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Asymmetrising sets in trees

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Abstract

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A set A of vertices of a graph G is *asymmetrising* if the identity is the only automorphism of G which stabilises A; G is *asymmetrisable* if it contains such a set. In this paper we investigate the existence of asymmetrising sets in trees.

We show that it is sufficient to consider rooted trees: given any (finite or infinite) tree T there is a vertex w such that T is asymmetrisable if and only if the rooted tree (T, w) is asymmetrisable. In the case of finite trees there is an n^3-algorithm for deciding the existence of an asymmetrising set and for determining the number of similarity classes of such sets. In the infinite case we obtain a characterisation of the asymmetrisable trees without endpoints. It follows from this characterisation that any regular tree, as well as any endpoint-free tree whose cardinal does not exceed the continuum, is asymmetrisable.

1. Introduction, definitions

A construction frequently encountered in graph theory is the extension of a graph by a new vertex (*one-point extension*). Let X be a graph, A a set of vertices of X, z a vertex not in X. The extension Y is obtained by adding to X all edges joining z to the vertices in A. Consider the automorphism group of Y. Clearly, the subgroup of Aut Y which fixes z is the same as the stabiliser Aut(X, A) of A in Aut X. Of particular interest are those extensions whose automorphism group coincides with Aut(X, A), in other words those for which z is a fixed point (*fixed point extensions*).

A general question which can be raised in this context is whether a given subgroup of Aut X can be the stabiliser of some set $A \subset V(X)$. In the situation which arises most frequently, the subgroup is Aut X itself and the existence of A is trivial. We shall deal with the opposite extreme, where the given subgroup is

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trivial. Call a set $A \subset V(X)$ *asymmetrising* if the identity is the only automorphism of X which stabilises A. Clearly some graphs (e.g. asymmetric ones) have asymmetrising sets while others do not (e.g. the complete bipartite graphs K_{mn} with $m+n \geq 4$). One may therefore pose the *asymmetrisation problem*: Given a graph X determine whether it has an asymmetrising set. In other words, this is the same as asking whether the vertices of X admit a bicolouration such that the only colour-preserving automorphism of X is the identity.

In this paper we propose to investigate the asymmetrisation problem for trees. Questions of complexity will be of no concern as we shall deal primarily with infinite trees. One feature which makes the asymmetrisation problem particularly attractive in the case of trees is that—with the exception of one rather special situation—every one-point extension is a fixed point extension. In a tree, asking for an asymmetrising set therefore amounts to asking for an asymmetric one-point extension. The exception may occur when the tree T contains a vertex which pairwise separates the vertices in A. The subtree of T generated by A then consists of paths having an endpoint and nothing else in common, and A is called *starlike*. Apart from this case, the group of the extension, Aut Y, and the stabiliser Aut(T, A) are isomorphic.

In order to obtain an asymmetrising set for a tree T the natural approach is to root T at some vertex w and to try to construct an asymmetrising set for the rooted tree (T, w). It is therefore important to know whether one can conclude the existence of an asymmetrising set of T when such a set is known for (T, w). As is shown by the simple example of $K_{1,3}$ rooted at one of its endpoints, this need not be so when the root is arbitrary. Part of the paper is devoted to showing that in any tree one can always find a vertex w such that T has an asymmetrising set whenever there is one for (T, w) (Remark 2.4(i), Theorem 4.2, Proposition 5.2).

The basic distinction to make is between trees containing a ray (one-way infinite path) and those which are rayless. The latter are not considered in this paper, except the finite trees for which we obtain—not very surprisingly—a polynomial time algorithm solving the asymmetrisation problem (Section 3). Infinite rayless trees are dealt with in [5]. The main part of the paper is Section 4 which is concerned with trees that have no endpoints, i.e., are the union of their double rays (two-way infinite paths). We show that such a tree T either has no asymmetrising set or else the number of essentially different asymmetrising sets (which cannot be transformed into one another by automorphisms) is $2^{|T|}$. From this we obtain a characterisation of endpoint-free trees having an asymmetrising set from which in turn some significant sufficient conditions for the existence of such sets can be deduced. For example, all regular trees and all endpoint-free trees of order $\leq 2^{\aleph_0}$ have asymmetrising sets. For regular trees this result has already been obtained by Babai [1]. Finally, Section 5 deals with trees which contain a ray but no double ray. There remains the general case of trees which have double rays and endpoints. This case is not considered in this paper; for a treatment we refer the reader to [5].

Not unexpectedly, certain set theoretic hypotheses are related to the existence of asymmetrising sets. For example, the continuum hypothesis is equivalent to the statement that there is no asymmetrising set in the tree formed by $\aleph_2$ rays having precisely their endpoints in common. We have avoided such questions and are strictly working in ZFC.

Notation, Definitions 1.1. (i) Throughout, T will be a (finite or infinite) tree. We denote its vertex-set by V, occasionally by $V(T)$. The set of neighbours of $x \in V$ will be denoted by $V(x; T)$. If T' is a subgraph of T, we denote by $T \setminus T'$ the subgraph of T which consists of the edges of T not belonging to T', together with their incident vertices.

A *ray* is a 1-way infinite path, a *double ray* a 2-way infinite path. T is *rayless* if it contains no ray. A tree containing a ray but no double ray (i.e., having exactly one end in the sense of Halin [2]) will be called *one-ended*.

Given a tree T, the *core* T_* of T is defined as follows: if T is rayless, then $T_* = \emptyset$; if T is one-ended, then T_* is an arbitrary (but fixed) ray starting at some endpoint of T; if T has a double ray, then T_* is the union of all double rays. Note that any one-ended tree has an endpoint.

(ii) Given a rooted tree (T, w) denote by $\leqslant_w$ the natural partial order on V in which w is the least element. For $A \subset V$ and $x \in V$ put

$$A|_x := \{y \in A : x \leqslant_w y\}.$$

By T_x we denote the subtree of T induced by $V|_x$, called the *restriction of* T to x. With this notation, $A|_x = A \cap V(T_x)$. $U_x := V(x; T)|_x = V(x; T_x)$ is the set of *upper neighbours* of x. Of course, $U_w = V(w; T)$. Every $x \in V \setminus \{w\}$ has a unique *lower neighbour*, i.e., a vertex v such that $x \in U_v$.

The distance of a vertex $x \in V$ to the root w will be called the *height* of x, denoted by $h(x)$.

(iii) By Aut T we denote the *automorphism group* of T. If G is a subgroup of Aut T and $A \subset V$ then the *G-similarity class* (i.e., G-orbit) of A is $\{\sigma A : \sigma \in G\}$. Two subsets A, B of V are *G-similar* if there is a $\sigma \in G$ such that $\sigma A = B$. G-similarity of two vertices $x, y \in V$ is defined analogously. If $G = \text{Aut}\, T$ the reference to G will be omitted. The term 'similarity class of T' will be used to mean '(Aut T)-similarity class of some subset of V'. The *similarity number* of T, denoted by $s(T)$, is the cardinality of the set of all similarity classes of T.

The *stabiliser* of a set $A \subset V$ is the group $\text{Aut}(T, A) := \{\sigma \in \text{Aut}\, T : \sigma A = A\}$. A is *stable if* $\text{Aut}(T, A) = \text{Aut}\, T$. We shall say that A is an *asymmetrising set* of T if $\text{Aut}(T, A)$ consists of the identity alone. By $A(T)$ we denote the collection of all asymmetrising sets of T. T is *asymmetrisable* if it has an asymmetrising set. Note that $\text{Aut}(T, V \setminus A) = \text{Aut}(T, A)$ for any $A \subset V$. Hence the complement of an asymmetrising set is likewise asymmetrising.

The *asymmetrising number* of T, denoted by $a(T)$, is the maximum number of mutually nonsimilar asymmetrising subsets of V; in other words, $a(T)$ is the number of similarity classes of asymmetrising sets in T. It is immediate from the definition that $a(T) \leqslant s(T)$.

(iv) Given a rooted tree (T, w) its *automorphism group* is

$$\mathrm{Aut}(T, w) := \{\sigma \in \mathrm{Aut}\, T: \sigma w = w\},$$

i.e., the stabiliser of $\{w\}$. We shall usually write G_w for $\mathrm{Aut}(T, w)$. The various concepts introduced in (iii) can be transferred to rooted trees by relativisation to G_w. Thus, two vertices or sets of vertices are *similar* in (T, w) if they are G_w-similar in T. A *similarity class of* (T, w) is the G_w-similarity class of some subset of V; the number of such similarity classes is the *similarity number* of (T, w), denoted by $s_w(T)$. $A \subset V$ is an *asymmetrising set* of (T, w) if $G_w \cap \mathrm{Aut}(T, A) = 1$. By $\mathscr{A}_w(T)$ we denote the set of asymmetrising sets of (T, w). (T, w) is *asymmetrisable* if $\mathscr{A}_w(T) \neq \emptyset$. The *asymmetrising number* of (T, w), $a_w(T)$, is the number of G_w-similarity classes of asymmetrising sets of (T, w). For $x \in V$ we abbreviate $s_x(T_x)$, $a_x(T_x)$ and $\mathscr{A}_x(T_x)$ by $s(x)$, $a(x)$ and $\mathscr{A}(x)$, respectively. In particular $s(w) = s_w(T)$, etc. In a rooted tree no set of vertices can be similar to its complement; hence $a_w(T)$ is never odd.

Let $x \in V \backslash \{w\}$. An important parameter of x is its *multiplicity* in (T, w), defined to be the number of vertices in the G_w-orbit of x which have the same lower neighbour as x. Clearly, G_w-similar vertices have the same multiplicity.

Observe that

$$\sigma[A|_x] = (\sigma A)|_{\sigma x} \quad \text{for any } x \in V,\, A \subset V,\, \sigma \in G_w. \tag{1}$$

Hence two upper neighbours x, y of a vertex u are similar in (T, w) if and only if (T_x, x) and (T_y, y) are isomorphic (as rooted trees). This means that the multiplicity $m(x)$ counts how many times the tree (T_x, x) appears among the upper neighbours of u.

(v) Given a set S and a cardinal number n, $\binom{S}{n}$ will denote the set of all n-element subsets of S. If S is itself a cardinal, $S = m$, say, then $\binom{S}{n}$ has the usual meaning as binomial coefficient, with the convention that $\binom{m}{n} = m^n$ if $n \leqslant m$ and m is infinite. Then $\binom{m}{n} = |\binom{S}{n}|$.

The set of all natural numbers will be denoted by ω. If κ is a cardinal, $\Pi(\kappa)$ denotes the set of cardinals $\leqslant \kappa$, and $\bar{\kappa} := |\Pi(\kappa)|$.

2. Preliminary results

Most of our results are based on the following simple observations.

Remarks 2.1. (i) *Let (T, w) be a rooted tree, T' a subtree of T such that $V(T') \cap V(T \backslash T')$ consists of a single vertex w'. Then given any asymmetrising set A of (T, w), $A \cap V(T')$ is an asymmetrising set of (T', w').*

This is a consequence of the fact that any automorphism of (T', w') can be extended to an automorphism of (T, w) by taking the identity on the complement of $V(T')$.

For any $x \in V$, $V(T_x) \cap V(T \setminus T_x) = \{x\}$. Hence we obtain that *if* $A \in \mathscr{A}_w(T)$, *then* $A|_x \in \mathscr{A}(x)$.

(ii) Given a rooted tree (T, w) let u, u' be two distinct upper neighbours of some $v \in V$. *If A is an asymmetrising set of (T, w), and u and u' are G_w-similar, then $A|_u$ and $A|_{u'}$ are not G_w-similar.*

The reason for this is that if $\sigma[A|_u] = A|_{u'}$ for some $\sigma \in G_w$, then $\sigma u = u'$ and the two sets $A|_u$ and $A|_{u'}$ are mapped into each other by the involution $\alpha_\sigma \in G_v \cap G_w$ defined by

$$\alpha_\sigma x := \begin{cases} \sigma x & \text{if } x \in V(T_u), \\ \sigma^{-1} x & \text{if } x \in V(T_{u'}), \\ x & \text{otherwise.} \end{cases}$$

Hence α_σ stabilises A without being the identity, contrary to the hypothesis that A is an asymmetrising set.

(iii) As an immediate consequence of the two preceding remarks we obtain that *if (T, w) is asymmetrisable, then*

$$m(x) \leq a(x) \quad \textit{for any } x \in V \setminus \{w\}. \tag{2}$$

(iv) Let T' be a stable subtree of T. Given $A \in \mathscr{A}(T)$ and $A' \in \mathscr{A}(T')$ then $B = A \setminus V(T') \cup A'$ is again an asymmetrising set of T, for if an automorphism σ of T stabilises B, then it stabilises A', hence $\sigma|_{T'} = \mathrm{id}_{T'}$, and consequently σ also stabilises A, so that $\sigma = \mathrm{id}_T$. In other words, *if T is asymmetrisable, then any asymmetrising set of T' can be extended to an asymmetrising set of T* (conditional extendability). Also it is clear that if two extensions

$$A \setminus V(T') \cup A' \quad \text{and} \quad A \setminus V(T') \cup A''$$

with $A', A'' \in \mathscr{A}(T')$ are (Aut T')-similar, then A' and A'' are (Aut T)-similar. Therefore

$$a(T) > 0 \text{ implies } a(T) \geq a(T'). \tag{3}$$

Conditional extendability of asymmetrising sets also occurs in an important situation where the subtree is *not* stable.

Lemma 2.2. *Suppose (T, w) is asymmetrisable, and let $u \in V$. Then any asymmetrising set of (T_u, u) can be extended to an asymmetrising set of (T, w).*

Proof. It suffices to consider the case where $u \in U_w$. Let A and A_u be asymmetrising sets of (T, w) and (T_u, u), respectively. If $A_u = \sigma(A|_x)$ for some $\sigma \in G_w$ and $x \in U_w$, then replace $A|_x$ by $A|_u$, and $A|_u$ by A_u; otherwise, replace $A|_u$ by A_u. The set so constructed is then clearly an asymmetrising set of (T, w), and its restriction to u is A_u. □

The above observations find a quantitative expression in the following recursion theorem which expresses the asymmetrising number $a_w(T)$ in terms of the asymmetrising numbers and multiplicities of the restrictions of (T, w) to the neighbours of w.

Theorem 2.3. *Given a rooted tree (T, w), let $\{M_i: i \in I\}$ be the set of G_w-similarity classes of the neighbours of w. For each $i \in I$ let $m_i = |M_i|$ and $a_i = a(x)$, $x \in M_i$. Then the asymmetrising number of (T, w) is $a_w(T) = 2 \prod_{i \in I} \binom{a_i}{m_i}$.*

Proof. Let $i \in I$ and consider any representative v of the class M_i. Then by definition, a_i is the number of similarity classes of asymmetrising sets of the rooted tree (T_v, v). Let $\mathcal{A}_i$ be the set of all G_w-similarity classes of asymmetrising sets $X \in \mathcal{A}(x)$, where $x \in M_i$. Then clearly $|\mathcal{A}_i| = a_i$. Let $\mathcal{B} := \{A \in \mathcal{A}_w(T): w \notin A\}$. It is immediate that for any $A \in \mathcal{B}$ the set Q_i of G_w-similarity classes of the resrictions $A|_x$, $x \in M_i$, is a member of $\binom{\mathcal{A}_i}{m_i}$, and that the map $A \mapsto (Q_i)_{i \in I}$ is a bijection of $\mathcal{B}$ onto $\prod_{i \in I} \binom{\mathcal{A}_i}{m_i}$.

To complete the proof note that $|A_w(T)| = 2|\mathcal{B}|$ since $A \cup \{w\} \in \mathcal{A}_w(T)$ for any $A \in \mathcal{B}$. □

Remarks 2.4. (i) Suppose that a tree T has no fixed point but contains a fixed edge $e = [u, v]$. Then e is unique and the two components T', T'' of $T \backslash e$ (containing u and v, respectively) are isomorphic as rooted trees. Hence

$$a_u(T) = a_v(T) = a_u(T')a_v(T'') = a_u(T')^2.$$

Among the trees satisfying this hypothesis are the finite and rayless trees without fixed points [6].

(ii) If w is a fixed point or an endpoint of a fixed edge of T, then for any $x, y \in V$,

$$x \leqslant_w y \text{ implies } a_x(T) \leqslant a_y(T).$$

This follows immediately from the fact that any automorphism of (T, y) is also an automorphism of (T, x), whence $\mathcal{A}_x(T) \subset \mathcal{A}_y(T)$.

Proposition 2.5. *Suppose a fixed-point-free tree T has a fixed edge $e = [u, v]$. Then*:

(i) $a_u(T) = \min_{x \in V} a_x(T)$;

(ii) $a(T) = \binom{\sqrt{a_u(T)}}{2}$ *or* $a_u(T)$, *according as $a_u(T)$ is finite or infinite.*

Proof. (i) is an obvious consequence of Remark 2.4(ii).

(ii) Let T^+ be the tree obtained by subdividing the edge e by a new vertex w.

Then w is a fixed point of T^+, and the asymmetrising sets of T are clearly those of T^+ which do not contain the vertex w. Thus

$$a(T) = \tfrac{1}{2}a(T^+) = \tfrac{1}{2}a_w(T^+). \tag{4}$$

On the other hand, the neighbours u, v of w in T^+ form a single Aut(T^+, w)-similarity class, hence by Theorem 2.3,

$$a(T^+) = a_w(T^*) = 2\binom{a_u(T_u^+)}{2} = 2\binom{a_u(T_u)}{2}.$$

Condition (ii) is then a consequence of Remark 2.4(i). □

3. Finite and rayless trees

Theorem 2.3 provides a simple recursive procedure for calculating the asymmetrising number of finite trees.

Algorithm 3.1. Let (T, w) be a finite rooted tree. Define a function $\tilde{a}(x)$, $x \in V$, by the following rules:

(i) Set $\tilde{a}(x) = 2$ for any endpoint x of T, $x \neq w$.

(ii) Select a vertex x such that $\tilde{a}(y)$ is already defined for every upper neighbour y of x. Let $M_1, \ldots, M_n$ be the G_w-similarity classes of upper neighbours of x, and set

$$\tilde{a}(x) = 2\prod_{i=1}^{n}\binom{a_i}{m_i},$$

where $m_i = |M_i|$, and $a_i = \tilde{a}(y)$, $y \in M_i$, $i = 1, \ldots, n$.

(iii) If $\tilde{a}(w)$ is defined, stop; otherwise go back to (ii).

Then $a_w(T) = \tilde{a}(w)$.

Determining the G_w-similarity classes M_i amounts to considering the upper neighbours $y \in U_x$ and testing the restrictions (T_y, y) for isomorphism. As this can be done in linear time [3–4], the algorithm for calculating the asymmetrising number is clearly polynomial.

Example 3.2. In this example, w is a fixed point of T (see Fig. 1). Therefore G_w = Aut T and we obtain the asymmetrising number of T itself: $a(T) = a_w(T) = 112$. Recall that any finite tree T either has a central vertex or a central edge and that these are fixed by Aut T. Thus $a(T)$ may be calculated by rooting T at the central vertex or at a new vertex subdividing the central edge (cf. Remark 2.4(i)).

In the following, repeated use will be made of an operation which we call summation of rooted trees. Given a family (T_i, w_i), $i \in I$, of rooted trees, the *sum*

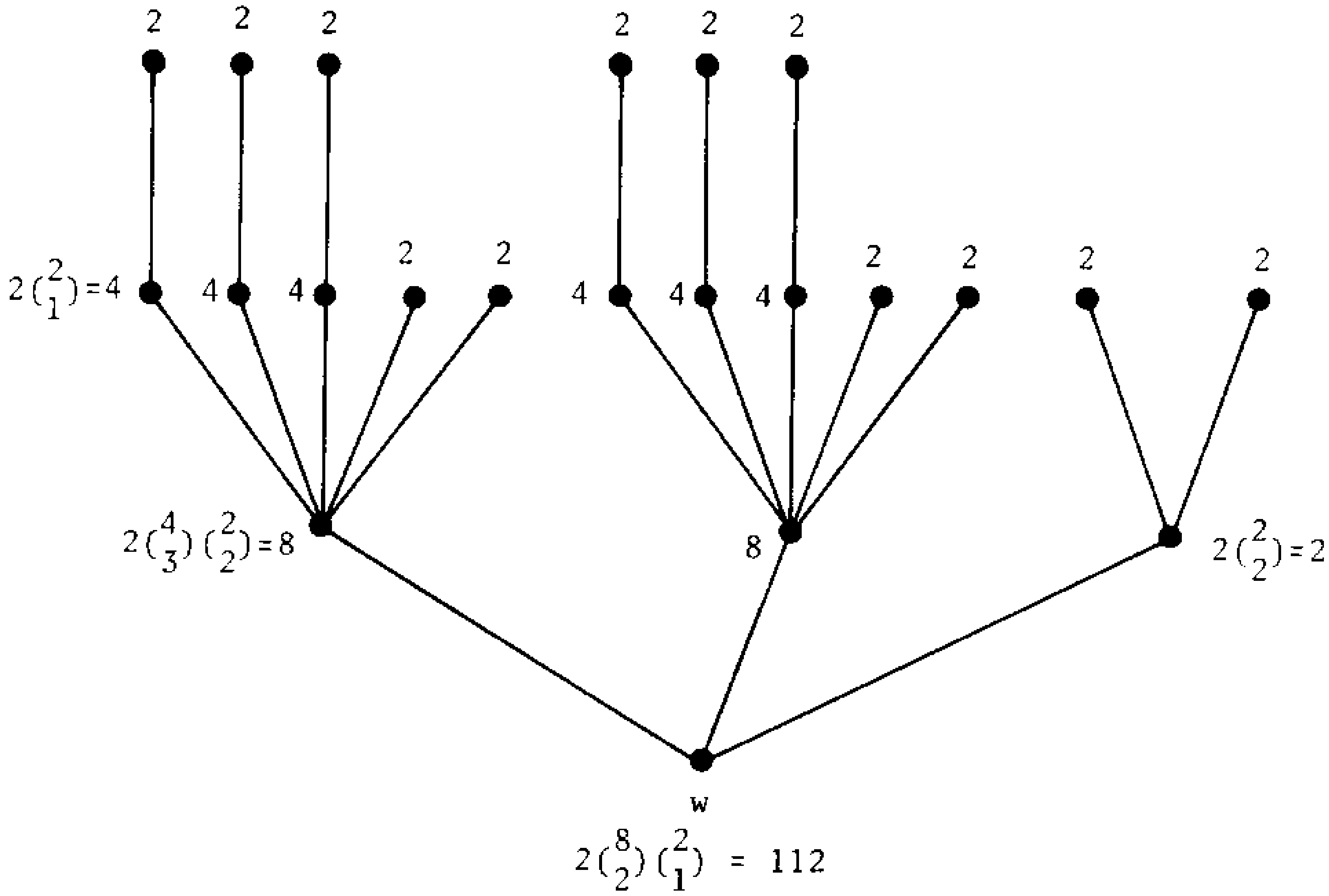

Fig. 1.

is the rooted tree (T, w) obtained by forming the disjoint union of the trees (T_i, w_i), and joining each w_i to a new vertex w. That is to say, the restrictions of the sum to the neighbours of w are precisely the given trees (T_i, w_i). Note that the sum of any family of rayless trees is rayless.

Remark 3.3. The minimal positive value for the asymmetrising number of a rooted tree is 2. Trees for which this value is attained are easily constructed. Let (T, w) be any finite asymmetrisable tree. Consider the sum, (T^*, w^*), of $r := a_w(T)$ copies $(T_1, w_1), \ldots, (T_r, w_r)$ of (T, w). In (T^*, w^*) the vertices $w_1, \ldots, w_r$ form a G_{w^*}-similarity class and $m^*(w_i) = a_w(T) = r$ so that by Theorem 2.3, $a_{w^*}(T^*) = 2$. Observe that the correspondence $(T, w) \mapsto (T^*, w^*)$ is one–one in the sense of isomorphism.

For unrooted trees the minimal positive value for the asymmetrising number is 1. In view of (4) this can be achieved by taking two disjoint copies (T_1, w_1), (T_2, w_2) of a rooted tree (T, w) with $a_w(T) = 2$ and joining the two roots w_1, w_2 by a new edge.

We conclude this section with an observation concerning the asymmetrising number of infinite rayless trees.

Remark 3.4. *There exist arbitrarily large rayless rooted trees whose asymmetrising number is* 2; *also, there exist rayless trees with arbitrarily large asymmetrising number. More precisely we have the following*:

(i) *For any ordinal α there is a set $\mathcal{M}_\alpha$ of $\beth_{\alpha+1}$ non-isomorphic rayless rooted trees (T, w) such that $|T| = \beth_\alpha$ and $a_w(T) = 2$.*

(ii) *Given any cardinal* $\kappa > 0$ *there is a rayless rooted tree* (T, w) *such that* $a_w(T) = 2^\kappa$.

Proof. Induction on α. For the sake of convenience we start at $\alpha = -1$, taking for $\mathcal{M}_{-1}$ the set of all trees (T^*, w^*) introduced in Remark 3.3, with (T, w) ranging over all finite asymmetrisable rooted trees. Suppose $\mathcal{M}_\alpha = \{(T_i, w_i): i \in I\}$ has already been constructed. For $J \subset I$ such that $|J| = \beth_{\alpha+1}$ let (T_J, w_J) be the sum of two copies of every (T_j, w_j) with $j \in J$. Then by Theorem 2.3, $a_{w_J}(T_J) = 2$. We may therefore take

$$\mathcal{M}_{\alpha+1} := \{(T_J, w_J): J \subset I, |J| = \beth_{\alpha+1}\}.$$

If α is a limit ordinal take any sequence $(T_\beta, w_\beta)_{\beta<\alpha}$, where $(T_\beta, w_\beta) \in \mathcal{M}_\beta$; again by Theorem 2.3, the sum of two copies of each (T_β, w_β) is a rayless rooted tree with asymmetrising number 2. Take $\mathcal{M}_\alpha$ to be the set of all such sums.

(ii) Given a cardinal $\kappa > 0$ choose α so large that $\beth_{\alpha+1} \geqslant \kappa$. Take a subset $\mathcal{K}$ of $\mathcal{M}_\alpha$ with $|\mathcal{K}| = \kappa$ and let (Z, z) be the sum of one copy of each of the rooted trees belonging to $\mathcal{K}$. Then by Theorem 2.3, $a_z(Z) = 2^\kappa$. □

4. Upward extendable trees

In this section we consider trees T which are (essentially) equal to their core T_* (Definition 1.1(i)). Call a rooted tree (T, w) *upward extendable* if $U_x \neq \emptyset$ for any $x \in V$. That is to say, the partially ordered set $(V, \leqslant_w)$ has no maximal element. One sees immediately that the following are equivalent:

(i) (T, w) *is upward extendable*;

(ii) *either* T *has no endpoint or* w *is the only endpoint of* T;

(iii) *every vertex of* T *lies on a ray originating at* w;

(iv) T *contains a ray, and* $T \backslash T_*$ *is either empty or a path one of whose endpoints is* w.

Clearly the property of being upward extendable is inherited by all restrictions (T_x, x). Any one-ended upward extendable tree is a ray.

For these trees the situation is particularly satisfactory in as much as the asymmetrising number can be given in closed form rather than by a recursive procedure (Corollary 4.3). We also show that the existence of an asymmetrising set is independent of the choice of the root (Theorem 4.2). A technical result (Theorem 4.5) provides a characterisation of the upward extendable trees which are asymmetrisable, by relating the multiplicities of the restrictions (T_x, x), $x \in V\backslash\{w\}$, to their similarity numbers. We then derive some results about the similarity number of upward extendable trees (Propositions 4.7, 4.9), and use these to obtain the main theorem which gives a necessary and sufficient condition for asymmetrisability in terms of the multiplicities and the orders of the restrictions of (T, w) (Theorem 4.10).

The principal tool for dealing with asymmetrising sets in upward extendable trees is the observation that in such a tree any set may be 'blown up' without changing its stabiliser.

Lemma 4.1. *Let (T, w) be an upward extendable rooted tree, and $f:\omega \to \omega$ a strictly increasing function. For $A \subset V$ define*

$$A_f := \bigcup_{x \in A} J_x, \tag{5}$$

where J_x is the set of all successors of x whose height is $f(h(x))$. Then:

(i) *the map $A \mapsto A_f$ is one–one;*

(ii) *$\sigma A_f = (\sigma A)_f$ for any $\sigma \in G_w$;*

(iii) *A and A_f have the same stabiliser in G_w; in particular, if A is an asymmetrising set of (T, w), then so is A_f.*

Proof. Note that J_x is a non-empty set because $f(h(x)) \geqslant h(x)$.

(i) Suppose $A, B \subset V$ are such that $A_f = B_f$ and let $x \in A$. Then $\emptyset \neq J_x \subset B_f$ and hence $J_x \cap J_y \neq \emptyset$ for some $y \in B$, which implies that x and y are comparable with respect to $\leqslant_w$. Thus $A \subset B$ and by symmetry the two sets are equal.

(ii) Since $\sigma V(T_x) = V(T_{\sigma x})$ and $h(\sigma x) = h(x)$ for any $\sigma \in G_w$ and $x \in V$, we have that $\sigma J_x = J_{\sigma x}$ and therefore (ii).

(iii) This is an immediate consequence of (i) and (ii). □

We shall call A_f the *dilatation* of A by f. Note that if $f(n) = n$, then $A_f = A$ for any $A \subset V$.

The operation of dilatation may also be defined in a more general setting. Suppose (T, w) contains a ray, and let $w \in V(T_*)$, with the convention that if T is one-ended, then w is the unique endpoint of T_*. Abbreviate $V(T_*)$ by V_*. Given a strictly increasing $f:\omega \to \omega$, the dilatation of $A \subset V$ by f may be defined as

$$A_f := A \setminus V_* \cup \bigcup_{x \in A \cap V_*} (V_* \cap J_x). \tag{6}$$

Clearly (6) reduces to (5) if (T, w) is upward extendable. It is a straightforward exercise to check that the three statements of Lemma 4.1 still hold in the more general context. The essential point in the proof of (ii) is that the set V_* is stable with respect to G_w.

Theorem 4.2. *Let T be a tree containing a double ray, $w \in V(T_*)$. If (T, w) is asymmetrisable, then so is T, and $a(T) \geqslant 2^{|T_*|}$.*

Proof. Consider an asymmetrising set A of (T, w) such that $w \notin A$. By taking the complement if necessary, we may assume that $|V(T_*) \setminus A| = |T_*|$. We apply dilatation in the sense of (6) using the function $f(n) = 6n$. Since $w \in V(T_*)$ we have that $V_* = V(T_*)$. Given $X \subset V_* \backslash A$ put

$$B_X = X_f \cup A_f \cup \bigcup_{x \in A_f \cap V_*} V(x; T_*) \cup D_2,$$

where D_2 is the distance 2 neighbourhood of w in T_*, i.e.,

$$D_2 = \{v \in V_*: \text{dist}(v, w) \leqslant 2\}.$$

Note that $\{w\} \cup V(w; T_*) \subset D_2 \subset B_X$, and that the only part of B_X which depends on X is X_f.

By the dilatation all distances in T_* are multiplied by 6, i.e., if $S \subset V_*$, then S_f consists of vertices whose height is a multiple of 6. This implies that w, A_f, and X_f can be recognised within B_X by the following invariant description:

$$\begin{cases} w \text{ is the unique vertex in } B_X \cap V_* \text{ whose distance 2} \\ \text{neighbourhood in } T_* \text{ is contained in } B_X; \\ A_f = B_X \backslash V_* \cup \{x \in B_X \cap V_*: V(x; T_*) \subset B_X\} \backslash (\{w\} \cup V(w; T_*)); \\ X_f = \{x \in B_X \cap V_*: V(x; T_*) \cap B_X = \emptyset\}. \end{cases} \tag{7}$$

(6 is the smallest coefficient of dilatation that will guarantee the above characterisation of w in terms of D_2.)

Now suppose that $X, Y \subset V_* \backslash A$ are such that $\sigma B_X = B_Y$ for some $\sigma \in \text{Aut}\, T$. Both B_X and B_Y contain w and A_f; since T_* is stable under Aut T it follows from (7) that $\sigma w = w$ and $\sigma A_f = A_f$. Hence by Lemma 4.1, $\sigma A = A$ and therefore $\sigma = 1$, A being an asymmetrising set of (T, w). Putting $X = Y$ this implies that B_X is an asymmetrising set of T. On the other hand, for $X \neq Y$ we obtain that B_X and B_Y are nonsimilar in T. We have therefore constructed $2^{|V_* \backslash A|} = 2^{|T_*|}$ nonsimilar asymmetrising sets of T. □

Corollary 4.3. *If T is a tree of infinite order κ having fewer than κ endpoints, then $a(T) = 0$ or 2^κ. In particular this holds for upward extendable trees.*

For the proof we need the following lemma.

Lemma 4.4. *If a tree T of infinite order κ has fewer than κ endpoints, then T contains a ray and $|T_*| = \kappa$.*

Proof. Since any rayless tree is generated by its endpoints, i.e., is the union of all paths joining two endpoints, it follows that an infinite rayless tree and its set of endpoints have the same cardinality. Therefore T has a ray.

Suppose $|T_*| < \kappa$. Then $|T \backslash T_*| = \kappa$. $T \backslash T_*$ has at most $|T_*|$ components. Let Z_i, $i \in I$, be the infinite components of $T \backslash T_*$, Z their union, and F the union of the finite components of $T \backslash T_*$. Clearly $|F| \leqslant |T_*|$.

Being rayless, each Z_i has $|Z_i|$ endpoints and all but at most one of them are endpoints of T. Therefore T has at least $\sum_{i \in I} |Z_i| = |Z|$ endpoints. But $|Z| + |F| = |T \backslash T_*| = \kappa$ and $|F| < \kappa$, so that $|Z| = \kappa$, contrary to the hypothesis that T has fewer than κ endpoints. □

Proof of Corollary 4.3. Assume that T is asymmetrisable.

If T is one-ended, then it is countable by Lemma 4.4. Hence the subtree T_0 of T generated by the endpoints of T is finite. Obviously it is stable. $T \backslash T_0$ is a ray and likewise stable. Hence by Remark 2.1(iv), $a(T) \geqslant a(T \backslash T_0) = 2^{\aleph_0} = 2^{\kappa}$.

If T contains a double ray then the corollary follows immediately from Theorem 4.2 and Lemma 4.4. □

Theorem 2.3 says that (T, w) is asymmetrisable if and only if $m(x) \leqslant a(x)$ for any $x \in V \backslash \{w\}$. We now show that for upward extendable trees, $a(x)$ may be replaced by the similarity number $s(x)$.

Theorem 4.5. *An upward extendable tree (T, w) is asymmetrisable if and only if $m(x) \leqslant s(x)$ for any $x \in V \backslash \{w\}$.*

Proof. The necessity follows at once from Remark 2.1(iii) since $a(x) \leqslant s(x)$.

Sufficiency. Let $x \in V$ and denote by M_i, $i \in I$, the similarity classes of the upper neighbours of x. By hypothesis, $|M_i| = m(y) \leqslant s(y)$ for any $y \in M_i$, $i \in I$. Hence we can choose a family of sets A_y, $y \in M_i$, such that $A_y \subset V(T_y)$ and A_y and A_z are nonsimilar in (T_x, x) whenever $y \neq z$.

We now use dilatation. Let N_k, $k \in \omega$, be pairwise disjoint infinite subsets of ω such that $r \geqslant k$ for any $r \in N_k$, and consider a family $(f_x)_{x \in V}$ of strictly increasing functions $f_x : \omega \rightarrow \omega$ such that

$$f_x(n) \in N_{h(x)} \quad \text{for any } n \in \omega,\ x \in V. \tag{8}$$

Put

$$Z := \bigcup_{x \in V} \bigcup_{y \in U_x} Z_{xy},$$

where Z_{xy} is the dilatation of A_y by f_x in (T_x, x), i.e., $Z_{xy} = (A_y)_{f_x}$ in the notation of Lemma 4.1. On the other hand, it is clear from (8) and the disjointness of the sets N_k that

$$Z_{xy} = Z|_y \cap H(x), \tag{9}$$

where $H(x)$ is the set of all vertices in T whose height belongs to $N_{h(x)}$.

We claim that Z is an asymmetrising set of (T, w). Suppose that $\sigma \in G_w$ stabilizes Z and that $\sigma x = x$ for some $x \in V$. Let $y \in U_x$. Then by (9)

$$\sigma_x[(A_y)_{f_x}] = \sigma Z_{xy} = Z|_{\sigma y} \cap H(\sigma x) = Z|_{\sigma y} \cap H(x) = Z_{x, \sigma y} = (A_{\sigma y})f_x,$$

where $\sigma_x = \sigma \mid T_x$. Applying Lemma 4.1(i)(ii) to (T_x, x) we obtain that $\sigma_x A_y = A_{\sigma y}$, i.e., that A_y and $A_{\sigma y}$ are similar in (T_x, x). By the choice of the A_y's this means that $\sigma y = y$. Induction on the height of the vertices of T then shows that σ is the identity. □

Remark 4.6. Suppose an upward extendable tree (T, w) has an asymmetrising set. By Remark 2.1(i) the same is true for any restriction (T_x, x). Hence by

Theorem 4.2,

$$s(x) = a(x) = 2^{|T_x|} \quad \text{for any } x \in V. \tag{10}$$

In other words, *if an upward extendable tree contains a vertex x such that* $s(x) < 2^{|T_x|}$, *then* (T, w) *is not asymmetrisable.*

In view of this observation it will be of interest to have some information about the similarity number of an upward extendable tree, in particular, under what conditions it is equal to $2^{|T|}$. We begin with a result, similar in spirit to Theorem 2.3, relating the similarity number of (T, w) to those of the restrictions of T to the neighbours of w.

Proposition 4.7. *Given an upward extendable tree* (T, w) *let* $\{M_i : i \in I\}$ *be the set of* G_w*-similarity classes of the neighbours of* w*. For each* $i \in I$ *let* $m_i = |M_i|$, $s_i = s(u)$, $u \in M_i$, *and* $\mu_i = \min\{m_i, s_i\}$. *Then the similarity number of* (T, w) *is*

$$s_w(T) = \prod_{i \in I} (\tilde{m}_i s_i)^{\mu_i}.$$

Recall that if κ is a cardinal, then $\tilde{\kappa}$ is the number of cardinals $\leqslant \kappa$.

Proof. For $i \in I$ let T_i be the subtree of T spanned by w and the trees T_x, $x \in M_i$. Each T_i is stable under G_w and clearly $s_w(T) = \prod_{i \in I} s_w(T_i)$. It therefore suffices to show that $s_w(T_i) = (\tilde{m}_i s_i)^{\mu_i}$ for any $i \in I$.

Given $A \subset V(T_i)$, $i \in I$, define an equivalence $\sim_A$ on M_i by $x \sim_A y$ if and only if $A|_x$ is G_w-similar to $A|_y$. Call the cardinal numbers of the $\sim_A$-classes the *multiplicities* associated with A.

Denote the set of $\mathrm{Aut}(T_u, u)$-similarity classes of (T_u, u) by S_i. It is clear that the $\mathrm{Aut}(T_i, w)$-similarity class of a set $A \subset V(T_i)$ may be identified with the function f, defined on S_i, which for each similarity class $\gamma \in S_i$ counts the number of vertices $x \in M_i$ for which $A|_x$ is of similarity type γ. The nonzero values of f are the multiplicities associated with A. Thus the sum of the values of f, S_f, equals m_i. Hence

$$s_w(T_i) = |F|, \quad \text{where } F = \{f \in \Pi(m_i)^{\Sigma_i} : S_f = m_i\}.$$

Note that since T is upward extendable, $s_i = |\Sigma_i|$ is infinite (indeed $\geqslant 2^{\omega}$).

If m_i is finite, then any $f \in F$ has finite support, hence $s_w(T_i) = s_i$. On the other hand, $\mu_i = \min\{m_i, s_i\} = m_i$ and therefore $(\tilde{m}_i s_i)^{\mu_i} = s_i^{m_i} = s_i$. Thus $s_w(T_i) = (\tilde{m}_i s_i)^{\mu_i}$.

Now assume that m_i is infinite. Then

$$\begin{aligned}
s_w(T_i) &= |\{f \in \Pi(m_i)^{\Sigma_i} : S_f = m_i\}| \\
&= |\{f \in \Pi(m_i)^{\Sigma_i} : S_f \leqslant m_i\}| && (s_i \text{ being infinite}) \\
&= |\{f \in \Pi(m_i)^{P} : P \subset \Sigma_i, |P| \leqslant \mu_i\}| && (m_i \text{ being infinite}) \\
&= \sum_{\alpha \in \Pi(\mu_i)} \sum \left\{ \tilde{m}_i^{|P|} : P \in \binom{\Sigma_i}{\alpha} \right\} \\
&= \sum_{\alpha \in \Pi(\mu_i)} \tilde{m}_i^{\alpha} \cdot s_i^{\alpha} = (\tilde{m}_i s_i)^{\mu_i}. \qquad \square
\end{aligned}$$

Corollary 4.8. *If* (T, w) *is upward extendable and* $m(x) \leq s(x) = 2^{|T_x|}$ *for any* $x \in U_w$, *then* $s_w(T) = 2^{|T|}$.

Proof. Using the notation of 4.7, $m(x) \leq s(x)$ for any $x \in U_w$ implies that $\mu_i = m_i$ for every $i \in I$. Hence $s_w(T_i) = s_i^{m_i} = 2^{|T_x|m_i} = 2^{|T_i|}$, and

$$s_w(T) = \prod_{i \in I} 2^{|T_i|} = 2^{\Sigma |T_i|} = 2^{|T|}. \qquad \square$$

Proposition 4.9. *Let* (T, w) *be an upward extendable tree such that* $m(x) \leq 2^{|T_x|}$ *for any* $x \in V \backslash \{w\}$. *Then* $s_w(T) = 2^{|T|}$.

Proof. Suppose that $s_w(T) < 2^{|T|}$. If it were the case that $m(x) \leq s(x)$ for any $x \in V \backslash \{w\}$, then by Theorem 4.5, T has an asymmetrising set, and by Theorem 4.2, $s_w(T) = a_w(T) = 2^{|T|}$, a contradiction. Hence there is an $x_0 \in V \backslash \{w\}$ such that $s(x_0) < m(x_0) \leq 2^{\kappa_0}$, where $\kappa_0 = |T_{x_0}|$. Now repeat this argument with T_{x_0} etc., to obtain a countable sequence of vertices $w <_w x_0 <_w x_1 <_w \cdots$ such that

$$s(x_j) < m(x_j) \leq 2^{\kappa_j}, \quad \text{where } \kappa_j = |T_{x_j}|,\ j = 0, 1, \ldots \tag{11}$$

Abbreviate $m(x_j)$ and $s(x_j)$ by m_j and s_j, respectively.

Clearly $s_0 \geq s_1 \geq \cdots$. Hence there is a k such that $s_k = s_{k+1}$. By Proposition 4.7,

$$s_k \geq (\bar{m}_{k+1} s_{k+1})^{\min\{m_{k+1}, s_{k+1}\}} = (\bar{m}_{k+1} s_k)^{\min\{m_{k+1}, s_k\}}$$

which implies that $m_{k+1} < s_k = s_{k+1}$, a contradiction to (11). $\square$

Theorem 4.10. *An upward extendable tree* (T, w) *is asymmetrisable if and only if*

$$m(x) \leq 2^{|T_x|} \quad \textit{for any} \quad x \in V \backslash \{w\}. \tag{12}$$

Proof. The necessity follows from Theorem 4.5 and (10).

Sufficiency. Assume (12). Let $x \in V \backslash \{w\}$, $y \in V(T_x) \backslash \{x\}$. Then the multiplicity of y when calculated with respect to (T_x, x) is the same as when calculated with respect to (T, w). Hence by applying Proposition 4.9 to (T_x, x) we obtain that $s(x) = 2^{|T_x|}$, i.e., $m(x) \leq s(x)$, and the result follows from Theorem 4.5. $\square$

Corollary 4.11. *Let* (T, w) *be an upward extendable tree such that* $d_x \leq 2^{|T_y|}$ *for any* $x \in V$, $y \in U_x$, *where* d_x *is the degree of* x *in* T. *Then* T *is asymmetrisable.*

Proof. Immediate, since $m(y) \leq d_x$ for any $x \in V$, $y \in U_x$. $\square$

Corollary 4.12. (i) *If* T *is an upward extendable tree of order at most* $2^{\aleph_0}$, *then* T *is asymmetrisable.*

(ii) *Let* T *be a tree of infinite order* κ *such that for any* $x \in V$ *all components of* $T - x$ *are of order* $\geq \log \kappa$ (*where* $\log \kappa$ *is the least cardinal* λ *such that* $2^\lambda \geq \kappa$). *Then* T *is asymmetrisable.*

Proof. (i) If $|T| \leqslant 2^{\aleph_0}$, then $d_x \leqslant |T| \leqslant 2^{\aleph_0} \leqslant 2^{|T_y|}$ for any $x, y \in V$.

(ii) Clearly T has no endpoint. Choose any root w and let $x \in V$, $y \in U_x$. Then T_y is a component of $T - x$, hence $d_x \leqslant \kappa \leqslant 2^{\log \kappa} \leqslant 2^{|T_y|}$. By Corollary 4.11, (T, w) has an asymmetrising set, hence by Theorem 4.2 so does T. □

Among the trees which satisfy condition (ii) are those which are generated by their vertices of degree $\geqslant \log \kappa$, in particular, all *regular* trees of infinite degree. Regular trees of finite degree are countable, hence they are asymmetrisable by 4.12(i).

Example 4.13. In view of Remark 4.6 it is meaningful to ask whether there exist trees T without endpoints having fewer than 2^κ nonsimilar subsets, where $\kappa = |T|$. Very simple examples show that the answer is affirmative and that, in fact, $s(T)$ can be as small as $\tilde{\kappa}$ (of course, it cannot go below this).

For any infinite cardinal κ let (P_κ, w) be the sum of κ rays rooted at their endpoints, i.e., P_κ is the union of κ rays having exactly w in common. It will be seen that $s(P_\kappa)$ may attain extremal value, $\tilde{\kappa}$ or 2^κ, for arbitrarily large κ.

The neighbours of w form a single similarity class whence by Proposition 4.7 (dropping the subscripts i),

$$s(P_\kappa) = s_w(P_\kappa) = (s\tilde{m})^\mu = (2^{\aleph_0}\tilde{\kappa})^\mu, \quad \text{where } \mu = \min\{\kappa, 2^{\aleph_0}\}. \tag{13}$$

Taking κ so large that $\tilde{\kappa} \geqslant 2^{\aleph_0}$, (13) implies that

$$s(P_\kappa) = \tilde{\kappa}^{2^{\aleph_0}}, \tag{14}$$

and therefore

$$\tilde{\kappa} \leqslant s(P_\kappa) \leqslant 2^{\tilde{\kappa}}. \tag{15}$$

Specifically, if one takes $\kappa = \aleph_{\omega_\alpha}$, where $\aleph_\alpha = 2^\nu$ and $\nu \geqslant 2^{\aleph_0}$, then $\tilde{\kappa} = 2^\nu$ and (14) reduces to $s(P_\kappa) = \tilde{\kappa}$.

Assuming the continuum hypothesis one can be slightly more precise. If $\aleph_1 \leqslant \kappa < \aleph_{\omega_1}$, then $\tilde{\kappa} = \aleph_0$ and $\mu = \aleph_1$, hence by (13), $s(P_\kappa) = 2^{\aleph_1} > 2^{\tilde{\kappa}}$. We can therefore say that (15) holds if and only if $\kappa \geqslant \aleph_{\omega_1}$.

Examples where the upper bound in (15) is reached are obtained by taking κ to be of cofinality ω and satisfying $\aleph_\kappa = \kappa$. Such cardinals may be arbitrarily large. Clearly they satisfy $\kappa = \tilde{\kappa}$ and $\kappa < \kappa^{\aleph_1} \leqslant 2^\kappa$. Hence assuming the generalised continuum hypothesis, $\kappa^{\aleph_1} = 2^\kappa$ so that $s(P_\kappa) = 2^\kappa$.

In this context it is worth noting that because of the form of condition (12), the existence of an asymmetrising set in a given tree may depend on set theoretic assumptions. For example, the tree $P_{\aleph_2}$ is asymmetrisable if and only if the continuum hypothesis does not hold.

Remark 4.14. Is the asymmetrisability of a tree inherited by its core? The obvious approach, taking the part of an asymmetrising set of T that lies in the core, does not work because in general the automorphisms of the core are not extendable to all of T. Indeed, the answer to the question is negative. This is of some interest inasmuch as the core provides a lower bound for the asymmetrising number of T (Theorem 4.2). Consider the following example.

Given a cardinal $\kappa > 2^{\aleph_0}$ let (P_κ, w) be the tree of Example 4.13. Choose a vertex $x_\alpha \neq w$ on each of the κ rays which make up P_κ. The subscripts α may be considered to be the ordinals $<\kappa$. By Remark 3.4(ii), for any α there is a rayless tree (Q_α, y_α) such that $a(Q_\alpha) = 2^{\aleph_\alpha}$ and y_α is a fixed point of Q_α. Form (T, w) by identifying each x_α with y_α. Clearly $T_* = P_\kappa$, and by Theorem 4.10, P_κ is not asymmetrisable. On the other hand, it is easily seen that taking an asymmetrising set of each Q_α one obtains an asymmetrising set of T.

5. One-ended trees

In the preceding sections we have repeatedly made use of the fact that for the trees under consideration there is a vertex w such that

$$a(T) > 0 \text{ if and only if } a_w(T) > 0. \tag{16}$$

If T has a fixed element (vertex or edge), then any fixed vertex or any vertex incident with a fixed edge may be chosen as the root w. As already pointed out in Remark 2.4(i), fixed elements exist whenever T is rayless. On the other hand, if T contains a double ray, then any vertex of the core will satisfy (16) (Theorem 4.2). We now consider the one remaining case, where T is one-ended and has neither a fixed vertex nor a fixed edge. Note that in a one-ended tree any endpoint of a fixed edge is a fixed point. The assumption that T has no fixed elements thus amounts to saying that there are no fixed points. An example of a tree with these properties is the *inverse binary tree* B which is defined as follows: Choose a ray $R = \langle x_0, x_1, \ldots \rangle$ in the 3-regular tree T_3; then B is the subgraph of T_3 induced by

$$\bigcup_{k \in \omega} \{y : \operatorname{dist}(x_k, y) \leq k\}$$

see Fig. 2.

An important property of an (unrooted) one-ended tree T is that its vertex-set can be endowed with a natural partial order by *rooting T at infinity*. T being one-ended we have that for any $x \in V$ there is a unique ray $R_x \subset T$ which starts at x. The order on V is then given by setting

$$x \leq y \text{ if and only if } R_x \subset R_y.$$

Lemma 5.1. *If a one-ended tree T contains a ray $R = \langle w_0, w_1, \ldots \rangle$ such that any (T, w_n) has an asymmetrising set, then $a(T) \geq 2^{\aleph_0}$.*

Proof. Let $R = \langle w_0, w_1, \ldots \rangle$ be a ray in T such that (T, w_n) has an asymmetrising set A_n, $n \geq 0$, and let a subset $I \subset V(R)$ be given. We shall construct an asymmetrising set B_I of T such that $B_I \cap V(R) = I$.

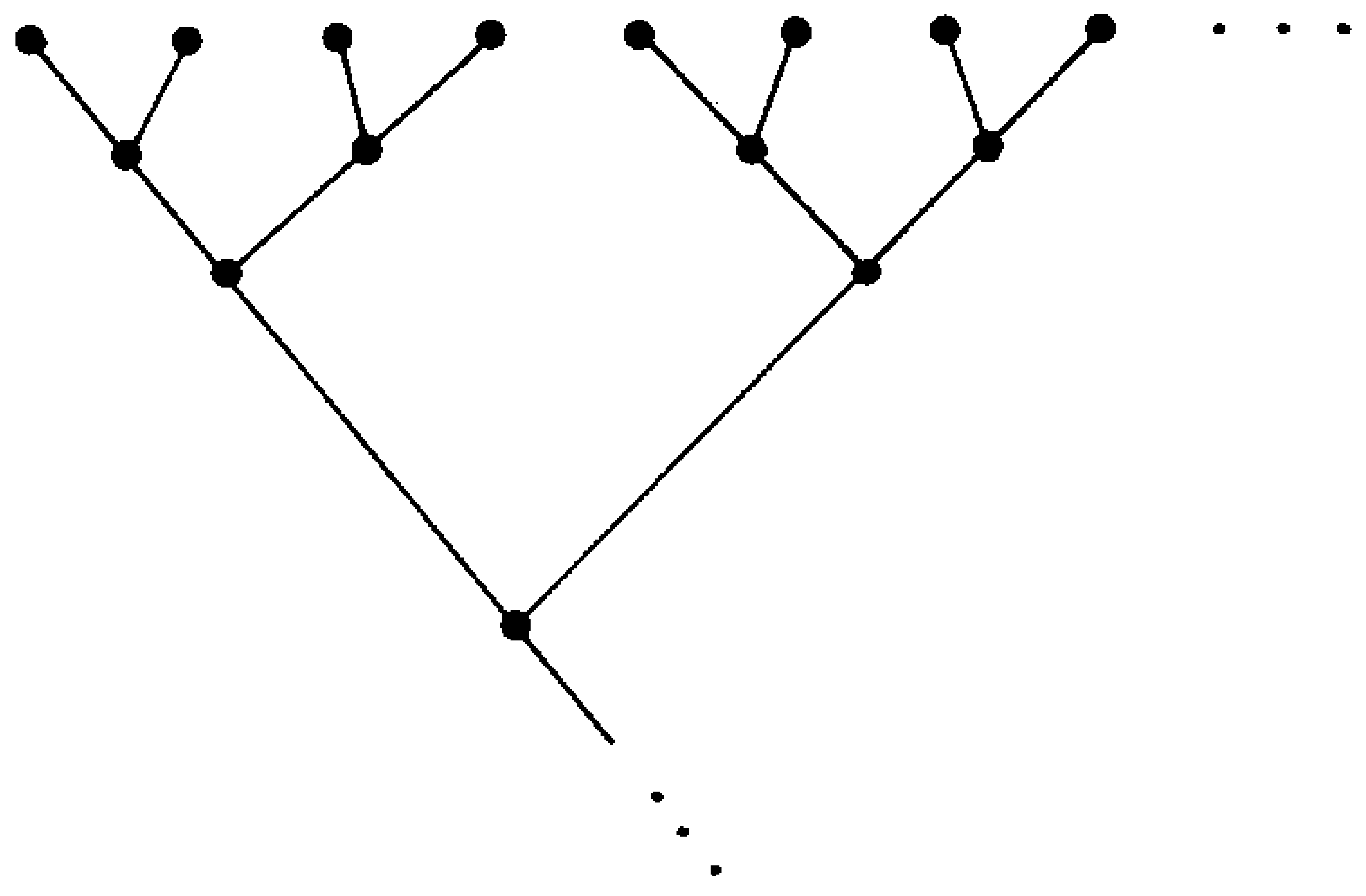

Fig. 2.

For $n \geq 0$ put $W_n := \{x \in V : x \geq w_n\}$ and let T_n be the subtree of T induced by W_n. We now construct an increasing sequence of sets $B_0, B_1, \ldots$ such that:

(i) $B_n \cap W_{n-1} = B_{n-1}$, $n \geq 1$;

(ii) $B_n \cap V(R) = I_n := \{w_i \in I : i \leq n\}$, $n \geq 0$;

(iii) B_n is an asymmetrising set of (T_n, w_n), $n \geq 0$.

Put $B_0 = \{w_0\}$ or $\emptyset$ according as w_0 does or does not belong to I, and suppose B_n has already been constructed. Since (T_n, w_n) is the restriction of (T, w_{n+1}) to w_n we can apply Lemma 2.2 (using the existence of A_{n+1}) to obtain an asymmetrising set A'_{n+1} of (T, w_{n+1}) such that

$$A'_{n+1} \cap W_n = B_n. \tag{17}$$

Put

$$B'_{n+1} := A'_{n+1} \cap W_{n+1}.$$

By Remark 2.1(i), B'_{n+1} is an asymmetrising set of (T_{n+1}, w_{n+1}) and by (17) it satisfies (i). Putting $B_{n+1} = B'_{n+1} \cup \{w_{n+1}\}$ or $B'_{n+1} \backslash \{w_{n+1}\}$ according as w_{n+1} does or does not belong to I we obtain a set satisfying (i)–(iii).

We claim that $B_I := \bigcup_{n \in \omega} B_n$ is an asymmetrising set of T. By one-endedness of T, given any $\sigma \in \text{Aut}\, T$ there is a natural number N such that $\sigma \in \text{Aut}(T, w_n)$ for all $n \geq N$, and therefore $\sigma_n := \sigma \mid W_n \in \text{Aut}(T_n, w_n)$. Thus, if σ stabilises B, then σ_n stabilises B_n and hence is the identity for all $n \geq N$, i.e., $\sigma = 1$.

Since no automorphism of T can map a subset of $V(R)$ onto a different subset of $V(R)$, it follows that if I, I' are two distinct subsets of $V(R)$, then B_I and $B_{I'}$ are nonsimilar with respect to $\text{Aut}\, T$. We have therefore constructed $2^{\aleph_0}$ nonsimilar asymmetrising sets of T. $\square$

Proposition 5.2. *Let T be a one-ended tree without fixed point. Then the following are equivalent*:

(i) *there is a $w \in V$ such that (T, w) is asymmetrisable*;

(ii) *T contains a ray $R = \langle w_0, w_1, \ldots \rangle$ such that any (T, w_n) is asymmetrisable*;

(iii) $a(T) \geq 2^{\aleph_0}$.

Proof. (iii)$\Rightarrow$(i) is trivial; (ii)$\Rightarrow$(iii) is Lemma 5.1.

(i)$\Rightarrow$(ii). It suffices to show that if v and w are adjacent vertices such that $v<w$ and (T, w) has an asymmetrising set, then so does (T, v). This will imply that for any vertex x of the ray starting at w, (T, x) has an asymmetrising set.

For $x \in V\backslash\{v\}$ denote by $m_v(x)$ the multiplicity of x in (T, v). By Theorem 2.3, (T, v) is asymmetrisable provided

$$m_v(x) \leqslant a_x(T_{vx}) \quad \text{for any } x \in V(v; T), \tag{18}$$

where T_{vx} is the restriction of (T, v) to x, i.e., the subtree induced by $\{y \in V : x \leqslant_v y\}$.

If $x \in V(v; T)\backslash\{w\}$, then the restrictions of (T, v) and (T, w) to x coincide. Also, v is the lower neighbour of x in both (T, v) and (T, w). Hence (18) reduces to $m(x) \leqslant a(x)$, and this is satisfied because (T, w) is asymmetrisable.

To show that (18) also holds for $x = w$ we use that T is fixed point free. Thus there is a $\sigma \in \operatorname{Aut} T$ such that $\sigma v \neq v$. The key observation is that σT_{vw} is not only a restriction of (T, v) but also of (T, w); indeed, $\sigma T_{vw} = T_{v,\sigma w} = T_{\sigma w}$. Hence

$$m_v(w) = m(\sigma w) \leqslant a_{\sigma w}(T_{\sigma w}) = a_{\sigma w}(\sigma T_{vw}) = a_w(T_{vw}),$$

the inequality arising, as before, from the hypothesis that (T, w) is asymmetrisable. □

Corollary 5.3. *If T is any asymmetrisable tree, then $a(T) \geqslant 2^{|T_*|}$.*

Proof. If T is rayless, then $T_* = \emptyset$, hence $a(T) \geqslant 2^{|T_*|}$ holds trivially. If T is one-ended, then $a(T) \geqslant 2^{\aleph_0} = 2^{|T_*|}$ by Lemma 5.1. If T has a double ray, this is Theorem 4.2. □

Corollary 5.4. *The asymmetrising number of any tree is either finite or uncountable.*

Proof. Suppose that $a(T) = \aleph_0$ for some T. Then by Corollary 5.3, T is rayless. By [4] it has a fixed point or a fixed edge, and by subdividing the latter we may assume w.l.o.g. that T has a fixed point w (subdivision of a fixed edge does not alter an infinite asymmetrising number; see (4)). Thus $a_w(T) = a(T) = \aleph_0$.

We will obtain a contradiction by showing that T contains a ray $\langle w_0, w_1, \ldots\rangle$ such that $a(w_n) = \aleph_0$ for all n. Put $w_0 = w$ and suppose w_n has already been defined. Since T is asymmetrisable we have that $m(x) \leqslant a(x) \leqslant \aleph_0$ for any $x \in V$. Let $W := \{x \in U_{w_n} : m(x) < a(x) < \aleph_0\}$; that is to say, $x \in W$ if and only if

$$2 \leqslant \binom{a(x)}{m(x)} < \aleph_0.$$

If $a(x) < \aleph_0$ for all $x \in U_{w_n}$, then by Theorem 2.3 (applied to (T_{w_n}, w_n)), $a(w_n)$ is either finite or uncountable, depending on whether W consists of finitely or infinitely many $\mathrm{Aut}(T_{w_n}, w_n)$-similarity classes, a contradiction. Hence w_n has an upper neighbour w_{n+1} with $a(w_{n+1}) = \aleph_0$. $\square$

In fact, it can be shown that if $a(T)$ is uncountable, then it is of the form 2^κ for some κ [5, Corollary 4.7].

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