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Reconstruction of infinite graphs

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Abstract

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The paper recalls several known results concerning reconstruction and edge-reconstruction of infinite graphs, and draws attention to some possibly interesting unsolved problems.

1. Introduction

In this paper, graphs are understood to be simple, i.e. without loops or multiple edges, and the letters G, H always denote graphs. Digraphs are understood to have no loops and no pair of edges with the same tail and the same head, but two distinct vertices of a digraph may be joined by two edges if these are oriented in opposite directions.

If $\xi \in V(G)$ then $G - \xi$ is called a *vertex-deleted* subgraph of G. The 'Reconstruction Problem' asks whether a graph is determined up to isomorphism if we know its vertex-deleted subgraphs up to isomorphism, i.e. whether a graph is in this sense 'reconstructible' from its vertex-deleted subgraphs. To express this more carefully, we make the following definitions: A *hypomorphism* of G onto H is a bijection $\phi: V(G) \rightarrow V(H)$ such that $G - \xi \cong H - \phi(\xi)$ for every $\xi \in V(G)$ (where $\cong$ means 'is isomorphic to'). Two graphs G, H are *hypomorphic* (in symbols, $G \simeq H$) if there exists a hypomorphism of G onto H. A graph G is *reconstructible* if every graph hypomorphic to G is isomorphic to G. The terms 'hypomorphism', 'hypomorphic' and 'reconstructible' are defined in the same way for digraphs, but using the notion of isomorphism appropriate to digraphs. The well-known 'Reconstruction Conjecture' can be formulated in either of the following equivalent ways.

Reconstruction Conjecture (first version). If G, H are hypomorphic finite graphs with at least 3 vertices then $G \cong H$.

Reconstruction Conjecture (second version). Every finite graph with at least 3 vertices is reconstructible.

The analogous conjecture concerning digraphs is false: Stockmeyer [19–20] has shown that there are infinitely many mutually non-isomorphic finite digraphs which are not reconstructible.

It is somewhat easier to see that the analogue for infinite graphs of the Reconstruction Conjecture is not in general true. Probably the simplest counterexample is the fact that $T_\alpha \simeq 2T_\alpha$ but $T_\alpha \not\cong 2T_\alpha$ for any infinite cardinal α, where T_α denotes a tree in which every vertex has valency α and $2T_\alpha$ is the union of two disjoint trees isomorphic to T_α. Nevertheless, there are some challenging problems about reconstructibility of infinite graphs, such as the following.

Problem 1. Is every locally finite connected infinite graph reconstructible?

Problem 2. If two infinite trees are hypomorphic, are they necessarily isomorphic?

Problem 3. A conjecture of Halin: If two infinite graphs are hypomorphic, then each is isomorphic to a subgraph of the other.

We remark that two hypomorphic connected graphs need not be isomorphic: the complements of T_α, $2T_\alpha$ provide an immediate counterexample, and Andreae [2] has provided counterexamples in which the graphs concerned and their complements are all connected. We remark also that two hypomorphic locally finite forests need not be isomorphic: see [10].

Problems 1, 2 and 3 are probably very difficult, but this paper will suggest some further problems (all unsolved so far as I know), some of which might be more accessible.

For further background concerning reconstruction, including basic elementary results quoted here without specific references, the reader may consult [5, Section 10.2] [6, 9, 15] and [16, Section 11(a)].

2. Some problems about reconstruction of graphs which are not necessarily locally finite

Definitions. A *neighbour* of a vertex ξ of G is a vertex adjacent to ξ. The set of neighbours of ξ in G will be denoted by $N_G(\xi)$; and $v_G(\xi)$ will denote the *valency* $|N_G(\xi)|$ of ξ in G. We shall write $V_\alpha(G) = \{\xi \in V(G)\colon v_G(\xi) = \alpha\}$; and $E_{\alpha,\beta}(G)$ will denote the set of those edges of G which join elements of $V_\alpha(G)$ to elements of $V_\beta(G)$. If G is finite, the *valency sequence* of G is the sequence obtained by listing the valencies of its vertices in nondecreasing order, the *valency sequence* of a vertex ξ of G is the sequence of valencies of its neighbours listed in nondecreasing order and the *valency sequence sequence* of G is the sequence of sequences obtained by listing the valency sequences of the vertices of G in lexicographical order.

A graph is *2-connected* if it is connected and has no cut-vertices. A *block* of G is a maximal 2-connected subgraph of G.

If ξ is a vertex of a digraph D then $\mathrm{ov}_D(\xi)$ will denote the outvalency of ξ in D, i.e. the number of edges of D with tail ξ, and $\mathrm{iv}_D(\xi)$ will denote the invalency of ξ in D, i.e. the number of edges with head ξ.

Partial results concerning the Reconstruction Conjecture often take one of two forms: either they assert reconstructibility of particular kinds of finite graphs or they assert that hypomorphic finite graphs (with at least 3 vertices) must have some common properties, e.g.

Proposition 1. *Hypomorphic finite graphs with at least* 3 *vertices have the same valency sequence.*

Proposition 2. *Hypomorphic finite graphs with at least* 3 *vertices have the same valency sequence sequence.*

Proposition 3. *Hypomorphic finite graphs with at least* 3 *vertices have the same number of components.*

These three propositions are all easy to prove. However, easily proved results concerning reconstruction of finite graphs may suggest analogous questions about infinite graphs which are appreciably harder. For example, the following theorem of Andreae [4] is substantially harder to prove than Proposition 1.

Theorem 1. *If G, H are hypomorphic infinite graphs then $|V_\alpha(G)| = |V_\alpha(H)|$ for every cardinal number α.*

I am not aware of any known result concerning infinite graphs on the lines of Proposition 2, and therefore suggest the following problem.

Problem 4. If G, H are hypomorphic infinite graphs, does there necessarily exist a bijection $\phi : V(G) \to V(H)$ such that $|N_G(\xi) \cap V_\alpha(G)| = |N_H(\phi(\xi)) \cap V_\alpha(H)|$ for every $\xi \in V(G)$ and every cardinal number α?

Since some infinite graphs are not reconstructible, a suitable counterexample might answer Problem 4 in the negative. In that event, one might ask the following somewhat less ambitious question.

Problem 5. If G, H are hypomorphic infinite graphs and α, β are cardinal numbers, are $|E_{\alpha,\beta}(G)|$ and $|E_{\alpha,\beta}(H)|$ necessarily equal?

If G, J are graphs and $\xi \in V(G)$ let $v_G(\xi, J)$ denote the number of subgraphs S of G such that $\xi \in V(S)$ and $S \cong J$. Thus $v_G(\xi) = v_G(\xi, K_2)$, where K_2 is a complete graph with two vertices, and therefore Proposition 1 is a special case of the following proposition, whose proof is also easy.

Proposition 4. *If G, H, J are finite graphs, $G \simeq H$ and $|V(J)| < |V(G)|$ then there exists a bijection $\phi: V(G) \to V(H)$ such that $v_G(\xi, J) = v_H(\phi(\xi), J)$ for every $\xi \in V(G)$.*

Problem 6. Is the conclusion of Proposition 4 necessarily true when G, H are hypomorphic infinite graphs and J is a graph such that $|V(J)| < |V(G)|$ (or in the more restricted case in which G, H are hypomorphic infinite graphs and J is a finite graph)?

Manvel [12] has proved the following theorem about finite digraphs on the lines of Proposition 1.

Theorem 2. *If C, D are hypomorphic finite digraphs with at least 5 vertices then there exists a bijection $\phi: V(C) \to V(D)$ such that $\mathrm{ov}_C(\xi) = \mathrm{ov}_D(\phi(\xi))$ and $\mathrm{iv}_C(\xi) = \mathrm{iv}_D(\phi(\xi))$ for every $\xi \in V(C)$.*

Problem 7. Is the conclusion of Theorem 2 necessarily true when C, D are hypomorphic infinite digraphs?

Problem 8. Is something on the lines of Proposition 2, but involving outvalencies and invalencies, true for finite and/or infinite digraphs?

The following theorem of Andreae [4] resembles Proposition 3 but is substantially harder to prove.

Theorem 3. *If G, H are hypomorphic infinite graphs each of which has at least one vertex of finite valency, then G and H have the same number of components.*

(Since $T_\alpha \simeq 2T_\alpha$ when α is infinite, the hypothesis that G, H have at least one vertex of finite valency is needed.)

This might prompt the following question.

Problem 9. Must two hypomorphic infinite graphs have the same number of blocks? If not, can this conclusion be drawn subject to some additional hypothesis like that of Theorem 3?

3. Locally finite graphs

Definitions. If ξ is a vertex of G, let $S_n(\xi, G)$ denote the subgraph of G induced by those vertices which are in the same component of G as ξ and at distance $\leqslant n$ from ξ. If (R, ρ) is a rooted graph (i.e. R is a graph and ρ is a vertex of R), let $\alpha_n(G, R, \rho)$ denote the number of vertices ξ of G such that there exists an isomorphism of $S_n(\xi, G)$ onto R which maps ξ to ρ.

We shall say that G is *homeomorphic to* T_3 if G is a tree, $V(G) = V_2(G) \cup V_3(G)$ and there is no infinite path P in G such that $V(P) \subseteq V_2(G)$. More informally, this means that G can be obtained from T_3 (a tree in which every vertex has valency 3) by inserting a finite number n_λ of 2-valent vertices into each edge λ (where n_λ may depend on λ and may be 0).

A type of condition on a graph which tends to play a role in infinite graph theory is one which says, loosely speaking, that a graph has a specified finite number p of infinite wings (commonly called 'ends') branching out of a finite centre. To make this idea precise, we define a graph G to be *p-coherent*, where p is a positive integer, if G can be expressed as the union of a finite subgraph and p disjoint infinite subgraphs but cannot be expressed as the union of a finite subgraph and $p+1$ disjoint infinite subgraphs. We shall only be concerned with applying this condition to locally finite graphs (i.e. graphs in which every vertex has finite valency). We observe that a locally finite tree T is p-coherent if and only if there are exactly p one-way infinite paths starting at each vertex in T. For infinite cardinal numbers α, it will be convenient to define an α-coherent graph to be one which can be expressed as the union of α but not fewer 1-coherent subgraphs.

Bondy and Hemminger [6] proved the following theorem.

Theorem 4. *If p is an integer $\geqslant 2$ then every p-coherent locally finite tree is reconstructible.*

This was proved by considering the cases $p \geqslant 3$, $p = 2$ separately, and the proof was somewhat longer and more difficult in the case $p = 2$.

Thomassen [23] and Andreae [1] proved the following theorems respectively.

Theorem 5. *Every 1-coherent locally finite tree is reconstructible.*

Theorem 6. *Every $\aleph_0$-coherent locally finite tree is reconstructible.*

Andreae [1] also pointed out that Theorems 4, 5 and 6 together have the following consequence.

Theorem 7. *A locally finite tree is reconstructible if it has no subgraph homeomorphic to T_3.*

In [17–18], I obtained the following generalisation of Theorem 4.

Theorem 8. *If p is an integer $\geqslant 2$ then every p-coherent locally finite connected graph is reconstructible.*

This was proved for $p \geqslant 3$ in [17] and for $p = 2$ in [18]. As in the case of Theorem 4, the proof was longer and more complicated when $p = 2$. We recall that a p-coherent graph might be thought of as having p 'infinite wings' branching out of a 'finite centre'. When $p \geqslant 3$, one can at least vaguely identify the position of the 'finite centre', but a 2-coherent graph could have a shape something like an infinitely long sausage, extending to infinity in both directions, and what one then regards as the location of the 'finite centre' of the graph (from which its two 'infinite wings' branch out) may be a more or less arbitrary choice. This inability to identify, even vaguely, the 'finite centre' in some 2-coherent graphs deprives us of a useful tool in proving reconstructibility and accounts for the more elaborate arguments needed when $p = 2$.

Theorem 7 followed from Theorems 4, 5 and 6. Therefore a tempting target might be to prove generalisations of Theorems 5 and 6 which, when combined with Theorem 8, would answer in the affirmative the following question.

Problem 10. Is a locally finite connected graph necessarily reconstructible if it has no subgraph homeomorphic to T_3?

Let us say that an infinite graph G is *narrow* if it can be expressed as the union of an infinite sequence $S_1, S_2, \ldots$ of finite subgraphs such that $S_i \cap S_j$ is empty when $|i - j| \geqslant 2$ and

$$|V(S_1 \cap S_2)| = |V(S_2 \cap S_3)| = |V(S_3 \cap S_4)| = \cdots .$$

This definition, which can also be expressed in other ways, embodies the idea that 'G does not get wider and wider as it goes off to infinity'. Clearly every narrow graph is locally finite. We observe also that, if p is a positive integer, a p-coherent locally finite tree is necessarily narrow since it can be expressed as the union of finite subgraphs $S_1, S_2, \ldots$ such that $S_i \cap S_j$ is empty when $|i - j| \geqslant 2$ and $|V(S_i \cap S_{i+1})| = p$ for every positive integer i.

Theorem 8 might suggest that we should try to generalise Theorem 5 by proving that every 1-coherent locally finite connected graph is reconstructible; but I suspect that this would be very difficult since 1-coherent graphs lack the kind of recognisable 'centre' which facilitates the proof of Theorem 8 when $p \geqslant 3$ and lack even those features of 2-coherent graphs which, somewhat less readily, enable Theorem 8 to be proved for $p = 2$. Indeed, proving reconstructibility of all

1-coherent locally finite connected graphs may well be of the same order of difficulty as Problem 1. However, since 1-coherent trees are narrow, another possible generalisation of Theorem 5 would be an affirmative answer to the following question.

Problem 11. Is every narrow connected graph reconstructible?

In fact, it would suffice to answer this question for 1-coherent narrow connected graphs, because it is easily proved that every narrow connected graph is p-coherent for some positive integer p, and when $p \geqslant 2$ Theorem 8 can be applied.

Although I have not yet found time to write out the details, it is probably easy to prove that, if p is a positive integer, every p-coherent locally finite graph that is not narrow has a subgraph homeomorphic to T_3. It may also be fairly easy to prove that a locally finite connected infinite graph has such a subgraph if, for all $p \leqslant \aleph_0$, it fails to be p-coherent. Therefore one might hope to obtain an affirmative answer to Problem 10 from affirmative answers to Problem 11 and the following problem.

Problem 12. Is every $\aleph_0$-coherent locally finite connected graph reconstructible?

Some of the questions raised in Section 2 can be answered for locally finite graphs. The following result is Theorem 1 of [3] (or perhaps, more accurately, a consequence of this theorem and the fact, proved in [6], that hypomorphic infinite graphs have the same finite components up to isomorphism).

Theorem 9. *If G, H are hypomorphic infinite locally finite graphs and (R, ρ) is a rooted graph and n is a positive integer then $\alpha_n(G, R, \rho) = \alpha_n(H, R, \rho)$.*

It is easily seen that this theorem implies affirmative answers to Problems 4 and 5 for locally finite graphs and to Problem 6 when G, H are locally finite and J is finite and connected. Moreover, the proof of Theorem 9 can be adapted to prove an analogous theorem concerning infinite locally finite digraphs, which implies affirmative answers to Problems 7 and 8 for such digraphs.

Theorem 9 can be generalised to some extent. Let $\mathscr{L}$ be the class of infinite locally finite graphs. Let a function f from $\mathscr{L}$ into the class of cardinal numbers be called *admissible* if it satisfies the conditions:

(i) if $G, H \in \mathscr{L}$ and $G \cong H$ then $f(G) = f(H)$;

(ii) if $G \in \mathscr{L}$ and $\omega \in V(G)$ then either $f(G) = f(G-\omega)$ or $f(G)$, $f(G-\omega)$ are both finite;

(iii) if $G \in \mathscr{L}$ and $\omega \in V(G)$ then for almost all $\alpha \in V(G-\omega)$ we have $f(G) + f(G-\omega-\alpha) = f(G-\omega) + f(G-\alpha)$.

(The expression 'almost all $\alpha \in V(G-\omega)$' means 'all but finitely many vertices $\alpha \in V(G-\omega)$'.) We can think of (ii) as saying that $f(G)$, $f(G-\omega)$ differ by at most a finite amount, which is true of many functions of interest since $G-\omega$ differs from G only by the absence of one vertex and finitely many edges. If the numbers involved are finite, the equation in (iii) can be written in the possibly more illuminating form

$$f(G)-f(G-\alpha)=f(G-\omega)-f(G-\omega-\alpha).$$

Thus (iii) may be thought of as saying that, with the possible exception of finitely many choices of α, removal of a vertex α has the same effect on $f(G-\omega)$ as on $f(G)$. This might seem likely to be true, for many naturally arising functions f, provided that α is not too near to ω in G. Thus both (ii) and (iii) seem fairly natural conditions. It is possible to prove the following theorem.

Theorem 10. *If f is an admissible function and G, $H \in \mathscr{L}$ and $G \simeq H$ then $f(G)=f(H)$.*

This contains Theorem 9, since $f(G)$ can be taken to be $\alpha_n(G, R, \rho)$. Moreover, Theorem 10 implies an affirmative answer to Problem 9 for infinite locally finite graphs because it is not hard to prove that f is admissible if $f(G)$ is the number of blocks of G for each $G \in \mathscr{L}$.

4. Edge-reconstruction

If $\lambda \in E(G)$ then $G-\lambda$ is called an *edge-deleted* subgraph of G. The 'Edge-Reconstruction Problem' asks whether a graph is 'reconstructible from its edge-deleted subgraphs'. To be more precise, an *edge-hypomorphism* of G onto H is a bijection $\phi: E(G) \rightarrow E(H)$ such that $G-\lambda \cong H-\phi(\lambda)$ for every $\lambda \in E(G)$. Two graphs G, H are *edge-hypomorphic* if there exists an edge-hypomorphism of G onto H. A graph G *is edge-reconstructible* if every graph edge-hypomorphic to G is isomorphic to G. The *Edge-Reconstruction Conjecture* is the conjecture that if G, H are edge-hypomorphic finite graphs with at least 4 edges then $G \cong H$, or, equivalently, that every finite graph with at least 4 edges is edge-reconstructible. A noteworthy result of Müller [13] (obtained by sharpening an idea of Lovász [11]) states that a finite graph with n vertices and more than $(n \log n)/\log 2$ edges is necessarily edge-reconstructible.

Infinite graphs are not all edge-reconstructible, although some ingenuity may be needed to prove this. Thomassen [21] has found examples of infinite graphs which are not edge-reconstructible and in [22] has found examples of infinite graphs without 0-valent vertices which are reconstructible but not edge-reconstructible, in contrast with the theorem [8] that every reconstructible finite graph with at least 4 edges and no 0-valent vertices is edge-reconstructible.

A graph G is said to be *bidegreed* if the set $\{v_G(\xi): \xi \in V(G)\}$ contains at most two distinct numbers. Myrvold, Ellingham and Hoffman [14] have proved that all bidgegreed finite graphs with at least 4 edges are edge-reconstructible. Some but not all of their arguments seem likely to adapt fairly readily to infinite but locally finite graphs, and so it might be interesting to investigate the following question.

Problem 13. Is every bidegreed infinite locally finite graph edge-reconstructible?

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