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f-Optimal factors of infinite graphs

F. Niedermeyer
Institut für Mathematik der Universität Hannover, Welfengarten 1, W-3000 Hannover 1, Germany

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Abstract

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A necessary and sufficient condition for the existence of perfect f-factors of countable graphs is proved. Further it is shown that every countable graph and every rayless graph possesses an f-optimal factor. Finally the structure of f-optimal factors of infinite graphs is examined.

Introduction

In this paper $G = (V, E)$ denotes always a graph, where $V \neq \emptyset$ is the set of vertices and $E \subseteq \{e \subseteq V : |e| = 2\}$ is the set of edges of G, and f a function from V into ω, the set of natural numbers.

We will deal with the question of whether there is a perfect f-factor of G, i.e. whether there is a subset F of E such that $d_F(v) = f(v)$ for each vertex v. If this question is answered negatively, we ask whether there is at least some F such that d_F approximates f optimally. We will consider only those subsets F of E for which $d_F(v)$ is finite for each $v \in V$. Such a set of edges we call a factor of G. For comparing two factors F and H with respect to the property of being optimal, we call F *better* than H if $|d_F(v) - f(v)| \leq |d_H(v) - f(v)|$ for every vertex v, and $|d_F(v) - f(v)| < |d_H(v) - f(v)|$ for at least one v.

In this paper we prove for countable G that G possesses a perfect f-factor iff G is *unobstructed,* which means that for every nonperfect f-factor F and for every unsaturated vertex v there exists an F-alternating path starting at v such that the symmetric difference of F and the edge set which is traversed by the path is better than F. Moreover, it is shown for countable G that G possesses a perfect f-factor iff for every nonperfect factor F there exists a factor better than F. This result is equivalent to the fact that G contains an optimal factor if G is countable. Further we will see: if G is rayless, then there even exists, for every non-optimal factor F, an optimal factor better than F. These results are generalizations of the corresponding statements for 1-factors which have been proved by Steffens [7] and Schmidt [6].

In Section 6 we will examine the structure of optimal factors. For finite graphs, Lovász and Plummer [4] introduced a decomposition of a graph's vertex set concerning optimal factors. It makes sense to consider this decomposition for arbitrary graphs. It will turn out that the properties of this decomposition which have been shown in [4] hold in infinite graphs, too.

1. Factors and f-factors

For any subset F of E,

$$d_F(v) := |\{e \in F : v \in e\}|$$

defines a function d_F from V into the class of cardinals. On the class of functions from V into the class of cardinals we define a weak order by

$$g \leqslant h \;:\Leftrightarrow\; g(v) \leqslant h(v) \text{ for all } v \in V$$

and write $g < h$ if $g \leqslant h$ and $g \neq h$. If misunderstandings are impossible, we denote by 1 the function with domain V and range $\{1\}$.

$F \subseteq E$ is said to be a *factor* of G if $d_F(v)$ is finite for each $v \in V$, and F is called an *f-factor* if $d_F \leqslant f$. A factor F will be called (f-)*perfect* if $d_F = f$. If F is an f-factor, a vertex $v \in V$ is said to be (f-F-)*saturated* (*unsaturated*, resp.) if $d_F(v) = f(v)$ ($d_F(v) < f(v)$, resp.). If F is a factor and if $g : V \to \omega$ is a function, we define a function $\delta_{F,g} : V \to \omega$ by

$$\delta_{F,g}(v) := |d_F(v) - g(v)|.$$

If $g = f$, we shall write 'δ_F' instead of '$\delta_{F,f}$'. A factor F_1 is said to be *better* than a factor F_2 (with respect to f) if

$$\delta_{F_1} < \delta_{F_2},$$

i.e. if the function f is approximated better by d_{F_1} than by d_{F_2}. A factor is called (f-)*optimal* if there is no better factor.

Our aim is to characterize graphs which have an optimal factor, and to describe the structure of optimal factors. Indeed, the complete bipartite graph $K_{\aleph_0,\aleph_1}$ and the function $f = 1$ show that not every graph has an optimal factor. Another important example is the bipartite graph $G^* = (V_0 \cup V_1, E^*)$, where

$$V_0 = \{(\alpha, 0) : \alpha < \aleph_1 \text{ is a limit ordinal}\},$$
$$V_1 = \aleph_1 \times \{1\},$$
$$E^* = \{\{(\alpha, 0), (\beta, 1)\} : \beta < \alpha\}.$$

We will see later that there is no 1-optimal factor of G^*.

A measure for the 'goodness' of a factor F is in some cases the *defect* of F,

$$\Delta(F, f) := \sum_{v \in V} \delta_{F,f}(v).$$

In Lovász and Plummer [4] only finite graphs are investigated, and there a factor is called 'optimal' if its defect is minimal. We will see later that for finite graphs this definition coincides with our definition.

At the end of this section we give some definitions concerning the terminology used in this paper. A sequence $\mathcal{P} = (v_i)_{0 \leq i < k}$ $(0 < k \leq \omega)$ is called a *walk* if $\{v_{i-1}, v_i\} \in E$ for $0 < i < k$. We define

$$V(\mathcal{P}) := \{v_i : 0 \leq i < k\} \quad \text{and} \quad E(\mathcal{P}) := \{\{v_{i-1}, v_i\} : 0 < i < k\}.$$

If $k < \omega$, $\mathcal{P}$ is said to have the *length* $k - 1$. $\mathcal{P}$ is called a *trail* (*path*, resp.) if no edge (vertex, resp.) will be traversed twice by $\mathcal{P}$. If $\mathcal{P} = (v_i)_{0 \leq i \leq k}$ and $\mathcal{Q} = (\omega_i)_{0 \leq i < l}$ are walks satisfying $k < \omega$, $l \leq \omega$, and $v_k = w_0$, then $\widehat{\mathcal{P}\mathcal{Q}}$ denotes the walk $(u_i)_{0 \leq i < k+l}$, where $u_i = v_i$ if $0 \leq i < k$, and $u_i = w_{i-k}$ if $k \leq i < k + l$. If $F_1, F_2 \subseteq E$, then a walk $\mathcal{P}$ is called *F_1-F_2-alternating* if the edges of $\mathcal{P}$ are alternately in $F_1 \setminus F_2$ and $F_2 \setminus F_1$. If $F \subseteq E$, let $F^C := E \setminus F$. A walk is called *F-alternating* if it is F-F^C-alternating. Note that an F_1-F_2-alternating walk is both F_1-alternating and F_2-alternating.

Given a function $g : V \to \omega$ and a vertex $v \in V$, we define new functions g_v and g^v on V as follows:

$$g_v(x) := \begin{cases} g(x) & \text{if } x \in V \setminus \{v\}, \\ g(x) - 1 & \text{if } x = v, \end{cases}$$

$$g^v(x) := \begin{cases} g(x) & \text{if } x \in V \setminus \{v\}, \\ g(x) + 1 & \text{if } x = v. \end{cases}$$

We can compose such functions; let $g^{uv} := (g^u)^v$, $g_v^u := (g^u)_v = (g_v)^u, \ldots$.

For any sets A, B, the symmetric difference $(A \setminus B) \cup (B \setminus A)$ will be denoted by $A \oplus B$.

2. F_1-F_2-trails

If F is a factor and $\mathcal{P} = (v_i)_{0 \leq i < k}$ an F-alternating trail, then $F \oplus E(\mathcal{P})$ is a factor, too. d_F and $d_{F \oplus E(\mathcal{P})}$ differ at most at the start- and the endpoint of $\mathcal{P}$. If for instance $1 < k < \omega$, $\{v_0, v_1\} \in F$, and $\{v_{k-2}, v_{k-1}\} \in F^C$, then $d_{F \oplus E(\mathcal{P})} = d_{Fv_0}^{v_{k-1}}$. F-alternating trails may be useful to 'improve' a factor F. This can be illustrated by the example G^* from the previous section: Let $F \subseteq E^*$ be a factor of G^* and let $v_0 \in V_0$ be a vertex such that $d_F(v_0) = 0$. Now assume that, for $0 \leq i \leq l$, v_i is already defined such that $(v_i)_{0 \leq i \leq l}$ is an F-alternating trail. If l is even, then $v_l \in V_0$. Choose $v_{l+1} \in V_1 \setminus \{v_i : i < l\}$ such that $\{v_l, v_{l+1}\} \in F^C$. This is possible since $d_{E^*}(v_l) = \aleph_0$ and F is a factor. If l is odd, then $v_l \in V_1$. If $d_F(v_l) > 0$, choose $v_{l+1} \in V_0$ such that $\{v_l, v_{l+1}\} \in F$. If on the other hand $d_F(v_l) = 0$, let $k := l + 1$

and stop this recursive definition. If the definition does not stop, let $k := \omega$. Let $\mathcal{P} := (v_i)_{0 \le i < k}$. $\mathcal{P}$ is an F-alternating trail, and we get

$$d_{F \oplus E(\mathcal{P})} = d_F^{v_0, v_{k-1}} \quad \text{if} \quad k < \omega, \quad \text{and} \quad d_{F \oplus E(\mathcal{P})} = d_F^{v_0} \quad \text{if} \quad k = \omega.$$

Therefore $\delta_{F \oplus E(\mathcal{P}),1} = (\delta_{F,1})_{v_0 v_{k-1}} < \delta_{F,1}$, if $k < \omega$, and $\delta_{F \oplus E(\mathcal{P}),1} = (\delta_{F,1})_{v_0} < \delta_{F,1}$, if $k = \omega$. So $F \oplus E(\mathcal{P})$ is better (with respect to $f = 1$) than F. By Fodor's Theorem (see e.g. [3, Theorem 11.2.6]) there exists, for every factor F of G^*, a vertex $v_0 \in V_0$ satisfying $d_F(v_0) = 0$. Thus no factor of G^* is 1-optimal.

The possibility of modifying a factor by an alternating trail will now be examined. The following lemma and definition will play a fundamental role.

Lemma 1. *Let F_1, F_2 be factors of G and let $\{v_0, v_1\} \in F_1 \setminus F_2$. Then there exists a trail $\mathcal{P} = (v_i)_{0 \le i < k}$ with the following properties:*
(i) $k > 1$,
(ii) *$\mathcal{P}$ is F_1-F_2-alternating,*
(iii) *$k = \omega$ or*

$$k < \omega, \qquad \{v_{k-2}, v_{k-1}\} \in F_1 \setminus F_2 \text{ and } d_{F_1}(v_{k-1}) > d_{F_2}^{v_0}(v_{k-1})$$

or

$$k < \omega, \qquad \{v_{k-2}, v_{k-1}\} \in F_2 \setminus F_1 \text{ and } d_{F_1}(v_{k-1}) < d_{F_2}^{v_0}(v_{k-1}).$$

Such a trail we call an F_1-F_2-trail.

Proof. The vertices v_i of $\mathcal{P}$ will be defined recursively. Assume that $v_0, \ldots, v_l$ ($l \ge 1$) are already defined such that $\mathcal{P}_l := (v_i)_{0 \le i \le l}$ is an F_1-F_2-alternating trail.

If l is odd, then $\{v_{l-1}, v_l\} \in F_1 \setminus F_2$. If $d_{F_1}(v_l) > d_{F_2}^{v_0}(v_l)$, let $k := l + 1$, $\mathcal{P} := \mathcal{P}_l$, and stop the recursive definition. Otherwise we have $d_{F_1}(v_l) \le d_{F_2}^{v_0}(v_l)$. Since

$$d_{F_1 \cap E(\mathcal{P}_l)}(v_l) = d_{F_2 \cap E(\mathcal{P}_l)}^{v_0}(v_l) + 1,$$

we can choose a vertex v_{l+1} such that $\{v_l, v_{l+1}\} \in (F_2 \setminus E(\mathcal{P}_l)) \setminus (F_1 \setminus E(\mathcal{P}_l)) = F_2 \setminus (E(\mathcal{P}_l) \cup F_1)$. Thus $\mathcal{P}_{l+1} = (v_i)_{0 \le i \le l+1}$ is an F_1-F_2-alternating trail.

If l is even, we stop the recursive definition if $d_{F_1}(v_1) < d_{F_2}^{v_0}(v_l)$, or we can find similarly to the first case an edge $\{v_l, v_{l+1}\} \in F_1 \setminus (E(\mathcal{P}_l) \cup F_2)$.

If the definition does not stop, let $k := \omega$ and $\mathcal{P} := (v_i)_{0 \le i < \omega}$. $\square$

The following lemma gives us some possibilities to apply the F_1-F_2-trails.

Lemma 2. *Let F_1, F_2 be factors of G and let $\mathcal{P} = (v_i)_{0 \le i < k}$ be an F_1-F_2-trail.*
(a) *If $\delta_{F_2}(v_0) < \delta_{F_1}(v_0)$ and $d_{F_1}(v_0) > d_{F_2}(v_0)$, then*

$$\delta_{F_1 \oplus E(\mathcal{P})}(v_0) = \delta_{F_1 v_{k-1}}(v_0) - 1 \quad \text{if } k < \omega,$$

$$\delta_{F_1 \oplus E(\mathcal{P})}(v_0) = \delta_{F_1}(v_0) - 1 \qquad \text{if } k = \omega.$$

(b) *If* $\delta_{F_1}(v_0) < \delta_{F_2}(v_0)$ *and* $d_{F_1}(v_0) > d_{F_2}(v_0)$, *then*

$$\delta_{F_2 \oplus E(\mathcal{P})}(v_0) = \delta_{F_2 v_{k-1}}(v_0) - 1 \quad \text{if } k < \omega,$$

$$\delta_{F_2 \oplus E(\mathcal{P})}(v_0) = \delta_{F_2}(v_0) - 1 \quad \text{if } k = \omega.$$

(c) *If* $k < \omega$, $v_0 \neq v_{k-1}$, *and* $\delta_{F_2}(v_{k-1}) \leqslant \delta_{F_1}(v_{k-1})$, *then*

$$\delta_{F_1 \oplus E(\mathcal{P})}(v_{k-1}) = \delta_{F_1}(v_{k-1}) - 1.$$

(d) *If* $k < \omega$, $v_0 \neq v_{k-1}$, *and* $\delta_{F_1}(v_{k-1}) \leqslant \delta_{F_2}(v_{k-1})$, *then*

$$\delta_{F_2 \oplus E(\mathcal{P})}(v_{k-1}) = \delta_{F_2}(v_{k-1}) - 1.$$

Before we present the simple proof of this lemma we want to give a corollary.

Corollary 1. *Let* $\mathcal{P} = (v_i)_{0 \leqslant i < k}$ *be an* F_1-F_2-*trail such that* $d_{F_1}(v_0) > d_{F_2}(v_0)$.
(a) *If* $\delta_{F_2}(v_0) < \delta_{F_1}(v_0)$ *and* $\delta_{F_2}(v_{k-1}) \leqslant \delta_{F_1}(v_{k-1})$, *then* $F_1 \oplus E(\mathcal{P})$ *is a factor better than* F_1.
(b) *If* $\delta_{F_1}(v_0) < \delta_{F_2}(v_0)$ *and* $\delta_{F_1}(v_{k-1}) \leqslant \delta_{F_2}(v_{k-1})$, *then* $F_2 \oplus E(\mathcal{P})$ *is a factor better than* F_2.

Note that the assumption $\delta_{F_2}(v_{k-1}) \leqslant \delta_{F_1}(v_{k-1})$ ($\delta_{F_1}(v_{k-1}) \leqslant \delta_{F_2}(v_{k-1})$, resp.) is fulfilled if F_2 is better than F_1 (F_1 is better than F_2, resp.).

Proof. (a) *Case* 1: $k = \omega$ *or* ($k < \omega$ *and* $v_0 \neq v_{k-1}$).
Since $\delta_{F_2}(v_0) < \delta_{F_1}(v_0)$ and $d_{F_1}(v_0) > d_{F_2}(v_0)$, we have $d_{F_1}(v_0) > f(v_0)$. Therefore

$$\delta_{F_1 \oplus E(\mathcal{P})}(v_0) = |d_{F_1 \oplus E(\mathcal{P})}(v_0) - f(v_0)| = |d_{F_1}(v_0) - 1 - f(v_0)|$$
$$= |d_{F_1}(v_0) - f(v_0)| - 1 = \delta_{F_1}(v_0) - 1.$$

Case 2: $k < \omega$ *and* $v_0 = v_{k-1}$.
Case 2.1: $\{v_{k-2}, v_{k-1}\} \in F_1 \setminus F_2$ *and* $d_{F_1}(v_{k-1}) > d_{F_2}^{v_0}(v_{k-1}) = d_{F_2}(v_{k-1}) + 1$.
Since $\delta_{F_2}(v_0) < \delta_{F_1}(v_0)$, we conclude that $d_{F_1}(v_0) > f(v_0) + 1$. Therefore

$$\delta_{F_1 \oplus E(\mathcal{P})}(v_0) = |d_{F_1 \oplus E(\mathcal{P})}(v_0) - f(v_0)| = |d_{F_1}(v_0) - 2 - f(v_0)|$$
$$= |d_{F_1}(v_0) - f(v_0)| - 2 = \delta_{F_1 v_{k-1}}(v_0) - 1.$$

Case 2.2: $\{v_{k-2}, v_{k-1}\} \in F_2 \setminus F_1$ *and* $d_{F_1}(v_{k-1}) < d_{F_2}^{v_0}(v_{k-1}) = d_{F_2}(v_{k-1}) + 1$.
This case cannot occur since $d_{F_1}(v_0) > d_{F_2}(v_0)$.
(b), (c), and (d) can be proved similarly. □

A first application of the 'technique of F_1-F_2-trails, we find in the proof of the following lemma.

Lemma 3. *Let F be an optimal factor and let $\mathcal{P}=(v_i)_{0\le i<k}$ be an F-alternating trail satisfying one of the following properties:*

(i) $\{v_0, v_1\}\in F$ *and* $d_F(v_0)>f(v_0)$, *or*

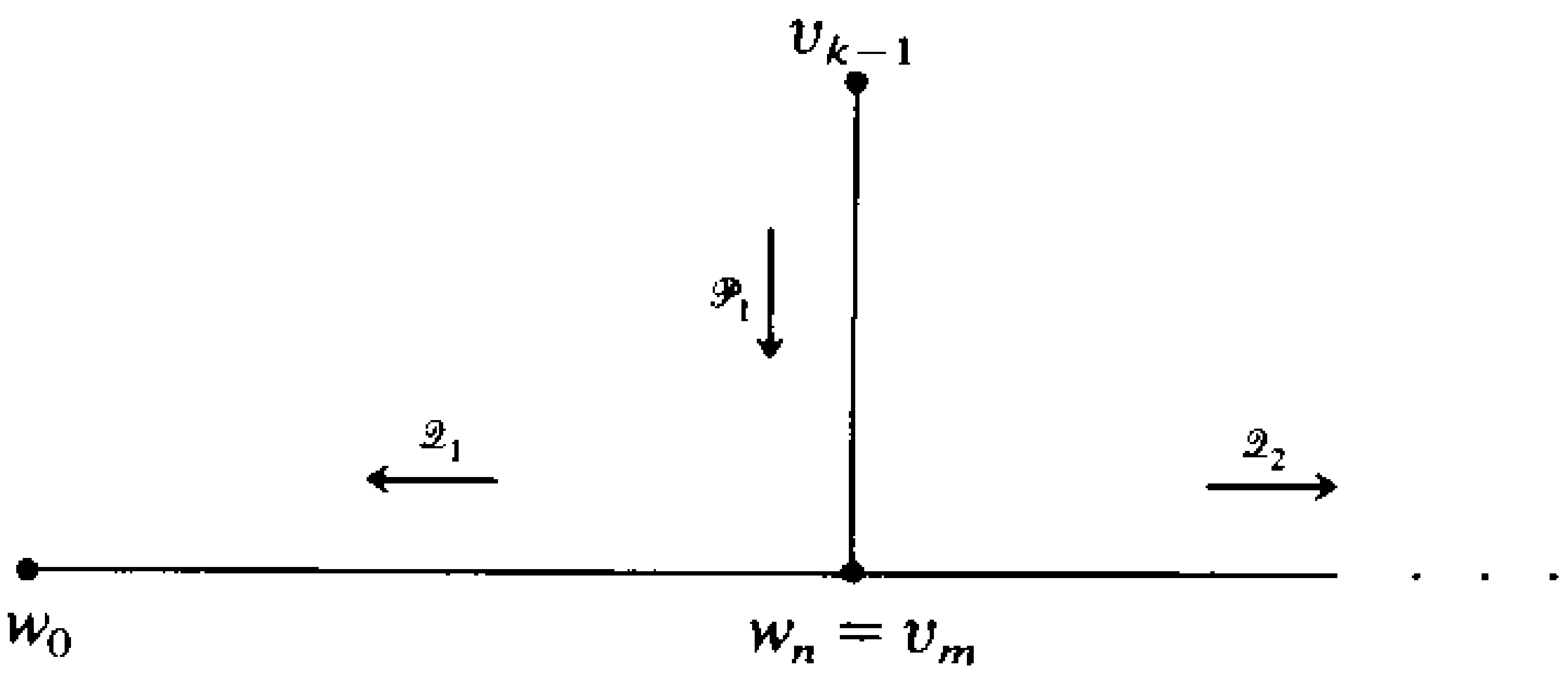
(ii) $\{v_0, v_1\}\in F^C$ *and* $d_F(v_0)<f(v_0)$.

Then $F\oplus E(\mathcal{P})$ is an optimal factor; further $k<\omega$ and $\delta_{F\oplus E(\mathcal{P})}=\delta_{Fv_0}^{v_{k-1}}$.

In other words: If an optimal factor is improved by an alternating trail at its startpoint it remains optimal, but it is debased at its endpoint.

Proof. Since F is optimal, $F\oplus E(\mathcal{P})$ is not better than F and therefore we conclude $k<\omega$ and $\delta_{F\oplus E(\mathcal{P})}=\delta_{Fv_0}^{v_{k-1}}$. To prove that $F\oplus E(\mathcal{P})$ is optimal, we assume for a contradiction that H is a factor better than $F\oplus E(\mathcal{P})$. Then there is some $w_0\in V$ such that $\delta_H(w_0)<\delta_{F\oplus E(\mathcal{P})}(w_0)$. If $d_{F\oplus E(\mathcal{P})}(w_0)<d_H(w_0)$. let $\mathcal{Q}=(w_i)_{0\le i<l}$ be an H-$(F\oplus E(\mathcal{P}))$-trail, and if $d_{F\oplus E(\mathcal{P})}(w_0)>d_H(w_0)$, let $\mathcal{Q}=(w_i)_{0\le i<l}$ be an $(F\oplus E(\mathcal{P}))$-H-trail. We have $V(\mathcal{P})\cap V(\mathcal{Q})\neq\emptyset$; otherwise $d_F(v)=d_{F\oplus E(\mathcal{P})}(v)$ for each $v\in V(\mathcal{Q})$, $\mathcal{Q}$ would be an H-F-trail (an F-H-trail, resp.), and, by Lemma 2, we would get $\delta_{F\oplus E(\mathcal{Q})}=\delta_{Fw_0w_{l-1}}$, if $l<\omega$, and $\delta_{F\oplus E(\mathcal{Q})}=\delta_{Fw_0}$, if $l=\omega$. Both contradict the assumption that F is optimal. Consequently there exists $m:=\max\{i: v_i\in V(\mathcal{Q})\}$. Choose n such that $w_n=v_m$ and let

$$\mathcal{Q}_1:=(w_{n-i})_{0\le i\le n},\qquad \mathcal{Q}_2:=(w_i)_{n\le i<l},\qquad \mathcal{P}_1:=(v_{k-i})_{1\le i\le k-m}.$$



Since F is optimal and, by Lemma 2,

$$\delta_{F\oplus E(\mathcal{P})\oplus E(\mathcal{Q})}=\delta_{F\oplus E(\mathcal{P})w_0w_{l-1}}=\delta_{Fv_0w_0w_{l-1}}^{v_{k-1}}\quad \text{if } l<\omega,$$

$$\delta_{F\oplus E(\mathcal{P})\oplus E(\mathcal{Q})}=\delta_{F\oplus E(\mathcal{P})w_0}=\delta_{Fv_0w_0}^{v_{k-1}}\quad \text{if } l=\omega,$$

we conclude that $v_{k-1}\notin\{w_0, w_{l-1}\}$. Therefore both $\widehat{\mathcal{P}_1\mathcal{Q}_1}$ and $\widehat{\mathcal{P}_1\mathcal{Q}_2}$ have positive length. We leave it to the reader to show that either $\mathcal{R}:=\widehat{\mathcal{P}_1\mathcal{Q}_1}$ or $\mathcal{R}:=\widehat{\mathcal{P}_1\mathcal{Q}_2}$ is an $(F\oplus E(\mathcal{P}))$-alternating trail such that $F\oplus E(\mathcal{P})\oplus E(\mathcal{R})$ is a factor better than F. This is a contradiction to the fact that F is optimal. $\square$

The following lemma will be applied in the proof of Theorem 1 and in Section 4.

Lemma 4. *Let F be a factor. Further let H be a factor such that at most one of the sets*

$$V_1(H) := \{v \in V: \delta_F(v) < \delta_H(v)\},$$
$$V_2(H) := \{v \in V: \delta_F(v) > \delta_H(v)\}$$

is infinite. Define

$$\vartheta(H) := \begin{cases} +\infty & \text{if } V_1(H) \text{ is infinite}, \\ -\infty & \text{if } V_2(H) \text{ is infinite}, \\ \sum_{v \in V} \delta_H(v) - \delta_F(v) & \text{else}. \end{cases}$$

If F is optimal, then $\vartheta(H) \geq 0$.

Proof. Let F be optimal. Suppose that there is a factor H with the property that at least one of the sets $V_1(H)$, $V_2(H)$ is finite and $\vartheta(H) < 0$. Then $\vartheta(H) \neq +\infty$ and therefore $V_1(H)$ is finite and

$$\vartheta_1(H) := \sum_{v \in V_1(H)} \delta_H(v) - \delta_F(v) = \sum_{v \in V \setminus V_2(H)} \delta_H(v) - \delta_F(v)$$

is nonnegative integer. Choose H such that $\vartheta_1(H)$ is minimal. Since $\vartheta(H) < 0$, we can choose $v_0 \in V_2(H)$. Then $d_H(v_0) \neq d_F(v_0)$.

Case 1: $d_H(v_0) < d_F(v_0)$.

Let $\mathcal{P} = (v_i)_{0 \leq i < k}$ be an F-H-trail. Since $v_0 \in V_2(H)$ and $d_H(v_0) < d_F(v_0)$, we have $d_F(v_0) > f(v_0)$ and hence $k < \omega$ by Lemma 3. Since F is optimal, we get $v_{k-1} \in V_1(H)$ by Corollary 1(a). In particular, $v_0 \neq v_{k-1}$ and from Lemma 2(d) follows

$$\delta_{H \oplus E(\mathcal{P})}(v_{k-1}) = \delta_H(v_{k-1}) - 1. \tag{1}$$

Therefore $v_{k-1} \notin V_2(H \oplus E(\mathcal{P}))$. Further it follows from

$$\delta_{H \oplus E(\mathcal{P})}(v_0) = \delta_H(v_0) \pm 1 \tag{2}$$

and $v_0 \in V_2(H)$, that $v_0 \notin V_1(H \oplus E(\mathcal{P}))$. Since, for each $v \notin \{v_0, v_{k-1}\}$, $v \in V_i(H \oplus E(\mathcal{P}))$ iff $v \in V_i(H)$ $(i = 1, 2)$, at most one of the sets $V_1(H \oplus E(\mathcal{P}))$, $V_2(H \oplus E(\mathcal{P}))$ is infinite and from (1) and (2) we get $\vartheta(H \oplus E(\mathcal{P})) \leq \vartheta(H) < 0$ and $\vartheta_1(H \oplus E(\mathcal{P})) = \vartheta_1(H) - \delta_H(v_{k-1}) + \delta_{H \oplus E(\mathcal{P})}(v_{k-1}) = \vartheta_1(H) - 1$. This contradicts the minimality of $\vartheta_1(H)$.

Case 2: $d_H(v_0) > d_F(v_0)$.

Analogously we can construct a contradiction. □

The following theorem characterizes in some cases the optimal factors of G—e.g. if G is finite.

Theorem 1. *Define*

$$\Delta_0 := \min\{\Delta(F, f): F \subseteq E \text{ is a factor}\}.$$

If Δ_0 is finite, then a factor F is optimal iff $\Delta(F, f) = \Delta_0$.

Proof. We only proof the nontrivial implication. Let F be an optimal factor. We have to show that $\Delta(F, f) \leqslant \Delta_0$. Choose a factor H satisfying $\Delta(H, f) = \Delta_0$. Since $\Delta_0 \in \omega$, the set $V_1(H)$ from Lemma 4 is finite. By Lemma 4, we get $0 \leqslant \sum_{v \in V} \delta_H(v) - \delta_F(v) = \Delta_0 - \Delta(F, f)$. Thus $\Delta(F, f) \leqslant \Delta_0$. □

Corollary 2. *If G is a finite graph, then a factor F is optimal iff it is optimal in the sense of Lovász and Plummer* [4], *i.e.* $\Delta(F, f) = \Delta_0$.

3. f-F-trails, f-obstructions, and stable f-factors

In this section we are going to examine f-factors. We give a necessary and sufficient condition for the existence of an f-perfect factor if G is a countable graph (Theorem 3). This criterion is a generalization of Steffens' criterion for the existence of an 1-perfect factor of a countable graph [7]. By the aid of this criterion Steffens proves in [7] for countable graphs the existence of an $\subseteq$-maximal matchable set $V' \subseteq V$ or, in our terminology, the existence of an 1-optimal 1-factor. We want to generalize this for f-factors, too (Theorem 6). The role of 'F-augmenting pathes' in [7] will be played in this paper by the 'f-F-trails' which have been introduced by Steffens in [8] (and are called 'F-augmenting trails' there).

Let F be an f-factor of G and let $\mathscr{P} = (v_i)_{0 \leqslant i < k}$ be a trail satisfying $k > 1$. $\mathscr{P}$ is called an *f-F-trail* if the following properties are fulfilled:

(i) $\mathscr{P}$ is F-alternating,

(ii) $d_F(v_0) < f(v_0)$,

(iii) $\{v_0, v_1\} \in F^C$,

(iv) if $k < \omega$, then $\{v_{k-2}, v_{k-1}\} \in F^C$ and $d_F(v_{k-1}) < f_{v_0}(v_{k-1})$.

It is easy to see that the following holds.

Lemma 5. *Let F be an f-factor and let $\mathscr{P} = (v_i)_{0 \leqslant i < k}$ be an f-F-trail. Then $F \oplus E(\mathscr{P})$ is an f-factor better than F. If $k < \omega$, then $d_{F \oplus E(\mathscr{P})} = d_F^{v_0 v_{k-1}}$, and if $k = \omega$, then $d_{F \oplus E(\mathscr{P})} = d_F^{v_0}$.*

An easy consequence of Lemma 1 is the following.

Lemma 6. *Let F_1 be an f-factor better than an f-factor F_2 and let $v_0 \in V$ such that $d_{F_2}(v_0) < d_{F_1}(v_0)$. Then there exists an f-F_2-trail starting at v_0.*

Proof. Choose an F_1-F_2-trail starting at v_0. This is an f-F_2-trail. □

Let F be an f-factor and let v_0 be an unsaturated vertex such that there is no f-F-trail starting at v_0. Then (F, v_0) is called an *(f-)obstruction*. If there is no f-obstruction in G, G is called *(f-)unobstructed*. The condition of being unobstructed corresponds to '*condition* (A)' for 1-factors in [7]. Since a perfect factor is better than any nonperfect factor, we have the following by Lemma 6.

Theorem 2. *If G possesses a perfect factor, then G is unobstructed.*

The converse of this theorem is trivial for finite graphs and false for arbitrary graphs. This is shown again by $K_{\aleph_0, \aleph_1}$ or G^*, described in Section 1. For countable graphs however the converse holds (Theorem 3). To show this, we need the concept of 'stable factors' which corresponds to that of 'independent matchings' in [7]. An f-factor F is called (f-)*stable* if there is no f-F-trail traversing edges of F. So, an f-factor is stable iff each f-F-trail has length 1. Note that $\emptyset$ and each perfect factor are stable. In the next lemmas we want to collect some properties of stable f-factors and obstructions.

Lemma 7. *Let F be a stable f-factor, let $H \supseteq F$ be an f-factor, and let $\mathcal{P}$ be an f-H-trail. Then $\mathcal{P}$ is an $(f - d_F)$-$(H \setminus F)$-trail in the graph $G \setminus F := (V, E \setminus F)$.*

Proof. Let $\mathcal{P} = (v_i)_{0 \leq i < k}$. Since $d_{H \setminus F} = d_H - d_F \leq f - d_F$, $H \setminus F$ is an $(f - d_F)$-factor of $G \setminus F$, and a vertex is f-H-saturated iff it is $(f - d_F)$-$(H \setminus F)$-saturated. It remains to show that there is no edge in $F \cap E(\mathcal{P})$. Suppose for a contradiction that $F \cap E(\mathcal{P}) \neq \emptyset$. Without loss of generality we can assume that $\{v_1, v_2\} \in F$; otherwise consider $H' = H \setminus \{\{v_{s-2}, v_{s-1}\}\}$ and $\mathcal{P}' = (v_i)_{s-1 \leq i < k}$, where $s := \min\{i: \{v_i, v_{i+1}\} \in F\}$. If there is some edge in $(H \setminus F) \cap E(\mathcal{P})$, let $t := \min\{i: \{v_i, v_{i+1}\} \in H \setminus F\}$. Otherwise let $t := k$. In both cases $(v_i)_{0 \leq i < t}$ is an f-F-trail which traverses an edge of F. This is a contradiction to the stability of F. □

Lemma 8. *Let F be an f-stable f-factor of G and let $J \subseteq F^C$ be an $(f - d_F)$*-stable *$(f - d_F)$-factor of the graph $G \setminus F$. Then $F \cup J$ is an f-stable f-factor of G.*

Proof. Since $d_{F \cup J} = d_F + d_J \leq d_F + (f - d_F) = f$, $F \cup J$ is an f-factor. Let $\mathcal{P}$ be an f-$(F \cup J)$-trail. We have to show that there is no $(F \cup J)$-edge in $E(\mathcal{P})$. Because of Lemma 7, $\mathcal{P}$ is an $(f - d_F)$-J-trail in $G \setminus F$. Thus $\mathcal{P}$ traverses no edge of F. Furthermore, $\mathcal{P}$ traverses no edge of J by the stability of J. □

Lemma 9. *If G is f-unobstructed and if F is a stable f-factor of G, then $G \setminus F$ is $(f - d_F)$-unobstructed.*

Proof. Let J be an $(f - d_F)$-factor of $G \setminus F$ and let $v_0 \in V$ be an $(f - d_F)$-J-unsaturated vertex. Since $J \cup F$ is an f-factor of G and v_0 is f-$(J \cup F)$-unsaturated and G is f-unobstructed, there exists an f-$(J \cup F)$-trail starting at v_0. Because of Lemma 7, this is already an $(f - d_F)$-J-trail in $G \setminus F$. Therefore (J, v_0) is no $(f - d_F)$-obstruction. □

Lemma 10. *G contains an $\subseteq$-maximal stable f-factor.*

Proof. Let $\mathcal{M}$ be the set of stable f-factors of G. By Lemma 7, $\bigcup_{\lambda<\kappa} F_\lambda$ is a stable f-factor whenever $(F_\lambda)_{\lambda<\kappa}$ is a $\subseteq$-chain of stable f-factors. Since $\mathcal{M} \neq \emptyset$ ($\emptyset \in \mathcal{M}$) the assertion follows by Zorn's Lemma. □

Note that each optimal f-factor is $\subseteq$-maximal stable. The converse is false as $K_{\aleph_0,\aleph_1}$ and $\emptyset$ show. We will see later that the equivalence holds for countable graphs. Because of Lemma 8 we have the following.

Lemma 11. *Let F be an $\subseteq$-maximal f-stable f-factor of G. Then $\emptyset$ is the only $(f-d_F)$-stable $(f-d_F)$-factor of $G\backslash F$.*

In some cases, the following two lemmas can give us information about the existence of stable f-factors which are non-empty.

Lemma 12. *Let H be an f-factor of G and let $x_0 \in V$. We define $\mathcal{A} := \{\mathcal{P}$: $\mathcal{P}$ is an H-alternating trail starting at x_0 with an H-edge$\}$ and $F := \{e \in H$: There exists a trail $\mathcal{P} \in \mathcal{A}$ terminating at a vertex incident with $e\}$. If $\mathcal{A}$ contains no trail of infinite length and if no trail $\mathcal{P} \in \mathcal{A}$ terminates with an H^C-edge at an f-H-unsaturated vertex, then F is a stable f-factor.*

Proof. Suppose F is not stable. Then there exists an f-F-trail $\mathcal{Q} = (w_i)_{0\leq i<k}$ traversing an edge $e = \{x, y\} \in F$. By definition of F, there exists a trail $\mathcal{P} = (x_i)_{0\leq i\leq l} \in \mathcal{A}$ with endpoint $x_l = x$, without loss of generality.

Case 1: $\{x_0, x_1\} = \{w_s, w_{s+1}\}$ *for some* s.

If $x_0 = w_s$ let $\mathcal{R} := (w_i)_{s\leq i<k}$, and if $x_0 = w_{s+1}$ let $\mathcal{R} := (w_{s+1-i})_{0\leq i\leq s+1}$. $\mathcal{R}$ is an F-alternating trail starting at x_0 with an H-edge. $\mathcal{R}$ has infinite length or terminates with an F^C-edge at an F-unsaturated vertex. For the vertices $x_0 = w_s, w_{s+1}, w_{s+2}, \ldots$, resp. $x_0 = w_{s+1}, w_s, w_{s-1}, \ldots$ it follows successively by the definition of F:

$$\{w_i, v\} \in H \text{ iff } \{w_i, v\} \in F.$$

So $\mathcal{R}$ is an H-alternating trail which starts at x_0 with an H-edge and has either infinite length or terminates with an H^C-edge at an H-unsaturated vertex. This is a contradiction to the assumption on $\mathcal{A}$.

Case 2: $\{x_0, x_1\} \notin E(\mathcal{Q})$.

Let $r := \min\{i \geq 1: x_i \in V(\mathcal{Q})\}$ and define

$$\mathcal{P}_1 := (x_i)_{0\leq i\leq r}, \qquad \mathcal{Q}_1 := (w_i)_{s\leq i<k}, \qquad \mathcal{Q}_2 := (w_{s-i})_{0\leq i\leq s},$$

where s is any index such that $w_s = x_r$. If either $\{x_{r-1}, x_r\} \in H$ and $s = 0$, or $\{x_{r-1}, x_r\} \in H$, $s > 0$, and $\{w_{s-1}, w_s\} \in F$, or $\{x_{r-1}, x_r\} \in H^C$, $s > 0$, and $\{w_{s-1}, w_s\} \in F^C$, let $\mathcal{R} := \widehat{\mathcal{P}_1\mathcal{Q}_1}$. Otherwise let $\mathcal{R} := \widehat{\mathcal{P}_1\mathcal{Q}_2}$. As in case 1 $\mathcal{R}$ is an H-alternating trail starting with an H-edge at x_0, and $\mathcal{R}$ has either infinite length or terminates with an H^C-edge at an f-H-unsaturated vertex. Again this is a contradiction to the assumption on $\mathcal{A}$. □

Lemma 13. *Let H be an f-factor and $x_0 \in V$. We define $\mathscr{B} := \{\mathscr{P}$: $\mathscr{P}$ is an H-alternating trail which starts at x_0, but not with an H-edge$\}$, $F := \{e \in H$: There exists a trail $\mathscr{P} \in \mathscr{B}$ terminating at a vertex incident with $e\}$. If $\mathscr{B}$ contains no trail of infinite length and if no trail $\mathscr{P} \in \mathscr{B}$ terminates with an H^C-edge at a vertex w satisfying $d_H(w) < f_{x_0}(w)$, then F is a stable f-factor.*

Note that the F in Lemma 12 and in Lemma 13 contains all H-edges incident with x_0.

Proof. As in the proof of Lemma 12 we assume that there is an f-F-trail $\mathscr{Q} = (w_i)_{0 \leq i < k}$ which traverses an edge $e = \{x, y\} \in F$. We conclude that $x_0 \notin \{w_0, w_{k-1}\}$ since otherwise $\mathscr{Q}$ (resp. $(w_{k-i})_{0<i\leq k}$) would be an f-H-trail in contradiction to the assumption on $\mathscr{B}$. By the definition of F, there exists $\mathscr{P} = (x_i)_{0 \leq i \leq l} \in \mathscr{B}$ with endpoint $x_l = x$, without loss of generality. Let $r := \min\{i \geq 0: x_i \in V(\mathscr{Q})\}$ and define

$$\mathscr{P}_1 := (x_i)_{0 \leq i \leq r}, \qquad \mathscr{Q}_1 := (w_i)_{s \leq i < k}, \qquad \mathscr{Q}_2 := (w_{s-i})_{0 \leq i \leq s},$$

where s is any index satisfying $w_s = x_r$. As in the proof of Lemma 12, either $\mathscr{R} := \widehat{\mathscr{P}_1 \mathscr{Q}_1}$ or $\mathscr{R} := \widehat{\mathscr{P}_1 \mathscr{Q}_2}$ is an H-alternating trail starting at x_0 with an H^C-edge which has infinite length or terminates with an H^C-edge at an f-H-unsaturated vertex, which is f_{x_0}-H-unsaturated too since $x_0 \notin \{w_0, w_{k-1}\}$. This is a contradiction to the assumption on $\mathscr{B}$. $\square$

Lemma 14. *Let $x_0 \in V$ such that $f(x_0) > 0$. If G is f-unobstructed and if G contains no non-empty stable f-factor, then the following holds:*
(a) *G is f_{x_0}-unobstructed.*
(b) *G is f^{x_0}-unobstructed.*

Proof. (a) Suppose that there is some f_{x_0}-obstruction (J, v_0). Since G is f-unobstructed, there exists an f-J-trail $\mathscr{P} = (v_i)_{0 \leq i < k}$. We have $k < \omega$ and $d_J(v_{k-1}) \geq f_{x_0 v_0}(v_{k-1})$ because otherwise $\mathscr{P}$ still would be an f_{x_0}-J-trail. Since $d_J(v_{k-1}) < f_{v_0}(v_{k-1})$, we get $v_{k-1} = x_0$ and

$$d_J(x_0) = f_{v_0}(x_0) - 1. \tag{1}$$

Let $H := J \oplus E(\mathscr{P})$. H is an f-factor and so we conclude that

$$d_H = d_J^{v_0 v_{k-1}} \leq f. \tag{2}$$

By (1) and (2) we get

$$d_H(x_0) = f(x_0).$$

Now we consider the set $\mathscr{A}$ from Lemma 12. Suppose there is a trail $\mathscr{Q} = (x_i)_{0 \leq i < l} \in \mathscr{A}$ satisfying $l = \omega$ or $l < \omega$, $\{x_{l-2}, x_{l-1}\} \in H^C$, and $d_H(x_{l-1}) < f(x_{l-1})$.

Case 1: $l < \omega$.

Then $x_{l-1} \neq x_0$ and, by (2), $d_{H\oplus E(\mathcal{Q})} = d_{Hx_0}^{x_{l-1}} \leq f_{x_0}$. Further $d_J < d_J^{v_0 x_{l-1}} = d_{Hx_0}^{x_{l-1}} = d_{H\oplus E(\mathcal{Q})}$ by (2). Therefore $H \oplus E(\mathcal{Q})$ is an f_{x_0}-factor better than J and $d_J(v_0) < d_{H\oplus E(\mathcal{Q})}(v_0)$.

Case 2: $l = \omega$.

Similarly to case 1 one can prove that $H \oplus E(\mathcal{Q})$ is an f_{x_0}-factor better than J and $d_J(v_0) < d_{H\oplus E(\mathcal{Q})}(v_0)$.

Now Lemma 6 yields an f_{x_0}-J-trail starting at v_0. This is a contradiction to the fact that (J, v_0) is an f_{x_0}-obstruction. Therefore the conditions of Lemma 12 are fulfilled and we obtain an f-stable f-factor F. F is not empty since $d_H(x_0) = f(x_0) > 0$. This contradicts the requirement of the lemma.

(b) Suppose that there is an f^{x_0}-obstruction (H, v_0).

Case 1: $v_0 = x_0$.

Then $d_H(x_0) = f(x_0)$ since otherwise (H, x_0) would be an f-obstruction, too. Thus, the members of the set $\mathcal{B}$ in Lemma 13 which have infinite length or terminate with an H^C-edge in an f-H-unsaturated vertex are exactly the f^{x_0}-trails starting at x_0. Therefore, since (H, x_0) is an f^{x_0}-obstruction, the conditions on the set $\mathcal{B}$ in Lemma 13 are fulfilled. By Lemma 13 and $d_H(x_0) = f(x_0) > 0$, we obtain a non-empty f-stable f-factor. This contradicts the requirement of the lemma.

Case 2: $v_0 \neq x_0$.

Then $d_H(x_0) = f(x_0) + 1$ because otherwise (H, v_0) would be an f-obstruction, too. Choose $e = \{y, x_0\} \in H$ and define $J := H \setminus \{e\}$. J is an f-factor. By interated application of the fact that G is f-unobstructed and of Lemma 5, we obtain an f-factor K which is better than J and saturates v_0 and y. K is an f^{x_0}-factor and x_0 is f^{x_0}-unsaturated. There is no f^{x_0}-K-trail $\mathcal{P} = (x_i)_{0\leq i<k}$, since otherwise $K \oplus E(\mathcal{P})$ is an f^{x_0}-factor better than H which saturates v_0 and Lemma 6 yields an f^{x_0}-H-trail starting at v_0, contrary to the assumption that (H, v_0) is an f^{x_0}-obstruction. Hence (K, x_0) is an f^{x_0}-obstruction. This is impossible by case 1. □

Lemma 15. *Let $x_0 \in V$. If G is f-unobstructed, then there exists an f-factor F saturating x_0 such that $G \setminus F$ is $(f - d_F)$-unobstructed.*

Proof. The lemma is proved by induction on $f(x_0)$. If $f(x_0) = 0$, let $F := \emptyset$. Now let $f(x_0) > 0$. By Lemma 9, 10, and 11, we can assume without loss of generality that G contains no non-empty stable f-factor. By the induction hypothesis and Lemma 14(a), we obtain an f_{x_0}-factor H such that $G \setminus H$ is $(f_{x_0} - d_H)$-unobstructed and $d_H(x_0) = f_{x_0}(x_0) = f(x_0) - 1$. Again we can assume without loss of generality that $G \setminus H$ contains no non-empty stable $(f_{x_0} - d_H)$-factor. Since x_0 is f-H-unsaturated and G is f-unobstructed, there exists an f-H-trail $\mathcal{Q} = (x_i)_{0\leq i<k}$. If there is some $i < k$ such that x_i is f_{x_0}-H-unsaturated, let

$$r := \min\{i: d_H(x_i) < f_{x_0}(x_i)\} + 1.$$

Otherwise let $r := \omega$. Note that $r > 1$ and $x_0 \neq x_{r-1}$. Let

$$\mathcal{Q}_1 := (x_i)_{0\leq i<r} \quad \text{and} \quad F := H \oplus E(\mathcal{Q}_1).$$

F is an f-factor which saturates x_0. We will show that $G\setminus F$ is $(f-d_F)$-unobstructed. For this let K be an $(f-d_F)$-factor of $G\setminus F$ and let v_0 be an $(f-d_F)$-K-unsaturated vertex. We must find an $(f-d_F)$-K-trail in $G\setminus F$ starting at v_0. Since

$$d_K(x_i)=0 \quad \text{for all } i<r-1,$$

it follows that $K\cap E(\mathcal{Q}_1)=\emptyset$. Hence K is a factor of $G\setminus H$.

Case 1: $r=\omega$.

Then $f-d_F=f-d_H^{x_0}=f_{x_0}-d_H$. Thus K is an $(f_{x_0}-d_H)$-factor of $G\setminus H$ and v_0 is $(f_{x_0}-d_H)$-K-unsaturated. Since $G\setminus H$ is $(f_{x_0}-d_H)$-unobstructed, there exists an $(f_{x_0}-d_H)$-K-trail $\mathcal{P}=(v_i)_{0\leq i<k}$ in $G\setminus H$. Since $d_K(x_i)=0$ for all $i<\omega$, we conclude that $E(\mathcal{P})\cap E(\mathcal{Q})=\emptyset$. Thus $\mathcal{P}$ is an $(f-d_F)$-K-trail in $G\setminus F$.

Case 2: *$r<\omega$ and $\{x_{r-2},x_{r-1}\}\in H$.*

Then $f-d_F=f-d_{Hx_{r-1}}^{x_0}=(f_{x_0}-d_H)^{x_{r-1}}$. Thus K is an $(f_{x_0}-d_H)^{x_{r-1}}$-factor of $G\setminus H$ and v_0 is $(f_{x_0}-d_H)^{x_{r-1}}$-K-unsaturated. Since $f_{x_0}(x_{r-1})-d_H(x_{r-1})>0$, $G\setminus H$ is $(f_{x_0}-d_H)^{x_{r-1}}$-unobstructed by Lemma 14(b). Consequently there exists an $(f_{x_0}-d_H)^{x_{r-1}}$-K-trail $\mathcal{P}=(v_i)_{0\leq i<k}$ in $G\setminus H$. Since $d_K(x_i)=0$ for all $i<r-1$, we have again $E(\mathcal{P})\cap E(\mathcal{Q}_1)=\emptyset$ and so $\mathcal{P}$ is an $(f-d_F)$-K-trail in $G\setminus F$.

Case 3: $r<\omega$ and $\{x_{r-2},x_{r-1}\}\in H^{\mathrm{C}}$.

Similarly to case 2 we find with the help of Lemma 14(a) an $(f_{x_0}-d_H)_{x_{r-1}}$-K-trail in $G\setminus H$ starting at v_0 which is an $(f-d_F)$-K-trail in $G\setminus F$. □

Now we are able to prove the converse of Theorem 2 for countable graphs.

Theorem 3. *If G is countable, G possesses a perfect f-factor iff G is f-unobstructed.*

Proof. Let $V=\{v_i: i<\omega\}$. By Lemma 15 we are able to construct recursively f-factors $F_0\subseteq F_1\subseteq F_2\subseteq\cdots$ such that, for $i<\omega$, the vertices $v_0,\ldots,v_i$ are saturated by F_i and $G\setminus F_i$ is $(f-d_{F_i})$-unobstructed. Then $F:=\bigcup_{i<\omega}F_i$ is a perfect f-factor. □

The following theorem and its two corollaries are generalizations of Aharoni's Theorems 4, 3, and 1 in [1].

Theorem 4. *Let E be the disjoint union of E_1 and E_2, and let $f_1,f_2:V\to\omega$ be functions satisfying $f_1+f_2=f$ such that $G_1:=(V,E_1)$ is f_1-unobstructed and $G_2:=(V,E_2)$ is f_2-unobstructed. Then G is f-unobstructed.*

Proof. Let $F\subseteq E$ be an f-factor and v_0 be an f-F-unsaturated vertex. We have to find an f-F-trail starting at v_0. Let us define recursively a sequence $V_0\subseteq V_1\subseteq V_2\subseteq\cdots$ of finite subsets of V and, for $j=1,2$, sequences $F_0^{(j)}\subseteq F_1^{(j)}\subseteq F_2^{(j)}\subseteq\cdots$ of f_j-factors of G_j such that, for all $i<\omega$, each $x\in V_i$ is f_j-$F_i^{(j)}$-saturated and $G_j\setminus F_i^{(j)}$ is $(f_j-d_{F_i^{(j)}})$-unobstructed.

Let $V_0 := \{v_0\}$ and choose $F_0^{(1)}$, $F_0^{(2)}$ in agreement with Lemma 15. Now suppose that V_i, $F_i^{(1)}$, and $F_i^{(2)}$ have been defined. Let

$$V_{i+1} := \{x \in V: \exists y \in V_i: \{x, y\} \in F \cup F_i^{(1)} \cup F_i^{(2)}\} \cup V_i.$$

By induction hypothesis and since F, $F_i^{(1)}$, and $F_i^{(2)}$ are factors, V_{i+1} is finite. By iterated application of Lemma 15 and induction hypothesis, we obtain an $(f_j - d_{F_i^{(j)}})$-factor H_j saturating in $G_j \setminus F_i^{(j)}$ all vertices of V_{i+1} such that $G_j \setminus (F_i^{(j)} \cup H_j)$ is $(f_j - d_{F_i^{(j)}} - d_{H_j})$-unobstructed $(j = 1, 2)$. Let $F_{i+1}^{(j)} := F_i^{(j)} \cup H_j$.

For $j = 1, 2$, define $F_j := \bigcup_{0 \le i < \omega} F_i^{(j)}$ and $H := F_1 \cup F_2$. H is an f-factor of G which saturates each vertex $x \in \bigcup_{0 \le i < \omega} V_i$. Now let $\mathscr{P} = (v_i)_{0 \le i < k}$ be an H-F-trail. It is easy to see that, for every $i < k$, $v_i \in V_i$ and therefore $d_H(v_i) = f(v_i)$. If either $k = \omega$ or $k < \omega$, $\{v_{k-2}, v_{k-1}\} \in H \setminus F$ and $d_H(v_{k-1}) > d_F^{v_0}(v_{k-1})$, then $\mathscr{P}$ is an f-F-trail and the lemma is proved. The case $k < \omega$, $\{v_{k-2}, v_{k-1}\} \in F \setminus H$, and $d_H(v_{k-1}) < d_F^{v_0}(v_{k-1})$ cannot occur since $d_F(v_{k-1}) \le f_{v_0}(v_{k-1}) = d_{Hv_0}(v_{k-1})$. □

Corollary 3. *If G is f-unobstructed and if $G^* = (V, E^*)$ is a graph such that $E^* \supseteq E$, then G^* is f-unobstructed.*

Proof. Let $E_1 := E$, $E_2 := E^* \setminus E$, $f_1 := f$, $f_2 := 0$. □

Corollary 4. *If G is f-unobstructed and if x is a vertex satisfying $f(x) > 0$, then there exists a vertex y adjacent to x such that $f(y) > 0$ and $G \setminus \{\{x, y\}\}$ is f_{xy}-unobstructed.*

Proof. By Lemma 15, there exists an f-factor F such that $d_F(x) = f(x)$ and $G \setminus F$ is $(f - d_F)$-unobstructed. Choose $y \in V$ such that $\{x, y\} \in F$. Now let $E_1 := E \setminus F$, $E_2 := F \setminus \{\{x, y\}\}$, $f_1 := f - d_F$, $f_2 := d_{Fxy}$. □

4. Optimal f-factors

We will prove that G possesses an optimal f-factor if G is countable. For that we must prove the following lemma.

Lemma 16. *Let J be an $\subseteq$-maximal stable f-factor. Further let*

$$W := \{v \in V: d_J(v) < f(v) \wedge \exists w \in V(d_J(w) < f(w) \wedge \{v, w\} \in E \setminus J)\},$$

$$f^*(v) := \begin{cases} f(v) - d_J(v) & \text{if } v \in W, \\ 0 & \text{if } v \in V \setminus W. \end{cases}$$

Then $G \setminus J$ is f^*-unobstructed.

Proof. Let H be an f^*-factor of $G \setminus J$ and let $x_0 \in V$ be a vertex satisfying $d_H(x_0) < f^*(x_0)$. We have to find an f^*-H-trail in $G \setminus J$ starting at x_0. Consider the set $\mathscr{B}$ and the factor F from Lemma 13, referring to the graph $G \setminus J$ and the function $f - d_J$. Since $x_0 \in W$, there exists $w \in W$ such that $\{x_0, w\} \in E \setminus J$. If $\{x_0, w\} \in H^C$ and $d_H(w) < f(w) - d_J(w)$, then (x_0, w) is an f^*-H-trail starting at x_0, and the lemma is proved. If on the other hand $\{x_0, w\} \in H$ or $d_H(w) = f(w) - d_J(w)$, then it follows from the definition of F that F is not empty. Hence F cannot be $(f - d_J)$-stable by Lemma 11. Therefore we conclude from Lemma 13 that $\mathscr{B}$ contains a trail $\mathscr{P}$ which has infinite length or terminates with an H^C-edge at an $(f - d_J)_{x_0}$-H-unsaturated vertex. So $\mathscr{P}$ is an $(f - d_J)$-H-trail in $G \setminus J$ starting at x_0. Clearly $\mathscr{P}$ cannot meet a vertex in $V \setminus W$. Thus $\mathscr{P}$ is an f^*-H-trail. □

Theorem 5. *Let F be an f-factor of G such that there is no f-factor better than F. Then F is already an optimal factor.*

Proof. Suppose there is some factor H better than F. Let $v_0 \in V$ such that $\delta_H(v_0) < \delta_F(v_0)$. Then we have $d_F(v_0) < d_H(v_0)$. Let $\mathscr{P} = (v_i)_{0 \leq i < k}$ be an H-F-trail. $F \oplus E(\mathscr{P})$ is better than F by Corollary 1(b). Thus $F \oplus E(\mathscr{P})$ is not an f-factor and therefore we get $k < \omega$ and

$$d_{F \oplus E(\mathscr{P})}(v_{k-1}) = d_F^{v_0 v_{k-1}}(v_{k-1}) > f(v_{k-1}).$$

Since $d_F(v_{k-1}) \leq f(v_{k-1})$ and $\delta_{F \oplus E(\mathscr{P})}(v_{k-1}) \leq \delta_F(v_{k-1})$, it follows that $v_0 = v_{k-1}$, $d_F(v_0) = f(v_0) - 1$, and $d_{F \oplus E(\mathscr{P})}(v_0) = f(v_0) + 1$. Hence $\delta_{F \oplus E(\mathscr{P})} = \delta_F$, and $F \oplus E(\mathscr{P})$ is not better than F. This is a contradiction. □

Theorem 6. *If G is countable, then G possesses an f-optimal factor. Moreover, each $\subseteq$-maximal stable f-factor is optimal.*

Proof. Let J be an $\subseteq$-maximal stable f-factor. Let W and f^* be defined as in Lemma 16. By Theorem 3 and Lemma 16, there exists a perfect f^*-factor H of $G \setminus J$. Assume that there is some f-$(J \cup H)$-trail $\mathscr{P} = (v_i)_{0 \leq i < k}$ in G. By Lemma 7, $\mathscr{P}$ is an $(f - d_J)$-H-trail in $G \setminus J$. Therefore we have $d_J(v_1) < f(v_1)$ and of course $d_J(v_0) < f(v_0)$ and $\{v_0, v_1\} \in J^C$. Thus $v_0 \in W$. Since $d_H(v_0) < f(v_0) - d_J(v_0) = f^*(v_0)$, we have a contradiction to the fact that H is a perfect f^*-factor.

Consequently, $J \cup H$ is stable. By the maximality of J, it follows that $H = \emptyset$. And therefore, no f-J-trail exists in G. By Lemma 6, there is no f-factor better than J. By Theorem 5, J is optimal. □

Corollary 5. *If G is countable, then G possesses a perfect f-factor iff for every nonperfect factor F there exists a factor better than F.*

For countable graphs it is not true that, for every factor F, there exists an f-optimal factor better than F. This is demonstrated by the following counterexample: Let $G = (V, E)$, where $V = \omega \times \{0, 1, 2\}$ and $E :=$

$\{\{(x,0),(x,1)\}: x \in \omega\} \cup \{\{(x,1),(y,2)\}: x, y \in \omega\}$. Further let $f = 1$ and $F := \{\{(x,1),(x,2)\}: x \in \omega\}$. Then there is no optimal factor better than F. But the following holds.

Lemma 17. *Let F be a factor of G. Further let $g: V \to \omega$ be a function such that $g(v) = f(v)$ for almost all $v \in V$ (i.e. $\{v \in V: g(v) \neq f(v)\}$ is finite), and let K be a g-optimal factor such that $d_K(v) = d_F(v)$ for almost all $v \in V$. Then there exists an f-optimal factor H satisfying $\delta_{H,f} \leqslant \delta_{F,f}$.*

Proof. To get a contradiction we assume that there is a sequence $(J_i)_{0 \leqslant i < \omega}$ of factors such that $\delta_{F,f} > \delta_{J_0,f} > \delta_{J_1,f} > \cdots$. If J is a factor satisfying $\delta_{J,f} \leqslant \delta_{F,f}$, it holds

$$\begin{aligned}
&\{v \in V: \delta_{K,g}(v) < \delta_{J,g}(v)\} \\
&\quad = \{v \in V: \delta_{F,f}(v) < \delta_{J,f}(v) \text{ and } f(v) = g(v) \text{ and } d_F(v) = d_K(v)\} \\
&\qquad \cup \{v \in V: \delta_{K,g}(v) < \delta_{J,g}(v) \text{ and } (f(v) \neq g(v) \text{ or } d_F(v) \neq d_K(v))\} \\
&\quad \subseteq \emptyset \cup \{v \in V: f(v) \neq g(v) \text{ or } d_F(v) \neq d_K(v)\}.
\end{aligned}$$

Hence the set $V_1(J)$ of Lemma 4, referring to g and K in place of f and F, is finite. Thus, by Lemma 4, triangle inequality, and $\delta_{F,f}(v) \geqslant \delta_{J,f}(v)$ for all $v \in V$, we conclude that

$$\begin{aligned}
0 &\leqslant \sum_{v \in V} \delta_{F,f}(v) - \delta_{J,f}(v) = \sum_{v \in V} |d_F(v) - f(v)| - |d_J(v) - f(v)| \\
&\leqslant \sum_{v \in V} |d_F(v) - d_K(v)| + |d_K(v) - g(v)| + |g(v) - f(v)| \\
&\qquad - |d_J(v) - g(v)| + |g(v) - f(v)| \\
&= 2 \sum_{v \in V} |g(v) - f(v)| + \sum_{v \in V} |d_F(v) - d_K(v)| - \sum_{v \in V} \delta_{J,g}(v) - \delta_{K,g}(v) \\
&\leqslant 2 \sum_{v \in V} |g(v) - f(v)| + \sum_{v \in V} |d_F(v) - d_K(v)| < \infty
\end{aligned}$$

Therefore $(\sum_{v \in V} \delta_{F,f}(v) - \delta_{J_i,f}(v))_{0 \leqslant i < \omega}$ is a strictly increasing and bounded sequence of natural numbers. This is impossible. □

5. Optimal factors of rayless graphs

We have seen that if G is countable, then G contains an optimal factor, but not necessarily an optimal factor better than a given factor. In 'rayless' graphs the situation is more pleasant. G is said to be *rayless* if no path in G has infinite length. We will show that if G is rayless and F a non-optimal factor of G, then there exists an optimal factor better than F. The corresponding result for 1-factors has been proved by Schmidt [6] by transfinite induction on the 'order' of G. The order $\sigma(G)$ of a rayless graph will be defined by transfinite recursion as follows.

If G is finite, then G is said to have order 0. Let $\alpha > 0$ be an ordinal. G is said to have order α if there is a finite set $W \subseteq V$ such that each component of $G \setminus W := (V \setminus W, \{e \in E: e \cap W = \emptyset\})$ has order less than α. If G has order α for some α, then let $\sigma(G)$ be the least ordinal α such that G has order α.

Lemma 18 (Schmidt [6]). *G is rayless iff $\sigma(G)$ is defined.*

Proof. Let $\sigma(G)$ be undefined. We construct recursively a sequence $v_0, v_1, v_2, \ldots$ of vertices and a sequence $C_0, C_1, C_2, \ldots$ of subgraphs of G such that for all $n \in \omega$ the following holds: C_n is connected, $\sigma(C_n)$ is undefined, v_n is a vertex of C_n, C_{n+1} is a component of $C_n \setminus \{v_n\}$, $\{v_n, v_{n+1}\} \in E$. Then $(v_i)_{0 \leq i < \omega}$ is a path in G and hence G is not rayless. $n = 0$: Since $\sigma(G)$ is undefined, there exists a component C_0 of $G \setminus \emptyset = G$ such that $\sigma(C_0)$ is undefined. Choose a vertex v_0 of C_0. Now assume that v_n and C_n are already defined. Since $\sigma(C_n)$ is undefined by induction hypothesis, there exists a component C_{n+1} of $C_n \setminus \{v_n\}$ such that $\sigma(C_{n+1})$ is undefined. Since C_n is connected by induction hypothesis, there exists a vertex v_{n+1} of C_{n+1}, such that $\{v_n, v_{n+1}\} \in E$.

The other implication will be proved by transfinite induction on $\sigma(G)$. If $\sigma(G) = 0$, then G is finite and therefore rayless. If $\sigma(G) > 0$, there exists a finite set $W \subseteq V$ such that $\sigma(C) < \sigma(G)$ for each component of $G \setminus W$. Since by induction hypothesis each component of $G \setminus W$ is rayless, G is rayless, too. □

Theorem 7. *If G is rayless and if F is a non-optimal factor of G, then there exists an optimal factor better than F.*

Proof. We will prove the theorem by transfinite induction on $\sigma(G)$. If $\sigma(G) = 0$, then G is finite and the assertion holds trivially. For the induction step let $\sigma(G) > 0$. There exists a finite $W \subseteq V$ such that, for each component C of $G \setminus W$, $\sigma(C) < \sigma(G)$. Let $\{C_i = (V_i, E_i): i \in I\}$ be the set of components of $G \setminus W$. Define

$$F_W := F \cap \{e \in E: e \cap W \neq \emptyset\} \quad \text{and} \quad F_i := F \cap E_i \quad \text{for } i \in I.$$

F is the disjoint union of the F_i's and F_W and, for each $i \in I$, F_i is a factor of C_i. Further let us define

$$g(v) := \begin{cases} f(v) - d_{F_W}(v) & \text{if } v \notin W \text{ and } d_{F_W}(v) < f(v), \\ 0 & \text{otherwise.} \end{cases}$$

If $i \in I$, let g_i be the restriction of g to V_i. By induction hypothesis there exists, for each $i \in I$, a g_i-optimal factor K_i of C_i satisfying $\delta_{K_i, g_i} \preccurlyeq \delta_{F_i, g_i}$. Let

$$K := \bigcup_{i \in I} K_i \quad \text{and} \quad J := K \cup F_W.$$

An easy computation shows $\delta_{J,f} \leqslant \delta_{F,f}$. Since $g(v)=0$ for all $v \in W$, K is a g-optimal factor of G. Since W and F_W are finite, we conclude that $g(v)=f(v)$ and $d_J(v)=d_K(v)$ for almost all $v \in V$. By Lemma 17, there exists an f-optimal factor H such that $\delta_{H,f} \leqslant \delta_{J,f}$. It follows that $\delta_{H,f} \leqslant \delta_{F,f}$. □

6. The Lovász-decomposition

For an arbitrary function $g: V \to \omega$ we define subsets $A(g)$, $B(g)$, $C(g)$, $D(g)$ of V as follows:

$$C(g) := \{v \in V: d_F(v) = g(v) \text{ for every } g\text{-optimal factor } F \text{ of } G\},$$

$$A(g) := \{v \in V \setminus C: d_F(v) \geqslant g(v) \text{ for every } g\text{-optimal factor } F \text{ of } G\},$$

$$B(g) := \{v \in V \setminus C: d_F(v) \leqslant g(v) \text{ for every } g\text{-optimal factor } F \text{ of } G\},$$

$$D(g) := V \setminus (A \cup B \cup C).$$

Let $A := A(f)$, $B := B(f)$, $C := C(f)$, $D := D(f)$. We call (A, B, C, D) the *Lovász-decomposition* of G. Note that $V = C$ iff G possesses a perfect factor or G contains no optimal factor. In Lovász and Plummer [4, pp. 388–404] this decomposition is considered for finite graphs.[1] It is a generalization of the 'Gallai–Edmonds-decomposition' for 1-factors. (See e.g. [4, pp. 93–102].) We will show that the properties of the Lovász-decomposition proved in [4] remain true for arbitrary graphs.

Lemma 19. *Let F be an optimal factor of G and let $x \in D$.*

(a) *If $d_F(x) > f(x)$, then there exists an F-alternating trail which starts and terminates at x with an F-edge and meets vertices of D only.*

(b) *If $d_F(x) < f(x)$, then there exists an F-alternating trail which starts and terminates at x with an F^C-edge and meets vertices of D only.*

Proof. (a) Since $x \in D$, there exists an optimal factor H satisfying $d_H(x) < f(x)$. Let $v_0 := x$ and let $\mathcal{P} = (v_i)_{0 \leqslant i < k}$ be an F-H-trail. Lemma 3 implies $k < \omega$ and

$$\delta_{F \oplus E(\mathcal{P})} = \delta_{F v_0}^{v_{k-1}}, \qquad \delta_{H \oplus E(\mathcal{P})} = \delta_{H v_0}^{v_{k-1}}. \tag{1}$$

Hence $v_0 = v_{k-1} = x$ because otherwise, by Lemma 2(c), (d),

$$\delta_{F \oplus E(\mathcal{P})}(v_{k-1}) = \delta_F(v_{k-1}) - 1 \quad \text{or} \quad \delta_{H \oplus E(\mathcal{P})}(v_{k-1}) = \delta_H(v_{k-1}) - 1$$

contrary to (1). Therefore $d_F(v_{k-1}) > d_H^{v_0}(v_{k-1})$ and $\{v_{k-2}, v_{k-1}\} \in F \setminus H$ by the definition of an F-H-trail. Now let $1 \leqslant i_0 \leqslant k-2$. We must show that $v_{i_0} \in D$. Assume that i_0 is odd. (The case that i_0 is even is handled analogously.) Thus $\{v_{i_0-1}, v_{i_0}\} \in F$. Let us define $\mathcal{P}' := (v_i)_{0 \leqslant i \leqslant i_0}$ and $\mathcal{P}'' := (v_{k-i})_{1 \leqslant i \leqslant k-i_0}$. By Lemma 3, $F \oplus E(\mathcal{P}')$ and $F \oplus E(\mathcal{P}'')$ are optimal factors. It follows that $d_F(v_{i_0}) \leqslant f(v_{i_0})$

[1] [4] deals with the more general case of the '(f, g)-factors'.

and $d_F(v_{i_0}) \geq f(v_{i_0})$ since otherwise $F \oplus E(\mathscr{P}')$ or $F \oplus E(\mathscr{P}'')$ is better than F, respectively. Hence $d_F(v_{i_0}) = f(v_{i_0})$, and therefore $d_{F\oplus E(\mathscr{P}')}(v_{i_0}) < f(v_{i_0})$ and $d_{F\oplus E(\mathscr{P}'')}(v_{i_0}) > f(v_{i_0})$. This implies $v_{i_0} \in D$.

(b) can be proved analogously. □

Theorem 8. *Let F be an optimal factor of G. Then the following holds:*

(a) *If $x \in A$ and $y \in A \cup C$, then $\{x, y\} \notin F$.*

(b) *If $x \in B$, $y \in B \cup C$, and $\{x, y\} \in E$, then $\{x, y\} \in F$.*

(c) *If $x \in D$, then $d_F(x) \in \{f(x) - 1, f(x), f(x) + 1\}$.*

(d) *If $x \in D$ and $y \in C$, then $\{x, y\} \notin E$.*

Proof. (a) Assume for a contradiction that $x \in A$, $y \in A \cup C$, and $\{x, y\} \in F$. By the definition of A and C, it follows that $d_F(x) \geq f(x)$ and $d_F(y) \geq f(y)$. We even have $d_F(x) = f(x)$ and $d_F(y) = f(y)$ because otherwise, by Lemma 3, $F \setminus \{\{x, y\}\}$ is an optimal factor which is better than F or which contradicts $x \in A$ or $y \in A \cup C$. Since $x \in A$, there exists an optimal factor H satisfying $d_H(x) > f(x)$. Then $\{x, y\} \notin H$ since otherwise $H \setminus \{\{x, y\}\}$ is an optimal factor which is better than H or which contradicts $y \in A \cup C$. Let $x_0 := y$, $x_1 := x$, and let $\mathscr{P} = (x_i)_{0 \leq i < k}$ be an F-H-trail. Define $\mathscr{P}' := (x_i)_{1 \leq i < k}$. By Lemma 3, $H \oplus E(\mathscr{P}')$ is optimal and $k < \omega$.

Case 1. $\{x_{k-2}, x_{k-1}\} \in F \setminus H$ *and* $d_F(x_{k-1}) > d^y_H(x_{k-1})$.

Then $x_{k-1} \notin \{x, y\}$. Since $H \oplus E(\mathscr{P}')$ is not better than H, we conclude that $d_H(x_{k-1}) \geq f(x_{k-1})$. Therefore $d_F(x_{k-1}) > f(x_{k-1})$. By Lemma 3 (referring to F and $(x_{k-i})_{1 \leq i \leq k}$), $F \oplus E(\mathscr{P})$ is optimal. Now $d_{F\oplus E(\mathscr{P})}(y) = d_F(y) - 1$ is a contradiction to $y \in A \cup C$.

Case 2: $\{x_{k-2}, x_{k-1}\} \in H \setminus F$ *and* $d_F(x_{k-1}) < d^y_H(x_{k-1})$.

Since $H \oplus E(\mathscr{P}')$ is not better than H, we conclude that $d_H(x_{k-1}) \leq f^x(x_{k-1})$. Therefore we get $x_{k-1} \notin \{x, y\}$ since otherwise $d_{H\oplus E(\mathscr{P}')}(x) = d_H(x) - 2 \leq f(x) - 1$ or $d_{H\oplus E(\mathscr{P}')}(y) = d_H(y) - 1 \leq f(y) - 1$ contrary to $x \in A$ and $y \in A \cup C$. Consequently we find $d_F(x_{k-1}) < f(x_{k-1})$. Now we get a contradiction as in case 1.

(b) Can be proved analogously.

(c) Let $x \in D$. If $d_F(x) > f(x)$, let $\mathscr{P}$ be an F-alternating trail in agreement with Lemma 19(a). Then $d_{F\oplus E(\mathscr{P})} = d_{Fxx}$. Since $F \oplus E(\mathscr{P})$ is not better than F, we conclude that $d_F(x) = f(x) + 1$. If $d_F(x) < f(x)$, analogously it follows that $d_F(x) = f(x) - 1$ from Lemma 19(b).

(d) Suppose that $\{x, y\} \in E$ for some $x \in D$ and some $y \in C$. By (c), there exists an optimal factor F such that $d_F(x) = f(x) - 1$. Then $\{x, y\} \in F$ since otherwise $F \cup \{\{x, y\}\}$ would be an optimal factor such that $d_{F\cup\{\{x,y\}\}}(y) = d_F(y) + 1 = f(y) + 1$, contradicting $y \in C$. Let $\mathscr{P}$ be an F-alternating trail in agreement with Lemma 19(b), and let $H := F \oplus E(\mathscr{P})$. Then $\{x, y\} \in H$ and $d_H(x) = f(x) + 1$. Hence $H \setminus \{\{x, y\}\}$ is an optimal factor, and we have $d_{H\setminus\{\{x,y\}\}}(y) = f(y) - 1$. This contradicts $y \in C$. □

We are going to examine the structure of optimal factors on the set D in detail. Obviously the following holds: If $x \in D$, then an f-optimal factor F satisfying $d_F(x) = f(x) + 1$ ($d_F(x) = f(x) - 1$, resp.) is f^x-optimal (f_x-optimal, resp.). The following lemma shows that the converse assertion holds as well.

Lemma 20. *Let $x \in D$. Then*:
(a) $x \in C(f^x)$ *and* $x \in C(f_x)$
(b) *Every f^x-optimal factor and every f_x-optimal factor is f-optimal.*

Proof. (a) To see that $x \in C(f^x)$, let F be an f^x-optimal factor. We have to show that $d_F(x) = f(x) + 1$.

(i) Assume $d_F(x) \leq f(x)$. Choose an f-optimal factor H satisfying $d_H(x) = f(x) + 1$ and an H-F-trail $\mathcal{P} = (v_i)_{0 \leq i < k}$ starting at $v_0 = x$. By Lemma 3, we have $k < \omega$ and

$$\delta_{H \oplus E(\mathcal{P}), f} = (\delta_{H,f})_{v_0}^{v_{k-1}} \quad \text{and} \quad \delta_{F \oplus E(\mathcal{P}), f^x} = (\delta_{F,f^x})_{v_0}^{v_{k-1}}. \tag{1}$$

If $v_0 \neq v_{k-1}$, then Lemma 2(c), (d) implies

$$\delta_{H \oplus E(\mathcal{P}), f}(v_{k-1}) = \delta_{H,f}(v_{k-1}) - 1 \quad \text{or} \quad \delta_{F \oplus E(\mathcal{P}), f}(v_{k-1}) = \delta_{F,f}(v_{k-1}) - 1,$$

contrary to (1). If $v_0 = v_{k-1}$, then Lemma 2(b) implies $\delta_{F \oplus E(\mathcal{P}), f^x}(x) = \delta_{F,f^x}(x) - 2$. Again this is a contradiction to (1).

(ii) Assume $d_F(x) > f(x) + 1$. By Lemma 17, there exists an f-optimal factor H satisfying $\delta_{H,f} < \delta_{F,f}$. By Theorem 8(c), we find $f(x) - 1 \leq d_H(x) < d_F(x)$. Let $\mathcal{P} = (v_i)_{0 \leq i < k}$ be an F-H-trail starting at $v_0 = x$. By Lemma 3 and Lemma 2(a), (c), we get

$$\delta_{F \oplus E(\mathcal{P}), f^x} = (\delta_{F,f^x})_x^{v_{k-1}} \quad \text{and} \quad \delta_{F \oplus E(\mathcal{P}), f}(v_{k-1}) < \delta_{F,f}(v_{k-1}).$$

Therefore $x = v_{k-1}$, $\delta_{F \oplus E(\mathcal{P}), f^x} = \delta_{F,f^x}$, and $d_{F \oplus E(\mathcal{P})}(x) \neq d_F(x)$. Thus $F \oplus E(\mathcal{P})$ is an f^x-optimal factor satisfying $d_{F \oplus E(\mathcal{P})}(x) = f(x)$. This is impossible by (i).

$x \in C(f_x)$ can be proved analogously.

(b) Suppose that there is some f^x-optimal factor F which is not f-optimal. By Lemma 17, there exists an f-optimal factor H such that $\delta_{H,f} < \delta_{F,f}$. Again by Lemma 17, there exists an f^x-optimal factor J such that $\delta_{J,f^x} \leq \delta_{H,f^x}$. Since $d_F(x) = d_J(x) = f(x) + 1$ by (a), it follows that $\delta_{J,f^x} \leq \delta_{F,f^x}$ and, by the f^x-optimality of F, $\delta_{J,f^x} = \delta_{F,f^x}$. Thus

$$\delta_{F,f}(y) = \delta_{H,f}(y) \text{ for } y \neq x \quad \text{and} \quad d_H(x) = f(x).$$

Now let K be an f-optimal factor satisfying $d_K(x) = f(x) + 1$ and let $\mathcal{P} = (v_i)_{0 \leq i < k}$ be a K-H-trail starting at $v_0 = x$. Then $k < \omega$ by Lemma 3. Since $\mathcal{P}$ is a K-H-trail, we have $d_K(v_{k-1}) \neq d_H^{v_0}(v_{k-1})$ and therefore $v_{k-1} \neq v_0$. By Corollary 1(a), it follows from the f-optimality of K that $\delta_{H,f}(v_{k-1}) > \delta_{K,f}(v_{k-1})$. By Lemma 2(d), we get

$$\delta_{H\oplus E(\mathscr{P}),f^x}(v_{k-1}) = \delta_{H\oplus E(\mathscr{P}),f}(v_{k-1}) < \delta_{H,f}(v_{k-1}) = \delta_{F,f^x}(v_{k-1}).$$

Thus $\delta_{H\oplus E(\mathscr{P}),f^x} < \delta_{F,f^x}$ contrary to the f^x-optimality of F.

One can show analogously that every f_x-optimal factor is f-optimal. □

Corollary 6. *Let $x \in D$. Then*:

(a) $A \subseteq A(f^x) \cup C(f^x)$ *and* $A \subseteq A(f_x) \cup C(f_x)$.

(b) $B \subseteq B(f^x) \cup C(f^x)$ *and* $B \subseteq B(f_x) \cup C(f_x)$.

(c) $C \subseteq C(f^x)$ *and* $C \subseteq C(f_x)$.

(d) $D \subseteq D(f^x) \cup C(f^x)$ *and* $D \subseteq D(f_x) \cup C(f_x)$.

Proof. (a), (b), and (c) are immediate consequences of Lemma 20(b).

(d) Let y be a vertex of D and let F be an arbitrary f^x-optimal factor. If $d_F(y) < f^x(y)$ ($d_F(y) > f^x(y)$, resp.), then we must find an f^x-optimal factor J satisfying $d_J(y) > f^x(y)$ ($d_J(y) < f^x(y)$, resp.).

(i) $d_F(y) < f^x(y)$. By Lemma 20(a) we conclude that $y \neq x$ and therefore, by Lemma 20(b) and Theorem 8(c), $d_F(y) = f(y) - 1$. Lemma 19 yields an F-alternating trail $\mathscr{P}$ such that $d_{F\oplus E(\mathscr{P})} = d_F^{yy}$. Define $J := F \oplus E(\mathscr{P})$.

(ii) The case $d_F(y) > f^x(y)$ can be handled similarly.

$D \subseteq D(f_x) \cup C(f_x)$ can be proved analogously. □

If $W \subseteq V$, let us define $G[W] := (W, \{\{v, w\} \in E: v, w \in W\})$. A subset U of W is said to be a *component* of W if $G[U]$ is a component of $G[W]$. By Corollary 6(d), Lemma 20(a), and Theorem 8(d), we have the following.

Lemma 21. *If D^* is a component of D and if $x \in D^*$*, then $D^* \subseteq C(f^x)$ *and* $D^* \subseteq C(f_x)$.

Lemma 22. *Let D^* be a component of D, let $x, y \in D^*$, and let F be an optimal factor such that $d_F(x) \neq f(x)$. Then there exist optimal factors H_1, H_2 such that $d_{H_1}(y) = f(y) + 1$, $d_{H_2}(y) = f(y) - 1$, and*

$$H_i \oplus F \subseteq \{\{v, w\}: v, w \in D^*\} \quad (i = 1, 2).$$

Proof. Let $D' \subseteq D^*$ be the set of all vertices $z \in D^*$ such that there is some optimal factor H satisfying $d_H(z) \neq f(z)$ and

$$H \oplus F \subseteq \{\{v, w\}: v, w \in D^*\}. \tag{1}$$

Because of Theorem 8(c) and Lemma 19, it remains to prove that $D' = D^*$. Assuming the contrary, there are $v \in D'$ and $w \in D^* \setminus D'$ such that $\{v, w\} \in E$. Let H be an optimal factor satisfying $d_H(v) \neq f(v)$ and (1). By Lemma 19, we can assume without loss of generality that $d_H(v) = f(v) - 1$. Since $w \notin D'$, we have

$d_H(w)=f(w)$. Further $\{v, w\} \in H$ since otherwise $H \cup \{\{v, w\}\}$ would be an optimal factor which shows $w \in D'$. Choose an H-alternating trail $\mathcal{P}=(v_i)_{0\le i<k}$ in agreement with Lemma 19(b) (referring to H and v). $\mathcal{P}$ does not meet w since otherwise $H \oplus E(\mathcal{P}')$ is an optimal factor showing $w \in D'$, where $\mathcal{P}'=(v_i)_{0\le i\le t}$ and $v_t = w$. Therefore $H \oplus E(\mathcal{P})$ is an optimal factor satisfying $d_{H\oplus E(\mathcal{P})}(v)=f(v)+1$, (1), and $\{v, w\} \in H \oplus E(\mathcal{P})$. Thus $(H \oplus E(\mathcal{P}))\setminus\{\{v, w\}\}$ is an optimal factor showing $w \in D'$ contrary to $w \notin D'$. □

Theorem 9. *Let D^* be a component of D and let F be an f-optimal factor. Then exactly one of the following four statements holds:*

(i) $\forall x \in D^*$: $d_F(x)=f(x)$, $\{y, a\} \in F$ *for exactly one* $y \in D^*$ *and exactly one* $a \in A$, $\forall x \in D^* \forall b \in B$: $\{x, b\} \in E \Rightarrow \{x, b\} \in F$.

(ii) $\forall x \in D^*$: $d_F(x)=f(x)$, $\forall x \in D^* \forall a \in A$: $\{x, a\} \notin F$, $\{y, b\} \in E \setminus F$ *for exactly one* $y \in D^*$ *and exactly one* $b \in B$.

(iii) $\exists y \in D^* \forall x \in D^*$: $d_F(x)=f^y(x)$, $\forall x \in D^* \forall a \in A$: $\{x, a\} \notin F$, $\forall x \in D^* \forall b \in B$: $\{x, b\} \in E \Rightarrow \{x, b\} \in F$.

(iv) $\exists y \in D^* \forall x \in D^*$: $d_F(x)=f_y(x)$, $\forall x \in D^* \forall a \in A$: $\{x, a\} \notin F$, $\forall x \in D^* \forall b \in B$: $\{x, b\} \in E \Rightarrow \{x, b\} \in F$.

Proof. Obviously at most one of the four possibilities can occur.

Case 1: *$d_F(y) \ne f(y)$ for some $y \in D^*$.*

Then $d_F(y)=f(y)+1$ or $d_F(y)=f(y)-1$ by Theorem 8(c).

Case 1.1: $d_F(y)=f(y)+1$.

Since F is f^y-optimal, it follows by Lemma 21 that, for all $x \in D^* \setminus \{y\}$, $d_F(x)=f(x)$.
Suppose there is some $x \in D^*$ and some $a \in A$ such that $\{x, a\} \in F$. By Lemma 21, Corollary 6(a), and Theorem 8(a), we get $a \in C(f^y)$. Hence $d_F(a)=f(a)$. Choose an f-optimal factor H satisfying $d_H(x)=f(x)+1$ in agreement with Lemma 22. Then $\{x, a\} \in H$ and $d_H(a)=d_F(a)=f(a)$. By Lemma 3, $H \setminus \{\{x, a\}\}$ is f-optimal. Since $d_{H\setminus\{\{x,a\}\}}(a)=f(a)-1<f(a)$, this is a contradiction to $a \in A$. Thus $\forall x \in D^* \forall a \in A$: $\{x, a\} \notin F$.

One can show similarly that every edge $\{x, b\}$ with $x \in D^*$ and $b \in B$ is contained in F. Therefore (iii) holds.

Case 1.2: $d_F(y)=f(y)-1$.

Analogously to case 1.1 (iv) can be shown.

Case 2: *$d_F(x)=f(x)$ for all $x \in D^*$.*

Case 2.1: *There is some $y \in D^*$ and some $a \in A$ such that $\{y, a\} \in F$.*

Since $y \in D$, there exists an f-optimal factor H satisfying $d_H(y)<f(y)$. By case 1.2, it follows that $d_H(x)=f_y(x)$ for all $x \in D^*$ and $\{y, a\} \notin H$. Let $v_0 := y$, $v_1 := a$ and let $\mathcal{P}=(v_i)_{0\le i<k}$ be an F-H-trail. Then, for $i>0$, $v_i \notin D^*$.

Assuming the contrary, choose $j>0$ minimal such that $v_j \in D^*$. If j is even, then $\{v_{j-1}, v_j\} \in H \setminus F$. By case 1.2, we get $v_{j-1} \notin A$. Therefore, by Theorem 8(d), $v_{j-1} \in B$. Define $\mathcal{P}' := (v_i)_{0\le i\le j}$. $H \oplus E(\mathcal{P}')$ is f-optimal by Lemma 3. Further

$d_{H\oplus E(\mathscr{P}')}(v_j)=f(v_j)-1$ and $\{v_{j-1}, v_j\}\notin H\oplus E(\mathscr{P}')$. This is a contradiction to case 1.2 (ref. to $H\oplus E(\mathscr{P}')$ and v_{j-1}). If j is odd, we get similarly a contradiction to case 1.1.

In particular we get $v_{k-1}\neq y$. Now we want to show that $J:=F\oplus E(\mathscr{P})$ is f-optimal. By Lemma 3, we have $k<\omega$.

Case a: $\{v_{k-2}, v_{k-1}\}\in F\setminus H$ *and* $d_F(v_{k-1})>d_H^y(v_{k-1})=d_H(v_{k-1})$.

Since $H\oplus E(\mathscr{P})$ is not better than H, $d_H(v_{k-1})\geqslant f(v_{k-1})$. Therefore $d_F(v_{k-1})>f(v_{k-1})$ and J is f-optimal by Lemma 3. (Consider $(v_{k-i})_{0<i\leqslant k}$!)

Case b: $\{v_{k-2}, v_{k-1}\}\in H\setminus F$ *and* $d_F(v_{k-1})<d_H^y(v_{k-1})=d_H(v_{k-1})$.

This case will be treated similarly. Since $d_J(y)=f(y)-1\neq f(y)$ and J is f-optimal, it follows by case 1.2: $\forall x\in D^*\forall v\in A$: $\{x, v\}\notin J$ and $\forall x\in D^*\forall v\in B$: $\{x, v\}\in E\Rightarrow\{x, v\}\in J$. Since $v_i\notin D^*$ for $i>0$, an edge $\{x, v\}\neq\{y, a\}$ with $x\in D^*$ is in J iff it is in F. Consequently (i) holds.

Case 2.2: *There is some* $y\in D^*$ *and some* $b\in B$ *such that* $\{y, b\}\in E\setminus F$.

Analogously to case 2.1 one can show (ii).

Case 2.3: $\forall x\in D^*\forall a\in A$: $\{x, a\}\notin F$ *and* $\forall x\in D^*\forall b\in B$: $\{x, b\}\in E\Rightarrow\{x, b\}\in F$. We must show that this case cannot occur. Choose an arbitrary vertex $y\in D^*$ and an f-optimal factor H satisfying $d_H(y)<f(y)$. By case 1.2, we conclude that $d_H(x)=f_y(x)$ for all $x\in D^*$. Furthermore, $\forall x\in D^*\forall a\in A$: $\{x, a\}\notin H$ and $\forall x\in D^*\forall b\in B$: $\{x, b\}\in E\Rightarrow\{x, b\}\in H$. Let $\mathscr{P}=(v_i)_{0\leqslant i<k}$ be an F-H-trail starting at $v_0=y$. By Lemma 3, we have $k<\omega$. Since $\mathscr{P}$ traverses only edges of $F\oplus H$, $\mathscr{P}$ contains no edge joining D^* and $A\cup B$. Since there is no edge joining D^* and C by Theorem 8(d), $\mathscr{P}$ meets vertices of D^* only. In particular $v_{k-1}\in D^*$. Therefore $d_F(v_{k-1})=f(v_{k-1})=d_H^y(v_{k-1})$. This is a contradiction since v_{k-1} is the endvertex of an F-H-trail. □

G is said to be *f-critical* if G is connected and $V=D$. For $x\in C\cup D$ let us define

$$f_B(x):=f(x)-|\{b\in B\colon \{x, b\}\in E\}|.$$

It follows easily from Theorem 8(b) and Theorem 9 that f_B defines a function on $C\cup D$ into ω. From Theorem 8 and Theorem 9 we get the following corollary.

Corollary 7. *If G contains at least one f-optimal factor, then $G[C]$ possesses a perfect f_B-factor and, for each component D^* of D, $G[D^*]$ is f_B*-critical.

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