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Asymmetrization of infinite trees

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Abstract

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An *asymmetrizing set* of a tree T is a set A of vertices of T such that the identity is the only automorphism of T which stabilizes A. The *asymmetrizing number* of T is the cardinality of the set of orbits of asymmetrizing sets of T. We complete the results of Polat and Sabidussi (1991) by characterizing the asymmetrizable trees containing a double ray, and by proving that the asymmetrizing number of such a tree T (resp. of any tree T containing a ray but no double ray) is the product of the asymmetrizing numbers of the components of $T \setminus T_*$, where T_* is the union of all double rays (resp. is some ray) of T. We show that, if the asymmetrizing number of a tree is infinite, then it is of the form 2^κ for some cardinal κ. Besides, given two cardinals κ and λ with $\kappa \leqslant \lambda$ we present, on the one hand, a set of 2^λ non-isomorphic rayless trees of cardinality λ and asymmetrizing number 2^κ, and on the other hand, a set of 2^κ non-isomorphic asymmetric (i.e. having no non-identity automorphism) rayless graphs of cardinality κ.

Introduction

Given a tree T, a set A of vertices of T is said to be *asymmetrizing* if the identity is the only automorphism of T which stabilizes A. The *asymmetrizing number* $a(T)$ of T is the cardinality of the set of orbits of asymmetrizing sets of T. In [3], also appearing in this volume, Polat and Sabidussi investigated the *asymmetrizing problem for trees,* i.e. to determine whether a tree has an asymmetrizing set.

The case of rayless trees (trees without infinite path) and that of trees which have no endpoints, i.e. which are the union of their double rays (two-way infinite paths), were completely settled in this paper. In particular, it was shown that an endpoint-free tree T is asymmetrizable, i.e. has an asymmetrizing set, if and only if its asymmetrizing number is $2^{|T|}$, and a very simple characterization of those asymmetrizable trees was given.

In this paper we solve the general case of trees containing a ray by reducing this case to that of rayless trees. We prove that the asymmetrizing number of a tree T containing a ray but no double ray (resp. containing a double ray and which is asymmetrizable) is the product of the asymmetrizing numbers of the components of the forest obtained by deleting the edges of some ray (resp. of the double rays) of T. Besides we get a characterization of the asymmetrizable trees containing a double ray, quite similar to that of endpoint-free trees. These results (Theorems 3.1 and 3.5) complete in some way the study of the asymmetrization problem. Moreover, as several theorems in graph theory, Theorem 3.5 turns out to be 'self-refining', i.e. it yields a result (Proposition 3.8) stronger than itself. Finally, with this theorem and some new results about rayless trees (Section 3), we prove that if the asymmetrizing number of a tree is infinite, then it is of the form 2^{κ} for some cardinal κ.

1. Notation and definitions

The terminology and notation will be for the most part that used in [3] with the following differences.

1.1. Given a tree T, the *core* T_* of T is defined as follows: if T is rayless, then $T_* = \emptyset$; if T is one-ended, then T_* is an arbitrary (but fixed) ray of T starting at a vertex which must be chosen among the set of vertices fixed by any autmorphism of T if this set is non-empty (this definition is slightly different from that in [3]); if T has a double ray, then T_* is the union of all double rays. If $T_* \neq \emptyset$, then, for $x \in V(T_*)$, we denote by T^x the component of $T \setminus T_*$ containing x.

1.2. If G is a subgroup of Aut T and $A \subseteq V$ then the *G-similarity class* (i.e., G-orbit) of A is $G[A] := \{\sigma A: \sigma \in G\}$. We shall say that A is a *G-asymmetrizing set* of T if $G \cap \mathrm{Aut}(T, A)$ consists of the identity alone. By $\mathscr{A}_G(T)$ we denote the set of G-asymmetrizing sets of T. T is *G-asymmetrizable* if it has a G-asymmetrizing set. Note that $\mathrm{Aut}(T, V \setminus A) = \mathrm{Aut}(T, A)$ for any $A \subseteq V$. Hence the complement of a G-asymmetrizing set is likewise G-asymmetrizing.

The *G-asymmetrizing number* of T, denoted by $a_G(T)$, is the maximum number of mutually non-G-similar G-asymmetrizing subsets of V. It is immediate from the definitions that $a_G(T) \leqslant 2^{|T|}$.

If $G = \mathrm{Aut}\, T$ the reference to G will be omitted. Thus we will write similar, asymmetrizing, $\mathscr{A}(T)$ and $a(T)$ for G-similar, G-asymmetrizing, $\mathscr{A}_G(T)$ and $a_G(T)$, respectively.

1.3. Given a rooted tree (T, w) its *automorphism group* is $\mathrm{Aut}(T, w) := \{\sigma \in \mathrm{Aut}\, T: \sigma w = w\}$, i.e., the stabilizer of $\{w\}$. If G is a subgroup of Aut T we shall usually write G_w for $G \cap \mathrm{Aut}(T, w)$. The case $G = \mathrm{Aut}\, T$ (hence $G_w = \mathrm{Aut}(T, w)$) is particularly important. With this assumption we set the following definitions and notations: two vertices or sets of vertices are *similar in* (T, w) if they are G_w-similar in T; $A \subseteq V$ is an *asymmetrizing set* of (T, w) if it is a

G_w-asymmetrizing set of T; (T, w) is *asymmetrizable* if T is G_w-asymmetrizable; we denote $\mathscr{A}_{G_w}(T)$ and $a_{G_w}(T)$ by $\mathscr{A}_w(T)$ and $a_w(T)$, respectively. Notice that, *if w is a fixed point of T, then* $\mathrm{Aut}(T, w) = \mathrm{Aut}\, T$, *hence* $a(T) = a_w(T)$.

1.4. Let (T, w) be a rooted tree, G a subgroup of $\mathrm{Aut}(T, w)$, and $x \in V \setminus \{w\}$. The *G-multiplicity* of x in (T, w), denoted by $m_G(x)$, is the number of vertices in the G_w-orbit of x which have the same lower neighbor as x. Clearly, G-similar vertices have the same G-multiplicity. If $G = \mathrm{Aut}(T, w)$, we will write *multiplicity* and $m(x)$ for G-multiplicity and $m_G(x)$, respectively.

2. Rayless trees

Proposition 2.1. *Let (T, w) be a rayless rooted tree such that $a_w(T)$ is infinite. Then $a_w(T) = 2^\kappa$ for some cardinal κ.*

Proof. Suppose $a_w(T) \neq 2^\kappa$ for any cardinal κ. We will define by induction a sequence $(w_n)_{n \geq 0}$ of vertices of T such that $w_{n+1} \in V(w_n; T_{w_n})$ and $a_{w_n}(T_{w_n})$ is infinite and $\neq 2^\kappa$ for any cardinal κ.

Let $w_0 := w$. Suppose that w_n is defined for some $n \geq 0$. Since $a_{w_n}(T_{w_n})$ is infinite and $\neq 2^\kappa$ for any κ, there is, by [3, Theorem 2.3], a neighbor x of w_n in T_{w_n} such that $a_x(T_x)$ is infinite and $\neq 2^\kappa$ for any κ. Set $w_{n+1} := \kappa$.

$\langle w_0, w_1, \ldots \rangle$ is then a ray of T, a contradiction since T is rayless. Thus $a_w(T) = 2^\kappa$ for some cardinal κ. $\square$

Corollary 2.2. *Let T be a rayless tree such that $a(T)$ is infinite. Then $a(T) = 2^\kappa$ for some cardinal κ.*

Proof. T has a fixed vertex or a fixed edge, since it is rayless (see [4]). Then let w be a fixed vertex of T if there is some, or an endpoint of its fixed edge otherwise. The result is then a consequence of 2.1 since $a(T) = a_w(T)$ by 1.3 and [3, Proposition 2.5]. $\square$

Lemma 2.3. *For any infinite cardinal λ there is a set $\mathbf{N}_\lambda$ of 2^λ non-isomorphic rayless rooted trees (T, w) such that $|T| = \lambda$ and $a_w(T) = 2$.*

Proof. In this proof, as well as in that of Proposition 2.4, we will use the following operation. Given a family $(T_i, w_i)_{i \in I}$ of rooted trees, we define the *sum* of this family as the rooted tree (T, w) obtained by forming the disjoint union of the trees (T_i, w_i), and joining each w_i to a new vertex w. Note that the sum of any family of rayless trees is rayless.

We first recall Remark 3.4(i) of [3]: *For any ordinal α there is a set $\mathbf{M}_\alpha$ of $\beth_{\alpha+1}$ non-isomorphic rayless rooted trees (T, w) such that $|T| = \beth_\alpha$ and $a_w(T) = 2$.*

This lemma is then a simple extension of that remark with $N_\lambda = M_\alpha$ when $\lambda = \beth_\alpha$. Suppose $\beth_\alpha < \lambda < \beth_{\alpha+1}$, and let $M_\alpha = \{(T_i, w_i): i \in I\}$. For $J \in \binom{I}{\lambda}$ let (T_J, w_J) be the sum of two copies of every (T_j, w_j) with $j \in J$. Then, by Theorem 2.2, $a_{w_J}(T) = 2$. We may therefore take $N_\alpha := \{(T_J, w_J): J \in \binom{I}{\lambda}\}$, since clearly $|N_\lambda| = 2^\lambda$. □

Proposition 2.4. *For any infinite cardinals κ and λ, with $\kappa \leqslant \lambda$, there is a set of 2^λ non-isomorphic rayless trees T such that $|T| = \lambda$ and $a(T) = 2^\kappa$.*

Proof. (a) We will first prove that, for any infinite cardinal κ there is a rayless tree T_κ such that $|T_\kappa| = \kappa$ and $a(T_\kappa) = 2^\kappa$.

Define (T_ω, w_ω) as the sum of the family $(P_n, a_n)_{n<\omega}$, where (P_n, a_n) is the path of length n rooted at one of its endpoints. Let $\kappa \geqslant \omega$. Suppose that (T_λ, w_λ) is defined for any cardinal $\lambda < \kappa$. Then:

- if κ is a limit cardinal, (T_κ, w_κ) is the sum of the family $(T_\lambda, w_\lambda)_{\omega \leqslant \lambda < \kappa}$;
- if $\kappa = \lambda^+$, then (T_κ, w_κ) is the sum of κ copies of (T_λ, w_λ).

In each case, T_κ is clearly rayless and of cardinality κ. Furthermore, $a(T_\kappa) = a_{w_\kappa}(T_\kappa)$ since w_κ is a fixed point of T_κ. We have then to prove that $a_{w_\kappa}(T_\kappa) = 2^\kappa$. The proof goes by induction on κ, using in each case Theorem 2.3 of [3].

$$a_{w_\omega}(T_\omega) = 2 \prod_{n<\omega} \binom{a_{w_n}(P_n)}{1} = 2 \prod_{n<\omega} 2^{n+1} = 2^\omega.$$

If κ is a limit ordinal $>\omega$, then

$$a_{w_\kappa}(T_\kappa) = 2 \prod_{\omega \leqslant \lambda < \kappa} a_{w_\lambda}(T_\lambda) = 2 \prod_{\omega \leqslant \lambda < \kappa} 2^\lambda = 2^\kappa.$$

If $\kappa = \lambda^+$, then

$$a_{w_\kappa}(T_\kappa) = 2a_{w_\lambda}(T_\lambda)^\kappa = 2(2^\lambda)^\kappa = 2^\kappa.$$

(b) Now let $\lambda \geqslant \kappa$, and let $N_\lambda = \{(T_i, w_i): i \in I\}$ be the set defined in the preceding lemma. Let $i \in I$, and denote by w_κ some vertex of T_κ. Now let $(T_{(i)}, w_{(i)})$ be the sum of (T_κ, w_κ) with two copies of (T_i, w_i). Then $|T_{(i)}| = \lambda$, and by [3, Theorem 2.3] and the fact that the vertex $w_{(i)}$ is a fixed point of $T_{(i)}$, $a(T_{(i)}) = a_{w_{(i)}}(T_{(i)}) = 2^\kappa$. Since $|I| = 2^\lambda$, the set $\{T_{(i)}: i \in I\}$ is then a solution to the statement. □

From the existence of asymmetrizing rayless trees with given cardinality and asymmetrizing number, we can easily obtain a result about the existence of asymmetric (i.e. having only the identity as automorphism, or equivalently such that the empty set is asymmetrizing) rayless graphs of given cardinality.

Proposition 2.5. *For any infinite cardinal κ there is a set of 2^κ non-isomorphic asymmetric rayless graphs of cardinality κ.*

Proof. By Proposition 2.4 there is an asymmetrizing rayless tree T of cardinality κ and asymmetrizing number 2^κ. Let $(A_\alpha)_{\alpha<2^\kappa}$ be a family of pairwise nonsimilar asymmetrizing sets of T. Denote by C a cycle $\langle w_0, w_1, w_2, w_3, w_4\rangle$ of length 5 with in addition the edge $[w_1, w_3]$. The identity is then the only automorphism of C fixing w_0. Now, for all $\alpha<2^\kappa$, let $(C_x^\alpha)_{x\in A_\alpha}$ be a family of pairwise disjoint graphs such that $V(C_x^\alpha \cap T)=\{x\}$ and (C_x^α, x) is isomorphic with (C, w_0); and finally let $G_\alpha := T \cup \bigcup_{x\in A_\alpha} C_x^\alpha$.

The graphs G_α are rayless, and pairwise non-isomorphic since so are the asymmetrizing sets A_α. Besides the restriction of any automorphism σ of G_α to the set $V(T)$ belongs to the stabilizer of A_α, and then it is the identity since A_α is asymmetrizing; hence $\sigma=1$ since $\sigma|\, V(C_x^\alpha)=1$ for any vertex x of T. □

3. Trees containing a ray

Theorem 2.3 of [3] completely settles the case of finite trees (see [3, Algorithm 3.1]), and in a least obvious way that of rayless infinite trees. We will now reduce the case of trees containing a ray to that of rayless trees. We will first improve substantially Proposition 5.2 of [3], which was the only result concerning one-ended trees without fixed point. Following what was done in [3], the vertex-set of a one-ended tree T will be endowed with the partial order defined by

$$x \leqslant y \text{ if and only if } R_x \subseteq R_y,$$

where, for a vertex x, R_x denotes the unique ray of T which starts at x.

Theorem 3.1. *Let T be a one-ended tree, and $R=\langle w_0, w_1, \ldots\rangle$ a ray of T. Then*

$$a_{w_0}(T)=\prod_{n\geqslant 0} a_{w_n}(T_n),$$

where, for $n\geqslant 0$, T_n denotes the component T^{w_n} of $T\backslash R$ containing w_n. Furthermore, $a(T)=a_{w_0}(T)$ if w_0 is a fixed point when T is not fixed-point free.

Proof. We introduce a few notations. For $n\geqslant 0$:
- T_n' is the subtree of T induced by the set of vertices $\geqslant w_n$;
- $T_{n+1}' := \bigcup\{T_{n+1}(x) : x\in U_{w_{n+1}}\backslash G_{w_{n+1}}[w_n]\}$;
- $T_0'' := T_0' = T_0$;
- $s_n := s_{w_n}(T_n)$, $s_n' := s_{w_n}(T_n')$, $s_n'' := s_{w_n}(T_n'')$, and $m_n := |G_{w_{n+1}}[w_n]|$.

Notice that, since the ray R is fixed by any element of G_{w_0}, if A is an asymmetrizing set of (T, w_0) then, for any $n\geqslant 0$, $A\cap V(T_n)\in \mathcal{A}_{w_n}(T_n)$; and conversely any set $\bigcup_{n\geqslant 0} A_n$ with $A_n\in \mathcal{A}_{w_n}(T_n)$, $n\geqslant 0$, is an asymmetrizing set of (T, w_0). Hence

$$a(T)\leqslant a_{w_0}(T)=\prod_{n\geqslant 0} a_n.$$

We are done if w_0 is a fixed point. Assume that T has no fixed point. We distinguish three cases.

Case 1: $a'_n = 0$ *for some* $n \geq 0$.

$a'_n = 0$ implies that there are $p \geq n$ and an automorphism s such that $s(w_i) = w_i$ if and only if $i \geq p$. Hence $a_p = 0$, and

$$0 \leq a(T) \leq a_{w_0}(T) = \prod_{n \geq 0} a_n = 0.$$

Case 2: $0 < a'_n < \omega$ *for every* $n \geq 0$.

Then clearly $0 < a_n < \omega$ for any $n \geq 0$. Thus $a_{w_0}(T) = \prod_{n \geq 0} a_n = 2^\omega$, since any positive asymmetrizing number is ≥ 2. Besides, for any $n \geq 0$, $a_{w_n}(T) = a'_n \prod_{i > n} a_i \neq 0$, thus $a(T) \geq 2^\omega$ by [3, Lemma 5.1]; hence $a(T) = a_{w_0}(T) = 2^\omega$.

Case 3. *All* a'_n *are* >0 *and at least one of them is infinite.*

Then there is a p such that a'_n is infinite for every $n \geq p$. Without loss of generality we can suppose that a_n is infinite for any n. We will prove that $a(T) \geq \prod_{n \geq 0} a_n$. We define by induction sets $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$ of cardinality a'_n and a_n of pairwise nonsimilar asymmetrizing sets of (T_n, w_n) and (T'_n, w_n), respectively.

Since a_0 is infinite, there are two disjoint sets $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$ of cardinality a_0 such that their union is a set of pairwise nonsimilar asymmetrizing sets of (T_0, w_0). Suppose that $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$ are defined for some $n \geq 0$. Let $\boldsymbol{C}_n$ be a set of cardinality a''_{n+1} of pairwise nonsimilar asymmetrizing sets of (T''_{n+1}, w_{n+1}).

If $m_n = 1$ then $a_{n+1} = a''_{n+1}$; let:

$$\boldsymbol{A}_{n+1} := \{A \cup C : A \in \boldsymbol{A}_n \text{ and } C \in \boldsymbol{C}_n\},$$

$$\boldsymbol{B}_{n+1} := \boldsymbol{C}_n.$$

Assume now that $m_n > 1$. Since a'_n is infinite, $a'_{n+1} = a'^{m_n}_n \cdot a''_{n+1}$. For every $x \in G_{w_{n+1}}[w_n]$ denote by σ_x an element of $G_{w_{n+1}}$ such that $\sigma_x(w_n) = x$. Since $|\boldsymbol{A}_n| = a'_n$ is infinite and $m_n \leq a'_n$ by Remark 2.1(iii) of [3], there is a family $(f_{\xi,n})_{\xi < a'^{m_n}_n}$ of injections from $G_{w_{n+1}}[w_n]$ into $\boldsymbol{A}_n$ such that their images are pairwise disjoint. Finally let

$$\boldsymbol{D}_n := \{\bigcup \{\sigma_x \circ f_{\xi,n}(x) : x \in G_{w_{n+1}}[w_n]\} \text{ and } \xi < a'^{m_n}_n\},$$

$$\boldsymbol{A}_{n+1} := \{C \cup D : C \in \boldsymbol{C}_n \text{ and } D \in \boldsymbol{D}_n\},$$

$$\boldsymbol{B}_{n+1} := \{A \cap V(T'_{n+1}) : A \in \boldsymbol{A}_{n+1}\}.$$

In both cases $\boldsymbol{A}_{n+1}$ and $\boldsymbol{B}_{n+1}$ are then sets of cardinality a'_{n+1} and a_{n+1} of pairwise nonsimilar asymmetrizing sets of (T'_{n+1}, w_{n+1}) and (T_{n+1}, w_{n+1}), respectively.

Let $B := \bigcup_{n \geq 0} B_n$, where $B_n \in \boldsymbol{B}_n$, $n \geq 0$. If $\sigma \in \mathrm{Aut}(T, B)$, then $\sigma B_0 = B_0$ since, by the definitions of $\boldsymbol{A}_0$ and $\boldsymbol{B}_0$, for any $n > 0$ and $x \in G_{w_n}[w_0]$, $B_n \cap V(T_x)$ is not similar with B_0. Thus $\sigma w_0 = w_0$, and this implies that σ fixes the ray R. Hence $\sigma B_n = B_n$ for any $n \geq 0$. Therefore the restriction of σ to $V(T'_n)$ is the identity, since B_n is an asymmetrizing set of T. Furthermore, by their definitions, two different such asymmetrizing sets are nonsimilar. Hence

$$a(T) \geq \prod_{n \geq 0} |\boldsymbol{B}_n| = \prod_{n \geq 0} a_n.$$

(c) Therefore

$$a(T) = a_{w_0}(T) = \prod_{n \geq 0} a_n.$$

And this completes the proof. $\square$

We get immediately the following.

Corollary 3.2. *Let T be a one-ended tree without fixed point. Then $a(T) = a_w(T)$ for any vertex w.*

3.3. To get a similar result for trees containing a double ray, we have to extend in a way some results of [3] about upward extendable trees. The next lemma is obtained by relativization of different results of Section 4 of [3] to a particular subgroup of automorphisms. Its proof, which is exactly the same as those given in [3], with G instead of Aut T, is thus omitted. We recall that, for a rooted tree (T, w), if G is a subgroup of Aut T, then $G_w := G \cap \mathrm{Aut}(T, w)$.

Given a tree T we will denote by $\mathcal{G}(T)$ the set of subgroups G of Aut T satisfying the following condition: if $\sigma \in \mathrm{Aut}\, T$ is such that σx belongs to the G-orbit $G[x]$ of x for every $x \in V$, then $\sigma \in G$.

Lemma 3.4. *Let (T, w) be an upward extendable tree, and $G \in \mathcal{G}(T)$. The following are equivalent*:

(i) *T is G_w-asymmetrizable*;
(ii) *T is G-asymmetrizable*;
(iii) $a_{G_w}(T) = 2^{|T|}$;
(iv) $a_G(T) = 2^{|T|}$;
(v) $m_G(x) \leq 2^{|T_x|}$ *for every* $x \in V \setminus \{w\}$.

Theorem 3.5. *Let T be a tree containing a double ray, such that (T^x, x) is asymmetrizable for every $x \in V(T_*)$; and let $w \in V(T_*)$. The following are equivalent*:

(i) *(T, w) is asymmetrizable*;
(ii) *T is asymmetrizable*;
(iii) $a_w(T) = \prod_{x \in V(T_*)} a_x(T^x)$;
(iv) $a(T) = \prod_{x \in V(T_*)} a_x(T^x)$;
(v) $m(x) \leq \prod_{y \in V(T_* \cap T_x)} a_y(T^y)$ *for every* $x \in V(T_*) \setminus \{w\}$.

Proof. (a) Let $x \in V(T_*)$. The subtree T^x is rayless, and asymmetrizable by hypothesis. Define the cardinal κ_x in the following way: If $a_x(T^x)$ is infinite, then by Proposition 2.1 it is of the form 2^κ for some cardinal κ; put $\kappa_x := \kappa$. If, otherwise, $a_x(T^x)$ is finite, then put $\kappa_x := 1$.

Define then the family $(T^{(x)})_{x\in V(T_*)}$ of pairwise disjoint trees such that, $T^{(x)}$ is the tree reduced to the vertex x if $\kappa_x = 1$, and is a regular tree of degree κ_x with $V(T^{(x)} \cap T_*) = \{x\}$, otherwise. For any $x \in V(T_*)$, $|T^{(x)}| = \kappa_x$ and $a_x(T^{(x)}) = 2^{\kappa_x}$ (by [1, 3.2] or by [3, 4.12] any regular tree T is asymmetrizable with $a(T) = 2^{|T|}$, moreover, this is a simple consequence of Lemma 3.4 with $G = \operatorname{Aut} T$, since condition (v) is trivially satisfied).

Now let $T' := T_* \cup \bigcup_{x\in V(T_*)} T^{(x)}$. Clearly $(T')_* = T'$,

$$|T'| = \sum_{x\in V(T_*)} \kappa_x \quad \text{and} \quad |(T')_x| = \sum_{y\in V(T_*\cap T_x)} \kappa_y \quad \text{for any } x \in V(T_*).$$

Finally let $G := \{\sigma \in \operatorname{Aut} T' \colon \sigma|\, V(T_*) = \sigma'|\, V(T_*) \text{ for some } \sigma' \in \operatorname{Aut} T\}$. G is an element of $\mathcal{G}(T')$. Indeed, if an automorphism σ of T' is such that $\sigma x \in G[x]$ for every $x \in V(T')$, then on the one hand, by the definition of G, $V(T_*)$ is invariant with respect to σ; and on the other hand, (T^x, x) and $(T^{\sigma x}, \sigma x)$ are isomorphic for any $x \in V(T_*)$. Hence there is $\sigma' \in \operatorname{Aut} T$ such that $\sigma|\, V(T_*) = \sigma'|\, V(T_*)$.

(b) Notice that $a_w(T) \leqslant \prod_{x\in V(T_*)} a_x(T^x)$. This is obvious since, for any $x \in V(T_*)$, the intersection of any asymmetrizing set of T with $V(T^x)$ is an asymmetrizing set of (T^x, x). More generally

$$a_x(T^x) \leqslant \prod_{y\in V(T_*\cap T_x)} a_y(T^y) \quad \text{for every } x \in V(T_*).$$

(c) If (T, w) is asymmetrizable then T' is G-asymmetrizable:

Suppose that T' is not G-asymmetrizable. Then, by 3.4(v), there is $x \in V(T')\setminus\{w\}$ such that

$$m_{G_w}(x) > 2^{|(T')_x|} = \prod_{y\in V(T_*\cap T_x)} 2^{\kappa_y} = \prod_{y\in V(T_*\cap T_x)} a_y(T^y).$$

Thus $x \in V(T_*)$ since T^y is asymmetrizable for every $y \in V(T_*)$. Hence

$$m(x) = m_{G_w}(x) > \prod_{y\in V(T_*\cap T_x)} a_y(T^y) \geqslant a_x(T_x)$$

by (b). Thus, by [3, Remark 2.1(iii)], (T, w) is not asymmetrizable.

(d) Also, if T' is G-asymmetrizable, then T is asymmetrizable with

$$a(T) = a_G(T') = \prod_{x\in V(T_*)} a_x(T^x):$$

We will associate with every $x \in V(T_*)$ two sets $\boldsymbol{R}(x)$ and $\boldsymbol{R}'(x)$.

Case 1: $\kappa_x \neq 1$.

Let $\boldsymbol{R}(x)$ (resp. $\boldsymbol{R}'(x)$) be a set of one representative of each similarity class of asymmetrizing sets of (T^x, x) (resp. $(T^{(x)}, x)$) which do not contain x. Since $\boldsymbol{R}'(x)$ and $\boldsymbol{R}(x)$ have the same cardinality 2^{κ_x}, there is a bijection f_x between these two sets. These bijections can be chosen so that, if x and y are similar in T, then, for any $A \in \boldsymbol{R}'(x)$ and $B \in \boldsymbol{R}'(y)$, A and B are G-similar in T' if and only if $f_x(A)$ and $f_y(B)$ are similar in T.

Case 2: $\kappa_x = 1$.

Let $\boldsymbol{R}'(x) = \{\emptyset\}$, and $\boldsymbol{R}(x)$ be the set having as single element an asymmetrizing set of (T^x, x) which does not contain x. And let f_x be the bijection between $\boldsymbol{R}'(x)$ and $\boldsymbol{R}(x)$. Let

$$\mathcal{A}' := \{A \in \mathcal{A}_G(T') : A \cap (V(T^{(x)}) \setminus \{x\}) \in \boldsymbol{R}'(x) \text{ for any } x \in V(T_*)\}.$$

For any G-asymmetrizing set A of T' there is obviously an $A' \in \mathcal{A}'$ which is G-similar with A. Hence $a_G(T')$ is equal to the cardinality of the set of G-similarity classes of elements of $\mathcal{A}'$.

For any $A \in \mathcal{A}'$ define

$$A_f := A \cap V(T_*) \cup \bigcup \{f_x(A \cap (V(T^{(x)}) \setminus \{x\})) : x \in V(T_*)\}.$$

Let A, $B \in \mathcal{A}'$ be such that $B_f = \sigma[A_f]$ for some $\sigma \in \operatorname{Aut} T$. Since T_* is stable with respect to $\operatorname{Aut} T$, we have $B \cap V(T_*) = \sigma[A \cap V(T_*)]$. Let $x \in V(T_*)$ and $y = \sigma x$. Then

$$f_y(B \cap (V(T^{(y)}) \setminus \{y\})) = \sigma[f_x(A \cap (V(T^{(x)}) \setminus \{x\}))],$$

thus, by the definition of f_x and f_y, there is an automorphism σ_{xy} between $(T^{(x)}, x)$ and $(T^{(y)}, y)$. Hence

$$\sigma^* := \sigma | V(T_*) \cup \bigcup \{\sigma_{x,\sigma x} : x \in V(T_*)\}$$

is an element of G such that $B = \sigma^*[A]$.

Therefore, on the one hand, taking $A = B$, we get $\sigma^* = 1$ since A is G-asymmetrizing. Thus $\sigma | V(T_*) = \sigma^* | V(T_*) = 1$, and for any $x \in V(T_*)$, $\sigma | V(T^x) = 1$ since $A_f \cap (V(T^x)$ is an asymmetrizing set of (T^x, x). Hence $\sigma = 1$, and A_f is an asymmetrizing set of T.

On the other hand, if A and B are non-G-similar elements of $\mathcal{A}'$, then A_f and B_f are nonsimilar asymmetrizing sets of T. Thus

$$a(T) \geq a_G(T') = 2^{|T'|} = \prod_{x \in V(T_*)} 2^{\kappa_x} = \prod_{x \in V(T_*)} a_x(T^x)$$

by 3.4 and (a), hence $a(T) = \prod_{x \in V(T_*)} a_x(T^x) = a_G(T')$ by (b).

(e) We can now prove the theorem. The implications (iv)$\Rightarrow$(ii)$\Rightarrow$(i)$\Leftarrow$(iii) are obvious. (i)$\Rightarrow$(ii)$\Rightarrow$(iv) are consequences of (c) and (d). (iv)$\Rightarrow$(iii) is a consequence of (b) and of the inequality $a(T) \leq a_w(T)$. (iii)$\Rightarrow$(v) since, by [3, Remark 2.1(iii)], if (T, w) is asymmetrizable, then by (b),

$$m(x) \leq a_x(T^x) \leq \prod_{y \in V(T_* \cap T_x)} a_y(T^y) \quad \text{for every } x \in V(T_*) \setminus \{w\}.$$

Conversely assume that

$$m(x) \leqslant \prod_{y \in V(T_* \cap T_x)} a_y(T^y) = 2^{|(T')_x|} \quad \text{for every } x \in V(T_*) \setminus \{w\}.$$

Since $(T(y), y)$ is asymmetrizing for every $y \in V(T_*)$, then $m_G(x) \leqslant 2^{|(T')_x|}$ for every $x \in V(T_*) \setminus \{w\}$. Hence, by Lemma 3.4, T' is G-asymmetrizable. Therefore T is asymmetrizable by (d). Thus (v) $\Rightarrow$ (ii). And this completes the proof. □

With this last result we are now able to prove the assertion announced at the end of [3].

Corollary 3.6. *If an asymmetrizable tree T contains a ray, then $a(T) = 2^\kappa$ for some cardinal κ.*

Proof. By Theorems 3.1 and 3.5 $a(T) = \prod_{x \in V(T_*)} a_x(T^x)$. The result is then a consequence of the facts that, if $a_x(T^x)$ is finite then $a_x(T^x) \geqslant 2$, and if $a_x(T^x)$ is infinite then $a_x(T^x) = 2^{\kappa_x}$ for some cardinal κ_x by Proposition 2.1. □

Remark 3.7. Theorem 3.5 could have been stated in a more general form by considering a subgroup $G \in \mathscr{G}(T)$ instead of Aut T itself. Obvious modifications in the given proof would have led to the extended result. But we will show that this generalization (Proposition 3.8) can easily be obtained by applying Theorem 3.5 to a modified tree. Notice that Theorem 3.5 itself was proved by applying Lemma 3.4 in that manner. Therefore, since Proposition 3.8 is also a generalization of that lemma, we can say that Lemma 3.4 and Theorem 3.5 are 'self-refining'.

Proposition 3.8. *Let T be a tree containing a double ray, and $G \in \mathscr{G}(T)$, such that T^x is G_x-asymmetrizable for every $x \in V(T_*)$; and let $w \in V(T_*)$. The following are equivalent:*

(i) *T is G_w-asymmetrizable;*
(ii) *T is G-asymmetrizable;*
(iii) $a_{G_w}(T) = \prod_{x \in V(T_*)} a_{G_x}(T^x)$;
(iv) $a_G(T) = \prod_{x \in V(T_*)} a_{G_x}(T^x)$;
(v) $m_{G_w}(x) \leqslant \prod_{y \in V(T_* \cap T_x)} a_{G_y}(T^y)$ *for every $x \in V(T_*) \setminus \{w\}$.*

Proof. (a) Let $(S^x)_{x \in V}$ be a family of pairwise disjoint rayless trees such that, $V(S^x \cap T) = \{x\}$, (S^x, x) is isomorphic with an element of the set N_λ defined in Lemma 2.3, where λ is a cardinal $> |T|$, and such that (S^x, x) and (S^y, y) are similar if and only if x and y are G-similar. Notice that $a_x(S^x) = 2$, and by the definition of N_λ, x is the only vertex of S^x whose degree is λ. Define then

$$T_G := T \cup \bigcup_{x \in V} S^x.$$

Clearly $(T_G)_* = T_*$, and $(T_G)^x := T^x \cup \bigcup_{y\in V(T_*)} S^y$ for every $x \in V(T_*)$. Besides the vertices of T are those of T_G whose degrees are λ. Hence, by the last property of the S^x, $\sigma x \in G[x]$ for any automorphism σ of T_G and any $x \in V$. Thus, by the definition of $\mathscr{G}(T)$, the restriction of σ to the set V is an element of G; and conversely any element σ of G can be extended to an automorphism σ_G of T_G by taking the identity on the complement of V.

(b) Let $A \in \mathscr{A}(T_G)$ and $\sigma \in G \cap \mathrm{Aut}(T, A \cap V)$. Then $\sigma_G \in \mathrm{Aut}(T_G, A)$, thus $\sigma_G = 1$; hence $\sigma = 1$ which implies that $A \cap V \in \mathscr{A}_G(T)$. Besides, if $B = \sigma A$ for some $\sigma \in \mathrm{Aut}\, T_G$ then $B \cap V = (\sigma|\, V)[A]$. Hence $a_G(T) \leqslant a(T_G)$.

Conversely if $A \in \mathscr{A}_G(T)$ and, for every $x \in V$, A_x is an asymmetrizing set of (S^x, x) which does not contain x, then

$$A_G := A \cup \bigcup_{x\in V} A_x \in \mathscr{A}(T_G).$$

Notice that, for any $x \in V$, $a_x(S^x) = 2$ implies that any two asymmetrizing sets of (S^x, x) not containing x are $\mathrm{Aut}(S^x, x)$-similar. Hence, if A, $B \in \mathscr{A}_G(T)$ are G-similar, then A_G and B_G are similar. Thus $a(T_G) \leqslant a_G(T)$.

Therefore we proved that T is G-asymmetrizable if and only if T_G is asymmetrizable, with $a_G(T) = a(T_G)$. Using exactly the same arguments we can prove that, on the one hand T is G_w-asymmetrizable if and only if (T_G, w) is asymmetrizable, with $a_{G_w}(T) = a_w(T_G)$; and on the other hand, for any $x \in V(T_*)$, $(T_G)^x$ is asymmetrizable since T^x is G_x-asymmetrizable, with $a_{G_x}(T^x) = a_x(T^x)$.

Consequently we get the equivalence of conditions (i), (ii), (iii) and (iv) of Proposition 3.8 by that of the respective conditions of Theorem 3.5.

Now T is G-asymmetrizable if and only if T_G is asymmetrizable, hence by Theorem 3.5 if and only if

$$m(x) \leqslant \prod_{y\in V((T_G)_*\cap(T_G)_x)} a_y((T_G)^y) \quad \text{for every } x \in V(T_*)\setminus\{w\};$$

thus, since $(T_G)_* \cap (T_G)_x = T_* \cap T_x$ and $m_{G_w}(x) = m(x)$, if and only if

$$m_{G_w}(x) \leqslant \prod_{y\in V(T_*\cap T_x)} a_{G_y}(T^y) \quad \text{for every } x \in V(T_*)\setminus\{w\}.$$

This proves the equivalence of conditions (i) and (v), and so completes the proof. $\square$

References

[1] L. Babai, Asymmetric trees with two prescribed degrees. Acta Math. Hungar. 29 (1977) 193–200.

[2] R. Halin, Über unendliche Wege in Graphen, Math. Ann. 157 (1964) 125–137.

[3] N. Polat and G. Sabidussi, Asymmetrising sets in trees, Discrete Math. 95 (this Vol.) 1991 271–289.

[4] N. Polat and G. Sabidussi, Fixed elements of infinite trees (DMS 89-1, Université de Montréal, 1989, preprint), Discrete Appl. Math., to appear.