KS (Ketonen-Solovay) and then prove that it is actually growing as fast as F_{ϵ_0}.

Lemma 2.4. *Let k and m be positive integers. Then there exists a least positive integer $n = \mathrm{KS}_{\epsilon_0}(k, m)$ such that for every mapping $d:[{n \atop 2}] \to \epsilon_0 \times \omega$ with $d(x, y) \in N(\gamma_k, x) \times [x]$ there exists an m-element set $M \in [{n \atop m}]$ such that for all $x<y<z$ in M*

(1) $d(x, y) = d(x, z)$,

(2) $\alpha \leq \alpha'$, where $d(x, y) = (\alpha, l)$ and $d(y, z) = (\alpha', l')$.

Proof. Again, we use a compactness argument. So one first shows that for every mapping $d:[{\omega \atop 2}] \to \epsilon_0 \times \omega$ such that $d(x, y) \in N(\gamma_k, x) \times [x]$ for all $x<y$ there exists an infinite set $F \in [{\omega \atop \omega}]$ such that (1) and (2) hold for all $x<y<z$ in F.

As in the proof of Lemma 1.1 assertion (1) readily follows from the Erdős-Rado canonization theorem, since the sets $N(\gamma_k, x)$ are finite. Assertion (2), then follows by noting that there does not exist any infinite descending chain of ordinals. □

Lemma 2.5. $\mathrm{KS}_{\epsilon_0}(k, m+3) \geq F_{\gamma_k}(m)$.

Proof. Let the mapping $d:[{\omega \atop 2}] \to \epsilon_0 \times \omega$ be defined by $d(x, y) = (0, 0)$ if $F_{\gamma_k}(x) \leq y$ and $d(x, y) = (\alpha, l)$ otherwise, where $\alpha \in N(\gamma_k, x)$ and $l \in [1, x-1]$ satisfy

$$F_\alpha^l(x) \leq y < F_\alpha^{l+1}(x).$$

One readily sees that α and l are defined properly. Let $M \in [{\omega \atop m+3}]$ satisfy (1) and (2) of Lemma 2.4 and let $x<y<z$ be the three largest elements of M. We show that $d(x, y) = (0, 0)$ or $d(y, z) = (0, 0)$ from which $\mathrm{KS}_{\epsilon_0}(m+3) \geq F_{\gamma_k}(m)$ follows. Assume to the contrary that $d(x, y) = d(x, z) = (\alpha, l)$ and $d(y, z) = (\beta, l')$ with $l, l' > 1$ and $\alpha \leq \beta$. Then

$$F_\alpha^l(x) \leq y < z < F_\alpha^{l+1}(x).$$

We apply F_α to this inequality. By assertion (1) of Observation 2.3 it follows that

$$z < F_\alpha^{l+1}(x) \leq F_\alpha(y). \tag{$*$}$$

By definition of $d(y, z)$ we know that

$$F_\beta^{l'}(y) \leq z. \tag{$**$}$$

But

$$F_\alpha(y) \leq F_\beta(y) \leq F_\beta^{l'}(y) \tag{$***$}$$

by Observation 2.3 and as $l' \geq 1$. Now $(*)$, $(**)$ and $(***)$ produce the obvious contradiction that $z<z$. □

Corollary 2.6. $\mathrm{KS}_{\epsilon_0}(m, m+3) \geq F_{\epsilon_0}(m)$.

This lower bound gives essentially the right order of magnitude, as the following result of Thumser [27] shows.

Theorem 2.7. $\mathrm{KS}_{\epsilon_0}(m, m) \leq F_{\omega^2_{m+1}}(g(m))$, *where g is an appropriate primitive recursive function and*

$$\omega^2_m = \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^2}}}}}\Big\}{m\ \omega\text{'s}}.$$

3. A general framework for developing fast growing Ramsey functions

In this Section we set up a general framework which is sufficient to exploit the ideas introduced so far. What we require are notational systems for initial segments of ordinals, in particular, appropriate selections of fundamental sequences.

3.1. *Some remarks about the second number class, fundamental sequences and hierarchies*

Throughout this chapter we are working within some fixed initial segment Δ of the second number class of ordinals, i.e., all ordinals strictly less than a given countable ordinal. To develop constructively noncollapsing hierarchy classes $\mathcal{F}^{\alpha}$ $(\alpha \in \Delta)$ of functions some requirements concerning the ordinals in Δ must be met. We tacitly suppose all ordinals in Δ to be smaller than the least nonconstructive ordinal ω_1 in the sense of Church and Kleene [3]; otherwise there would even arise problems in ordinal notation. Although there is no problem in defining the initial functions on which our hierarchy is based at successor ordinals (diagonalization will do!) there still remains the question of choosing appropriate fundamental sequences at limit stages. Kleene in his seminal paper [16] succeeded in defining function classes based on his system $\mathcal{O}$ of ordinal notations; nevertheless he did not touch on the problem mentioned before. Despite the facts that recursively well-ordered relations on ω already possess primitive recursive well-orderings of the same order type (Kleene [15]) and that for all known ordinal notations (cf. Schütte [24]) there seems to be a natural choice of fundamental sequences, a general method for obtaining such sequences has not been found yet (and it is most unlikely to be found).

3.2. *The Bachmann property*

3.2.1. *Fundamental sequences for $\lambda < \Delta$*

A sequence $(\lambda(n))_{n<\omega}$ of ordinals in Δ is called a fundamental sequence for λ if (1) $\lambda(n) < \lambda$, (2) $\lambda(n) < \lambda(n+1)$, *and* (3) $\lim_{n<\omega} \lambda(n) = \lambda$. We extend the definition of fundamental sequences to successor ordinals by $(\alpha+1)(n)=\alpha$ for $n<\omega$. For convenience we write $\beta(0)^n$ for $\beta(0)(0)\cdots(0)$, the n-fold interate of the fundamental sequence evaluated at zero.

3.2.2. *Bachmann property*

An assignment of fundamental sequences to all $\lambda<\Delta$, λ a limit ordinal, has the Bachmann property if for all limit ordinals λ, $\mu<\Delta$, and all positive integers $n<\omega$ the inequality $\lambda(n)<\mu\leqslant\lambda(n+1)$ implies that $\lambda(n)\leqslant\mu(0)$.

3.2.3. *Relation* $<_A$

For $\alpha<\beta<\Delta$ we let $\alpha<_A\beta$ if $\alpha=\beta(0)^n$ for some positive $n<\omega$.

3.2.4.

The relation $<_A$ is *built up* if for all limit ordinals $\lambda\in\Delta$, and all positive integers $n<\omega$ it follows that $\lambda(n)<_A\lambda(n+1)$.

Comment. The Bachmann property has been studied in Bachmann [1]. It requires the first term of the fundamental sequence for λ to be as large as possible and enables us to prove certain monotonicity properties of the hierarchy-generating functions thus assuring its noncollapsibility. The concept of built up relations $<_A$ is due to Schmidt [23]. She also discovered its equivalence to the Bachmann property.

Proposition 3.2.1 (Schmidt [23]). *An assignment of fundamental sequences to Δ has the Bachmann property if and only if $<_A$ is built up.*

Proof. By transfinite induction (see Rose [22]). □

Although there is no way to assign fundamental sequences to the whole second number class satisfying the Bachmann property, Schmidt [24] succeeded in constructing such fundamental sequences for each proper initial segment of the second number class.

Theorem 3.2.2 (Schmidt [23]). *For every proper initial segment Δ of the second number class there exists an assignment of fundamental sequences with the Bachmann property.*

It is well known that the Bachmann property implies certain monotonicity properties.

Proposition 3.2.3. *Let an initial segment Δ of the second number class and an assignment of fundamental sequences to Δ satisfying the Bachmann property be* *given. Let* $(H_\alpha)_{\alpha \in A}$ *be a sequence of functions* $H_\alpha : \omega \to \omega$ *with the following properties*:

(i) H_0 *is strictly monotone,*

(ii) $H_{\alpha+1}$ *is strictly monotone whenever* H_α *is strictly monotone,*

(iii) $H_\alpha(0) \leqslant H_{\alpha+1}(0)$ *and* $H_\alpha(x+1) < H_{\alpha+1}(x+1)$,

(iv) $H_\lambda(x) = H_{\lambda(x)}(x)$ *if* λ *is a limit ordinal.*

Then if follows for all $\alpha, \beta \in \Delta$ *that:*

(a) H_α *is strictly monotone,*

(b) *if* $\beta <_A \alpha$ *then* $H_\beta(0) \leqslant H_\alpha(0)$ *and* $H_\beta(x+1) < H_\alpha(x+1)$,

(c) *if* $\beta < \alpha$ *then* H_β *is eventually dominated by* H_α.

Proof. By induction on α this is fairly obvious, where we use Proposition 3.2.1 for part (b). □

3.3. *A strictly increasing hierarchy*

Let Δ be an initial segment of the second number class together with an assignment of fundamental sequences having the Bachmann property. According to Theorem 3.2.2 this is always achievable.

We define the family $(H_\alpha)_{\alpha<\Delta}$ of mappings $H_\alpha : \omega \to \omega$ as follows:

$$H_0(x) = x+1, \qquad H_{\alpha+1}(x) = H_\alpha^{x+1}(x), \qquad H_\alpha(x) = H_{\alpha(x)}(x)$$

for a limit ordinal α.

Proposition 3.3.1. *The functions* H_α *satisfy the conditions of Proposition* 3.2.3 *as well as* $H_\alpha(x) > x$.

Proof. (i) and (iv) are obvious, as is the first half of (iii) because by induction on α it follows that $H_\alpha(0) = 1$. To prove (ii) and the second part of (iii) we note that by transfinite induction on α it follows that $H_\alpha(x) > x$. For α a limit ordinal or $\alpha = 0$ this is obvious, so assume $\alpha = \lambda + 1$. By induction on $z \in \omega \backslash \{0\}$ we have $H_\lambda^z(x) > x$ because $H_\lambda^1(x) > x$ and

$$H_\lambda^{z+1}(x) = H_\lambda(H_\lambda^z(x)) > H_\lambda^z(x) > x,$$

by our main induction. Thus especially for $z = x+1$, $H_\alpha(x) = H_{\lambda+1}(x) = H_\lambda^{x+1}(x) > x$. Now we prove (ii) as follows:

$$\begin{aligned} H_{\alpha+1}(x+1) &= H_\alpha^{x+2}(x+1) = H_\alpha(H_\alpha^{x+1}(x+1)) \\ &> H_\alpha^{x+1}(x+1) > H_\alpha^{x+1}(x) = H_{\alpha+1}(x), \end{aligned}$$

where the last inequality follows from the assumption.

Finally, (iii) part two is proved by noting that

$$H_{\alpha+1}(x+1) = H_\alpha^{x+2}(x+1) = H_\alpha^{x+1}(H_\alpha(x+1)) > H_\alpha(x+1),$$

by our previous observation. □

Remark. The hierarchy corresponding to the family $(H_\alpha)_{\alpha<\Delta}$ is obtained by letting $\mathcal{H}^0 = \mathcal{H}^1$ be the set of elementary functions and by letting $\mathcal{H}^\alpha = E\{H_\beta \mid \beta < \alpha\}$, the elementary closure of functions preceding H_α. Properties (i)–(iv) guarantee that this hierarchy is strictly increasing in the sense that $\mathcal{H}^\alpha$ is a proper subset of $\mathcal{H}^\beta$ for all $2 \leqslant \alpha < \beta < \Delta$, cf., Rose [22, p. 45].

Recall that the families $(F_\alpha)_{\alpha<\omega}$ resp., $(F_\alpha)_{<\epsilon_0}$ introduced in Section 1 and 2 slightly differ from the family of the H_α's as defined above. However, it turns out that $H_\alpha(n) = F_\alpha(n+1) - 1$. So we define the F-functions in a general setting, viz.

$$F_0(n) = n + 1, \qquad F_{\alpha+1}(n) = F_\alpha^n(n), \qquad F_\alpha(n) = F_{\alpha[n]}(n),$$

α a limit ordinal, $\alpha[n] := \alpha(n-1)$ and $n \geqslant 1$. We define $F_\alpha(0) = 1$ for convenience.

Lemma 3.3.2. *For all $n < \omega$ and $\alpha < \Delta$ we have $H_\alpha(n) = F_\alpha(n+1) - 1$.*

Proof. By transfinite induction on α we have the following:

$\alpha = 0$: $F_0(n+1) = n + 2 = (n+1) + 1 = H_0(n) + 1$.

$\alpha = \lambda + 1$: we have $F_\lambda(n+1) = H_\lambda(n) + 1$ and using induction on z:

$$\begin{aligned} F_\lambda^{z+1}(n+1) &= F_\lambda(F_\lambda^z(n+1)) = F_\lambda(H_\lambda^z(n) + 1) \\ &= H_\lambda H_\lambda^z(n) + 1 = H_\lambda^{z+1}(n) + 1, \end{aligned}$$

especially

$$\begin{aligned} F_\alpha(n+1) &= F_{\lambda+1}(n+1) = F_\lambda^{n+1}(n+1) = H_\lambda^{n+1}(n) + 1 \\ &= H_{\lambda+1}(n) + 1 = H_\alpha(n) + 1. \end{aligned}$$

α a limit ordinal:

$$\begin{aligned} F_\alpha(n+1) &= F_{\alpha[n+1]}(n+1) = F_{\alpha(n)}(n+1) \\ &= H_{\alpha(n)}(n) + 1 = H_\alpha(n) + 1. \quad \square \end{aligned}$$

Remark. Generally $\alpha[n]$ fails to have the Bachmann property. However, from the lemma it follows that, in any case, the hierarchies belonging to the families $(H_\alpha)_{\alpha<\Delta}$ and $(F_\alpha)_{\alpha<\Delta}$ resp., are the same. For convenience we state our results in terms of the F_α's. This is possible as the family $(F_\alpha)_{\alpha<\Delta}$ satisfies properties (1), (2) and (3) of Observation 2.3.

Lemma 3.3.3. *$F_\alpha(k) \leqslant F_\alpha(l)$ for all $\alpha < \Delta$ and all positive integers $k \leqslant l$.*

Proof. By Propositions 3.3.1 and 3.2.2 H_α is strictly monotone and $F_\alpha(x) = H_\alpha(x-1) + 1$ for $x \geqslant 1$ by Lemma 3.3.2, so F_α has the property required in the lemma. $\square$

As in the previous section let us define for $\alpha < \Delta$ and $k \geq 1$,

$$N(\alpha, k) := \{\alpha[k]^i \mid i \geq 1\}.$$

Lemma 3.3.4. *Clearly, $N(\alpha, k)$ is a finite set, moreover, $F_\alpha(k) \leq F_\beta(k)$ for all $\alpha \in N(\beta, k)$, $\beta < \Delta$ and $k \geq 1$.*

Proof. Observe, that $F_\alpha(k) = H_\alpha(k-1) + 1 > k - 1 + 1 = k$ because of Lemma 3.3.2 and Proposition 3.3.1. As $\alpha = \beta[k]^i$, $i > 0$ we use induction on i.

$$i = 1: \qquad F_\beta(k) = \begin{cases} F_{\beta[k]}(k) \\ F^k_{\beta[k]}(k) \end{cases} \geq F_\alpha(k)$$

according to whether β is a limit ordinal or not ($\beta = 0$ being trivial). For the inductive step the same argument works. □

Lemma 3.3.5. *$N(\alpha, k) \subseteq N(\alpha, l)$ for all $\alpha \in \Delta$, $1 \leq k \leq l$.*

Proof. It suffices to prove that $N(\alpha, k) \subseteq N(\alpha, k+1)$. We use transfinite induction on α: $\alpha = 0$ is trivial. Suppose $\alpha = \beta + 1$.

$$\begin{aligned} N(\alpha, k) = N(\beta+1, k) &= N(\beta, k) \cup \{\beta\} \subseteq N(\beta, k+1) \cup \{b\} \\ &= N(\beta+1, k+1) = N(\alpha, k+1). \end{aligned}$$

Finally, let α be a limit ordinal. The inductive assumption gives

$$N(\alpha[k], k) \subseteq N(\alpha[k], k+1).$$

So it suffices to show that $\alpha[k] \in N(\alpha, k+1)$. Since $\alpha[k] < \alpha[k+1]$ choose $i \geq 1$ maximal with respect to $\alpha[k] < \alpha[k+1]^i \leq \alpha[k+1]$. $\alpha[k+1]^i$ being a successor ordinal implies either $\alpha[k+1]^{i+1} = \alpha[k]$ and we are through or otherwise contradicts the maximality of i. So suppose $\alpha[k+1]^i =: \lambda$ is a limit number. We show, that this case cannot occur. Otherwise, using the Bachmann property we would get

$$\alpha[k] \leq \lambda(0) = \lambda[1] = \alpha[k+1]^i[1] < \alpha[k+1]^{i+1},$$

which again would contradict the maximality of i. □

Let us summarize the preceding lemmas in the following theorem.

Theorem 3.3.6. *Let an arbitrary initial segment Δ of the second number class together with an assignment of fundamental sequences satisfying the Bachmann property be given. Define $\alpha[n+1] := \alpha(n)$, for $n \geq 0$ and define the family $(F_\alpha)_{\alpha<\Delta}$ by*

$$F_0(n) = n + 1, \qquad F_{\alpha+1}(n) = F^n_\alpha(n), \qquad F_\alpha(n) = F_{\alpha[n]}(n),$$

where $F_\alpha(0)=1$. Then $(F_\alpha)_{\alpha<\Delta}$ defines a strictly increasing hierarchy, extending Grzegorczyk's hierarchy, and satisfies:

(1) $F_\alpha(k) \leqslant F_\alpha(l)$ *for all* $1 \leqslant k \leqslant l < \omega$ *and* $\alpha < \Delta$.

(2) $F_\alpha(k) \leqslant F_\beta(k)$ *for all* $k < \omega$ *and* $\alpha \leqslant \beta < \Delta$.

(3) $N(\alpha, k) \subseteq N(\alpha, l)$ *for all* $1 \leqslant k \leqslant l < \omega$ *and* $\alpha < \Delta$.

4. Main theorem and applications

4.1. Main theorem

In order to state our theorem let us fix some segment $\Delta = \lambda + 1$ of the second number class together with an assignment of fundamental sequences ($\lambda_n := \lambda[n]$). Appealing to a coding system (recall our tacit assumption that Δ should be a constructive ordinal), thus all following mappings actually map into ω.

Lemma 4.1.1. *Let k and m be positive integers. Then there exists a smallest positive integer $n = \mathrm{KS}_\lambda(k, m)$ such that for every mapping $d : [{n \atop 2}] \to \lambda \times \omega$ with $d(x, y) \in N(\lambda_k, x) \times [x]$ there exists an m-element set $M \in [{n \atop m}]$ such that for all $x<y<z$ in M it follows that:*

(1) $d(x, y) = d(x, z)$,

(2) $\alpha \leqslant \alpha'$ *for* $d(x, y) = (\alpha, l)$, $d(y, z) = (\alpha', l')$.

Proof. Exactly as in Section 2. □

Theorem 4.1.2. $\mathrm{KS}_\lambda(k, m+3) \geqslant F_{\lambda_k}(m)$, $\mathrm{KS}_\lambda(m, m+3) \geqslant F_\lambda(m)$.

Proof. The essential conditions used in the proof of Lemma 2.5 are established in Theorem 3.3.6. We omit to repeat the arguments. □

4.2. Applications

4.2.1. KS_ω

Let $\lambda = \omega$ and define $\lambda(n) = \lambda[n+1] = n+1$. The function KS_ω fails to be primitive recursive. Comparing the definitions one readily observes that $\mathrm{KS}_\omega(1, m) = \mathrm{KM}(m)$. From Corollary 1.3 it follows that already $\mathrm{KS}_\omega(1, m)$ is not primitive recursive, whereas Theorem 4.1.2 just asserts that $\mathrm{KS}_\omega(1, m) \geqslant 2 \cdot m$. Thus one may wonder about the real growth rate of the function $\mathrm{KS}_\omega(k, m)$. It turns out that the lower bound of Theorem 4.1.2 is not too bad, as one may show that $\mathrm{KS}_\omega(k, m) \leqslant F_{\omega^{1+k}}(k \cdot m)$ (Thumser [27]).

4.2.2. KS_{ϵ_0}

Let $\lambda = \epsilon_0$ and associate fundamental sequences with λ as in Section 2 using the Cantor normal form; however, we let

$$(\alpha' + \omega^{\alpha_k} \cdot n_k)(n) = \alpha' + \omega^{\alpha_k} \cdot (n_k - 1) + \omega^{\alpha_k - 1}(n+1)$$

if α_k is a successor ordinal. A straightforward calculation shows that the above defined assignment has the Bachmann property, cf. also Rose [22, p. 78]. The sample functions $(F_\alpha)_{\alpha<\epsilon}$ as defined in Section 3 rely on the shifted assignment $\alpha[n+1]=\alpha(n)$. This square bracket assignment proves to be the assignment used in Section 2, as $\omega^{\beta+1}[n]=\omega^\beta\cdot(n-1)=\omega^\beta\cdot n$.

Although the predicate '$\mathrm{KS}_{\epsilon_0}(k,m)=n$' is primitive recursive (the number of color-checks can be bounded by an elementary function in k, m, n) and therefore is expressible in Peano arithmetic, there is no way of proving $\forall k, m\, \exists n \mathrm{KS}_{\epsilon_0}(k,m)=n$ in PA because of the rapid growth of KS_{ϵ_0}. As we mentioned in Section 2 no problems arise in coding ordinals less than ϵ_0 to get a system of notations satisfying the conditions of the last section: we simply use Cantor's normal form. Recalling $\epsilon_{0_k}=\epsilon_0[k]=\gamma_k$ we see in this particular case that

$$\mathrm{KS}_{\epsilon_0}(k, m+3)\geq F_{\gamma_k}(m).$$

We also know from Parsons [19], that the class of functions provably recursive in PA_{k-1}, where induction is restricted to formulas having at most $(k-1)$-nested quantifiers, is identical to the class $\mathscr{F}^{\gamma_k}$. So it is impossible to prove Lemma 4.1.1 within this restricted theory where k is supposed to be fixed.

4.2.3. KS_{Γ_0}

To extend our results further into the transfinite, let us choose a notational system sufficient to provide expressions for all ordinals less than the first strongly critical ordinal Γ_0. This ordinal was introduced by Feferman [6] and is analyzed in great detail in Schütte [24]. Γ_0 is also known to be the proof theoretical ordinal of Feferman's system of predicative analysis [5] as well as of Friedman's system ATR_0. It is shown in Friedman [8] that both systems are able to prove the same Π_1^1 sentences and a great deal of classical mathematics can be developed in either of them. The notational system to be employed is based upon some basic facts, which we want the reader to recall. Let $\Phi_\alpha:\mathrm{Ord}\to\mathrm{Ord}$ (Ord denotes the set of countable ordinals in any of the usual set theories) be given by

$$\Phi_0(\alpha)=\omega^\alpha$$

$$\Phi_\beta(\alpha)=\text{the } \alpha\text{th common fixed point of all } \Phi_\gamma,\ \gamma<\beta$$

(We have $\Phi_1(\alpha)=\epsilon_\alpha$).

Γ_0 can be characterized as the least ordinal satisfying $\Phi_{\Gamma_0}(0)=\Gamma_0$.
We want to make some comments concerning Fig. 1. All entries are so called principle ordinals, i.e. those $\alpha\in\mathrm{Ord}\,(\alpha\neq 0)$ which satisfy $\beta+\alpha=\alpha$ for all $\beta<\alpha$. They are listed in increasing order in the first row by the function $\Phi_0(\gamma)=\omega^\gamma$. Principal ordinals are indecomposable with respect to ordinal addition and each ordinal $\gamma\neq 0$ may be uniquely written as a decreasing, finite ordinal sum built up by principal ordinals (Cantor's normal form theorem). The

$$
\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & \cdots & \omega & \cdots & \epsilon_0 & \cdots \\
0 & 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^\omega & \cdots & \epsilon_0 & \cdots \\
1 & \epsilon_0 & \epsilon_1 & \epsilon_2 & \cdots & & & & & \\
\vdots & \vdots & & & & & & & & \\
\omega & \Phi_\omega(0) & \cdots & & & & & & & \\
\vdots & \vdots & & & & & & & & \\
\Gamma_0 & \Phi_{\Gamma_0}(0) = \Gamma_0 & & & & & & & &
\end{array}
$$

Fig. 1.

next theorem gives an appropriate choice and tells us which entries should be used in order to avoid infinite regress. To be more precise we take a closer look at the diagram and list all occurrences of some principal ordinal α, see Fig. 2. Up to some point δ we have $\Phi_\gamma(\alpha) = \alpha$ for all $\gamma < \delta$. These entries are of no help if we want a notational system, they would be denoted by themselves. So we take hold on the circled $\alpha = \phi_\delta(\beta)$ hoping $\delta < \alpha$ in order to continue. But

$$\alpha = \phi_\delta(\beta) \geqslant \phi_\delta(0) \geqslant \delta,$$

and $\delta < \alpha$ firstly may fail as soon as $\alpha = \delta$, which implies $\beta = 0$ and $\alpha = \phi_\alpha(0) = \Gamma_0$ by definition. Up to Γ_0 the process requires no infinite regress and can be used to prove Theorem 4.3.1. The existence of the circled α entry follows from the fact that $\phi_\alpha(\alpha) > \phi_\alpha(0) \geqslant \alpha$, where we use the normality (strict monotonicity and continuity) of $\phi_\alpha(\cdot)$ and $\phi_{(\cdot)}(\alpha)$. It should be noted however that generally $\phi_{(\cdot)}(\alpha)$, $\alpha \neq 0$, fails to be continuous: otherwise the inequalities $\phi_n(0) < \phi_n(1) < \phi_{n+1}(0)$ would imply $\phi_\omega(0) = \phi_\omega(1)$, contradicting the strict monotonicity of ϕ_ω.

The following theorem can be used to establish a convenient notational system and coding into ω, cf. Schütte [24].

$$
\begin{array}{cccccccccc}
 & 0 & 1 & 1 & \cdots & \beta & \cdots & \alpha & \cdots \\
0 & & & & & & & \alpha & \\
1 & & & & & & & \alpha & \\
2 & & & & & & & \alpha & \\
\vdots & & & & & \alpha & & \vdots & \\
\delta & & & & & & & & \\
\vdots & & & & & & & & \\
\alpha & & & & & & & & \\
\vdots & & & & & & & &
\end{array}
$$

Fig. 2.

Theorem 4.3.1. *For each ordinal* $0 \neq \gamma < \Gamma_0$ *there exist uniquely determined ordinals* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ $(n \geq 1)$ *where* $\alpha_i, \beta_i < \Phi_{\alpha_i}\beta_i$ $(i = 1, \ldots, n)$ *and* $\Phi_{\alpha_1}\beta_1 \geq \cdots \geq \Phi_{\alpha_n}\beta_n$ *and* $\gamma = \Phi_{\alpha_1}\beta_1 + \cdots + \Phi_{\alpha_n}\beta_n$.

Using this kind of Cantor normal form, we are now able to assign fundamental sequences to limit ordinals $\gamma < \Gamma_0$ by concentrating the definition always on the rightmost term $\Phi_{\alpha_n}\beta_n$. In detail we proceed as follows by transfinite induction (λ, μ are supposed to denote limit ordinals):

$$\begin{aligned}
&\Phi_0(\beta + 1)(n) = \Phi_0(\beta) \cdot (n + 2),\\
&\Phi_0(\lambda)(n) = \Phi_0(\lambda(n)),\\
&\Phi_{\alpha+1}(0)(n) = \Phi_\alpha^{n+1}(0),\\
&\Phi_{\alpha+1}(\beta + 1)(n) = \Phi_\alpha^{n+1}(\Phi_{\alpha+1}(\beta) + 1),\\
&\Phi_{\alpha+1}(\lambda)(n) = \Phi_{\alpha+1}(\lambda(n)),\\
&\Phi_\lambda(0)(n) = \Phi_{\lambda(n)}(0),\\
&\Phi_\lambda(\beta + 1)(n) = \Phi_{\lambda(n)}(\Phi_\lambda(\beta) + 1),\\
&\Phi_\lambda(\mu)(n) = \Phi_\lambda(\mu(n)).
\end{aligned}$$

Because of the normality of the functions $\Phi_{\lambda(\cdot)}$ the last definition provides us with a fundamental sequence converging to $\Phi_{\lambda(\mu)}$ assuming already one converging to μ. Let us justify the second-last definition (the other definitions may be established in a similar way).

Lemma 4.3.2. $\Phi_{\lambda(n)}(\Phi_\lambda(\beta) + 1) \rightarrow \Phi_\lambda(\beta + 1)$ *whenever* $\lambda(n) \rightarrow \lambda$, *where* $\rightarrow$ *denotes convergence.*

Proof. Let $\gamma := \sup_{n<\omega} \Phi_{\lambda(n)}(\Phi_\lambda(\beta) + 1)$. We claim that $\gamma \leq \phi_\lambda(\beta + 1)$. Certainly $\Phi_\lambda(\beta) + 1 < \Phi_\lambda(\beta + 1)$ since $\Phi_\lambda(\beta + 1)$ is principal. It follows that

$$\Phi_{\lambda(n)}(\Phi_\lambda(\beta) + 1) < \Phi_{\lambda(n)}(\Phi_\lambda(\beta + 1)) = \Phi_\lambda(\beta + 1)$$

for all $n < \omega$ because $\Phi_\lambda(\beta + 1)$ is a fixed point of all $\Phi_\delta(\delta < \lambda)$.

By definition of γ it follows that $\gamma \leq \Phi_\lambda(\beta + 1)$ establishing our claim. On the other hand $\gamma > \Phi_\lambda(\beta)$ and it remains to prove that γ is a fixed point of all $\Phi_\delta(\delta < \lambda)$. But

$$\begin{aligned}
\Phi_\delta(\gamma) &= \Phi_\delta(\sup \Phi_{\lambda(n)}(\Phi_\lambda(\beta) + 1))\\
&= \sup_{n<\omega} \Phi_\delta \Phi_{\lambda(n)}(\Phi_\lambda(\beta) + 1) = \sup_{n<\omega, \lambda(n)>\delta} \Phi_{\lambda(n)}(\Phi_\lambda(\beta) + 1) = \gamma,
\end{aligned}$$

where the second equality holds since Φ_δ is continuous. Therefore γ is a fixed point of all $\Phi_\delta(\cdot)$, $\delta < \lambda$, larger than their βth fixed point yielding $\Phi_\lambda(\beta + 1) \leq \gamma$, which together with our claim finishes the proof. $\square$

Schmidt [23] gives a slightly different, although equivalent definition.

Theorem 4.3.3. *The assignment of fundamental sequences to limit ordinals $\lambda < \Gamma_0$ given above determines a built up system of ordinal notation and therefore has the Bachmann property.*

It is straightforward but somewhat tedious to check the other requirements of notational systems we defined above. We would nevertheless like to mention that reasonable bounds for the cardinality of $N(\lambda_k, x)$ (where λ_k denotes the k-fold first argument iterate of $\Phi(\alpha, 0) := \Phi_\alpha(0)$ evaluated at 0) can be found. To deduce the following corollary one needs the analogue of Theorem 2.2 with respect to the system ATR_0, cf., Friedman, McAloon and Simpson [8].

Corollary 4.3.4. *The Ramsey function K_{Γ_0} grows faster than any recursive function f for which $\mathrm{ATR}_0 \vdash f$ is total.*

Analogous examples of even faster grower Ramsey functions may be constructed following the patterns of this paper.

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