Discrete Mathematics 95 (1991) 183–191
North-Holland

A partition relation for triples using a model of Todorčević*

E.C. Milner
University of Calgary, Alta., Canada

K. Prikry
University of Minnesota, Minneapolis, MN, USA

Received 22 September 1989
Revised 29 June 1990

Abstract

Milner, E.C. and K. Prikry, A partition relation for triples using a model of Todorčević, Discrete Mathematics 95 (1991) 183–191.

Todorčević has shown that there is a *ccc* extension $\mathcal{M}$ in which $MA_{\omega_1} + 2^{\omega} = \omega_2$ holds and also in which the partition relation $\omega_1 \to (\omega_1, \alpha)^2$ holds for every denumerable ordinal α. We show that the partition relation for triples

$$\omega_1 \to (\omega 2 + 1, 4)^3$$

holds in the model $\mathcal{M}$, and hence by absoluteness this is a theorem in ZFC.

1. Introduction

For an ordinal γ, a positive integer r and linear order types φ, ψ_i $(i < \gamma)$, the partition relation

$$\varphi \to (\psi_i)^r_{i<\gamma} \tag{1.1}$$

means that whenever $(S, <)$ is an ordered set of order type $\text{tp}(S) = \gamma$ and $\{K_i: i < \gamma\}$ is a partition of $[S]^r = \{X \subseteq S: |X| = r\}$, then there are $i < \gamma$ and $T \subseteq S$ such that $[T]^r \subseteq K_i$ and $\text{tp}(T) = \psi_i$. In the case when $\gamma = 2$ we write (1.1) as $\varphi \to (\psi_1, \psi_2)^r$, and the negation of this is expressed by replacing the arrow $\to$ by a nonarrow $\nrightarrow$.

Very few partition relations of this kind are known when $r \geq 3$ and the order types are not cardinal numbers. Such relations were discussed in some detail in [5], and we proved in that paper that

$$\varphi \to (\omega + k, 4)^3 \tag{1.2}$$

* Research supported by NSERC Grant No. A5198 and NSF Grant MCS 830361.

holds for any finite k and linear type φ which satisfies

$$\varphi \to (\omega)^1_\omega. \tag{1.3}$$

It was conjectured in [5] that the more general relation

$$\varphi \to (\alpha, m)^3$$

holds for all countable ordinals α and finite m, but we were unable to extend our method of proof of (1.2) to establish either of the next simplest cases of the conjecture

$$\omega_1 \to (\omega 2, 4)^3, \tag{1.4}$$

or

$$\omega_1 \to (\omega + 2, 5)^3. \tag{1.5}$$

In this paper we use some heavier artillery from [8] in order to prove that

$$\omega_1 \to (\omega 2 + 1, 4)^3, \tag{1.6}$$

which is slightly stronger than (1.4).

Our proof of (1.2) for the case $\varphi = \omega_1$ in [5] used the same type of argument employed by Baumgartner and Hajnal in [1]. Since

$$\omega_1 \to (\omega + k, 4)^3 \tag{1.7}$$

is an absolute statement relative to a *ccc* extension, it is sufficient to prove this under the additional assumption that Martin's Axiom MA_{ω_1} holds. However, we could not prove (1.4) by using the same combinatorial tools. Here we will prove (1.6) using a model of Todorčević. He proved [8] that there is a *ccc* extension $\mathcal{M}$ in which MA_{ω_1} holds, $2^\omega = \omega_2$, and also in which the relation

$$\omega_1 \to (\omega_1, \alpha)^2 \tag{1.8}$$

holds for all $\alpha < \omega_1$. We prove that (1.6) holds in the model $\mathcal{M}$ and so, by absoluteness, (1.6) is a theorem of ZFC. Note that (1.8) is independent of the axioms of ZFC since, by an earlier result of Hajnal [3], CH implies that $\omega_1 \nrightarrow (\omega_1, \omega + 2)^2$. Let us remark that we only use the special case of (1.8) when $\alpha = \omega 2 + 1$; we were unable to obtain anything better by using the full strength of (1.8). Also, we should point out that the proof used in [5] to prove (1.7) could, by an argument due to Baumgartner and Hajnal, be adapted to prove (1.2) by using MA in place of MA_{ω_1}. This argument does not allow us to extend in the same way our proof of (1.6) to the more general relation $\varphi \to (\omega 2 + 1, 4)^3$ for an order type satisfying (1.3). Thus, for example, whether or not the relation

$$\lambda \to (\omega 2, 4)^3$$

holds is still open, where λ is the order type of the reals.

2. Notation and preliminary lemmas

We use the standard notation $[X]^{<\omega}$ to denote the set of all finite subsets of X. If $(S, \leq)$ is a linearly ordered set and $x \in S$, then $S(\geq x) = \{y \in S: y \geq x\}$. Also, if X, Y are subsets of S, then we write $X < Y$ if $x < y$ holds for all $x \in X$ and $y \in Y$. If $\{K_i: i < \gamma\}$ is any partition of $[S]^r$, then we write

$$\psi \in \text{homog}(K_i)$$

if there is a subset $T \subseteq S$ such that $\text{tp}(T) = \psi_i$ and T is *homogeneous for the class* K_i, i.e. if $[T]^r \subseteq K_i$. For finite r, s and sets A, B, we denote by $[A]^r \otimes [B]^s$ the set of all subsets $X \subseteq A \cup B$ such that $|A \cap X| = r$ and $|B \cap X| = s$.

We need the following easily proved consequence of Ramsey's Theorem.

Lemma 2.1. *If r_i, k_i are finite and $f_i : [\omega]^{r_i} \to k_i$ $(i < \omega_1)$, then there is a uniform ultrafilter $\mathcal{U}$ on ω which contains an f_i-homogeneous set for each $i < \omega_1$.*

Proof. Let $\alpha < \omega_1$ and suppose that we have already constructed f_β-homogeneous sets U_β for $\beta < \alpha$ so that the intersection of any finite number of these is infinite. Then there is an infinite set X such tht $X \setminus U_\beta$ is finite for each $\beta < \alpha$, and by Ramsey's Theorem [6] there is an f_α-homogeneous set $U_\alpha \subseteq X$. The sets U_α $(\alpha < \omega_1)$ generate an ultrafilter $\mathcal{U}$. □

We also need the following special case of Solovay's Lemma (see e.g. [4, p. 287]).

Lemma 2.2. *Assume* MA_{ω_1}. *If the sets $A_i \in [\omega]^\omega$ $(i < \omega_1)$ have the property that the intersection of any finite number of them is infinite, then there is an infinite set $X \subseteq \omega$ such that $X \setminus A_i$ is finite for all $i < \omega_1$.*

3. A proof of (1.6)

We will use the same convention that was used in [5]; the letters A and B (possibly with suffixes or superfixes or primed) will always denote subsets of ω_1 which have respectively order types ω and ω_1 under the induced ordering.

As already observed in Section 1, it will be enough to prove that (1.6) holds in the model $\mathcal{M}$, i.e. we may and do assume that MA_{ω_1} holds and also that (1.8) holds with $\alpha = \omega 2 + 1$. Let $K_0 \cup K_1$ be any partition of $[\omega_1]^3$. We have to show that either

$$\omega 2 + 1 \in \text{homog}(K_0) \tag{3.1}$$

or

$$4 \in \text{homog}(K_1) \tag{3.2}$$

Lemma 3.1. *Let $A < B$ and assume that*:
(i) $[A]^2 \otimes [B]^1 \subseteq K_0$,
(ii) *the set* $\{a \in A: \{a\} \cup s \in K_1\}$ *is finite for each* $s \in [B]^2$, *and*
(iii) $(\forall x \in A \cup B)(\forall B_1 \subseteq B)(\exists B_2 \subseteq B_1)(\forall s \in [B_2]^2)\{x\} \cup s \in K_0$.
Then either (a) $\omega \in \text{homog}(K_1)$, *or* (b) *there is* $Z \subseteq A \cup B$ *such that* $\text{tp}(A \cap Z) = \omega$, $\text{tp}(B \cap Z) = \omega + 1$ *and* $[Z]^3 \subseteq K_0$ (*i.e.* (3.1) *or* (3.2) *holds*).

Proof. We will assume that (a) is false and deduce that (b) holds. Let $B = \{b_\alpha: \alpha < \omega_1\}$, where $b_0 < b_1 < \cdots$. We claim that there are $\alpha < \omega_1$, $X \in [A]^\omega$ and $Y \in [\{b_\beta: \beta < \alpha\}]^\omega$ such that

$$[X]^1 \otimes [Y \cup \{b_\alpha\}]^2 \subseteq K_0, \tag{3.3}$$

and

$$[Y]^2 \otimes [\{b_\alpha\}]^1 \subseteq K_0. \tag{3.4}$$

The lemma follows from the claim since, by Ramsey's Theorem and the assumption that (a) is false, we can assume that X and Y are both K_0-homogeneous. Then by (i), (3.3) and (3.4) the lemma holds with $Z = X \cup Y \cup \{b_\alpha\}$.

Let $\alpha < \omega_1$ be fixed. We try to construct the sets X, Y to satisfy (3.3) and (3.4) in ω steps as follows: Let $n < \omega$ and suppose that we have already constructed n-element sets $X_n \subseteq A$ and $Y_n \subseteq \{b_\beta: \beta < \alpha\}$ so that

$$[X_n]^1 \otimes [Y_n \cup \{b_\alpha\}]^2 \subseteq K_0, \tag{3.5}$$

and

$$[Y_n]^2 \otimes [\{b_\alpha\}]^1 \subseteq K_0 \tag{3.6}$$

both hold. If possible we now select $x_n \in A \backslash X_n$ and $y_n \in \{b_\beta: \beta < \alpha\} \backslash Y_n$ so that (3.5) and (3.6) remain true with X_n, Y_n replaced respectively by $X_n \cup \{x_n\}$ and $Y_n \cup \{y_n\}$. If it is not possible to choose suitable x_n and y_n, the construction terminates and we define

$$n_\alpha = n, \qquad X^\alpha = X_n, \qquad Y^\alpha = Y_n. \tag{3.7}$$

If, for some α, this construction continues for infinitely many steps, then our claim is established. So we can assume that n_α, X^α, and Y^α are defined as above for all $\alpha < \omega_1$. Now by Fodor's theorem [2] there are a stationary set $S \subseteq \omega_1$, an integer $n \in \omega$, and fixed n-element sets X_n and Y_n, so that (3.7) holds for each $\alpha \in S$.

By a finite number of applications of the hypothesis (iii), it follows that there is an uncountable set $T \subseteq S$ such that

$$[X_n \cup Y_n]^1 \otimes [\{b_\alpha: \alpha \in T\}]^2 \subseteq K_0. \tag{3.8}$$

Choose γ, $\alpha \in T$ with $\gamma < \alpha$. By the hypothesis (ii), the set

$$F = \{x \in A: \{x\} \cup s \in K_1 \quad \text{for some } s \in [Y_n \cup \{b_\gamma, b_\alpha\}]^2\}$$

is finite and so we can choose $a \in A \setminus (F \cup X_n)$. Since α belongs to S it follows that (3.5) and (3.6) both hold, and these also hold with b_γ in place of b_α since γ belongs to S. From these facts and (3.7) and by our choice of the element a, it is now a simple matter to check that (3.5) and (3.6) both hold with X_n replaced by $X_n \cup \{a\}$ and Y_n replaced by $Y_n \cup \{b_\gamma\}$. But this contradicts the fact that $n_\alpha = n$, since for α the above construction could be continued for at least one more step. □

The next lemma shows how we use the hypothesis that

$$\omega_1 \to (\omega_1, \omega 2 + 1)^2 \tag{3.9}$$

holds in $\mathcal{M}$.

Lemma 3.2. *Assume that* MA_{ω_1} *and* (3.9) *both hold. Let* $A, B \subseteq \omega_1$ *be such that* $A < B$ *and* $[A]^2 \otimes [B]^1 \subseteq K_0$. *Then either* (3.1) *or* (3.2) *holds.*

Proof. Let $\mathcal{U}$ be a uniform ultrafilter on A. For $s \in [B]^2$ and $i = 0$ or 1, define $A_i(s) = \{a \in A: \{a\} \cup s \in K_i\}$. Then either $A_0(s)$ or $A_1(s)$ belongs to $\mathcal{U}$. Suppose that there is a subset $Y \subseteq B$ having order type $\mathrm{tp}(Y) = \omega 2 + 1$ and such that $A_1(s) \in \mathcal{U}$ for all $s \in [Y]^2$. Then either $[Y]^3 \subseteq K_0$ and (3.1) holds, or there is $t \in [Y]^3 \cap K_1$. But in this latter case we can choose $a \in \bigcap \{A_1(s): s \in [t]^2\}$, and then $[\{a\} \cup t]^3 \subseteq K_1$ and (3.2) holds. Therefore, we may assume that, whenever $Y \subseteq B$ has order type $\omega 2 + 1$, then there is some pair $s \in [Y]^2$ such that $A_0(s)$ belongs to $\mathcal{U}$. By (3.9) it follows that there is $B' \subseteq B$ such that $A_0(s)$ belongs to $\mathcal{U}$ for all $s \in [B']^2$. Also, since MA_{ω_1} holds, it follows by Lemma 2.2 that there is an infinite set $A' \subseteq A$ such that $A' \setminus A_0(s)$ is finite for all $s \in [B']^2$. Thus, replacing A by A' and B by B', we may assume that condition (ii) of Lemma 3.1 is satisfied.

Fix an element $x \in A \cup B$ and a subset $B_1 \subseteq B$ and consider the partition $[B_1]^2 = L_0 \cup L_1$ in which $s \in L_0$ if and only if $\{x\} \cup s \in K_0$. If there is $Y \subseteq B_1$ such that Y has order type $\omega 2 + 1$ and $[Y]^2 \subseteq L_1$, then, by the same argument that was used in the preceding paragraph, it follows that either $[Y]^3 \subseteq K_0$, or there is $t \in [Y]^3$ such that $[\{x\} \cup t]^3 \subseteq K_1$; in either case the lemma follows. Therefore, we can assume that there is no such Y and so by (3.9) again, it follows that there is $B_2 \subseteq B_1$ such that $\{x\} \cup s \in K_0$ for all $s \in [B_2]^2$. In other words, the condition (iii) of Lemma 3.1 also holds. But condition (i) of that lemma holds by hypothesis, and so the result follows. □

Specker [7] proved that the partition relation

$$\omega^2 \to (\omega^2, m)^2 \tag{3.10}$$

holds for any integer m. The next lemma is a strengthening of this under the assumption that MA_{ω_1} holds.

Lemma 3.3. *Assume* MA_{ω_1}. *Let* W_i $(i<\omega)$ *be pairwise disjoint infinite sets,* $W=\bigcup\{W_i: i<\omega\}$ *and let* $f_\alpha:[W]^2\to 2(\alpha<\omega_1)$. *Then there is an uncountable set* $B\subseteq\omega_1$ *such that either* (i) *there are* $l_0<l_1<l_2<\cdots<\omega$ *and sets* $H_i\in[W_{l_i}]^\omega$ $(i<\omega)$ *such that* $f_\alpha(s)=0$ *for all* $\alpha\in B$ *and* $s\in\bigcup\{[H_i]^1\otimes[H_j]^1: i<j<\omega\}$, *or* (ii) *for any positive integer* m, *there are* m *integers* $l_0<l_1<\cdots<l_{m-1}$ *and* m *sets* $H_i\in[W_{l_i}]^m$ $(i<m)$ *such that* $f_\alpha(s)=1$ *for all* $\alpha\in B$ *and* $s\in\bigcup\{[H_i]^1\otimes[H_j]^1: i<j<m\}$.

Remark. Specker's Theorem corresponds to the case when $W_i=\{\xi:\omega i\leqslant\xi\leqslant\omega(i+1)\}$ $(i<\omega)$ and the f_α $(\alpha<\omega_1)$ are all equal, say $f_\alpha=f$. In this case, if (i) holds then, by Ramsey's Theorem, either there is an infinite set X contained in some H_i such that $f(s)=1$ for all $s\in[X]^2$ or, for each i there is an infinite set X contained in some H_i such that $f(s)=1$ for all $s\in[X]^2$ or, for each i there is a set $H_i'\in[H_i]^\omega$ such that $f(s)=0$ for all $s\in[H]^2$, where $H=\bigcup\{H_i: i<\omega\}$ has order type ω^2. On the other hand, if (ii) holds, then there is a set Y of cardinality m such that $|Y\cap H_i|=1$ $(i<m)$ and $[Y]^2\subseteq K_1$. Of course, the relation (3.10) is absolute and so it follows that this holds in ZFC. In fact, the proof of the lemma given below contains a ZFC proof of Specker's Theorem (see Remark (*) in the following proof).

Proof of Lemma 3.3. Without loss of generality we may assume that $W_i=\{\langle i,j\rangle: i<j<\omega\}$ $(i<\omega)$. For each $\alpha<\omega_1$ define a function $g_\alpha:[\omega]^4\to 2^3$ as follows: If $s=\{a,b,c,d\}$ where $a<b<c<d<\omega$, then we set $g_\alpha(s)=\langle g_\alpha^0(s), g_\alpha^1(s), g_\alpha^2(s)\rangle$, where

$$g_\alpha^0(s)=f_\alpha(\{\langle a,b\rangle,\langle c,d\rangle\}),$$
$$g_\alpha^1(s)=f_\alpha(\{\langle a,c\rangle,\langle b,d\rangle\}),$$
$$g_\alpha^2(s)=f_\alpha(\{\langle a,d\rangle,\langle b,c\rangle\}).$$

By Lemma 2.1, there is a uniform ultrafilter $\mathcal{U}$ on ω which contains a g_α-homogeneous set for each $\alpha<\omega_1$, i.e. there are $G_\alpha\in\mathcal{U}$ such that g_α is constant on $[G_\alpha]^4$. By MA_{ω_1} and Lemma 2.2, there is an infinite set $G\subseteq\omega$ such that $G\backslash G_\alpha$ is finite for each $\alpha<\omega_1$. It follows that there are a finite set $F\subseteq\omega$ and an uncountable set $B\subseteq\omega_1$ such that $G'=G\backslash F\subseteq G_\alpha$ holds for all $\alpha\in B$. (Remark (*): In the case when the f_α are all equal, then we may set $G=G'=G_\alpha$, and we do not need Lemma 2.2.) Replacing B by an uncountable subset if necessary, we may assume that the value of g_α on $[G']^4$ is constant for all $\alpha\in B$, say $g_\alpha(s)=\langle i_0,i_1,i_2\rangle$ for all $\alpha\in B$ and $s\in[G']^4$.

Let L_n $(n<\omega)$ be infinite pairwise disjoint subsets of G' and assume that $l_i<l_j$ holds for $i<j<\omega$, where $l_n=\min(L_n)$. Put $H_n=\{\langle l_n,y\rangle: y\in L_n\backslash\{l_n\}\}$ $(n<\omega)$.

If $i_0=i_1=i_2=0$, then (i) holds. For, if $h=\langle l_m,y_m\rangle\in H_m$, $h'=\langle l_n,y_n\rangle\in H_n$ and $m<n$, then $f_\alpha(\{h,h'\})=0$.

If one or more of i_0, i_1, i_2 is equal to 1, then (ii) holds. We will verify this for the case when $i_2=1$; the other cases are similar. Choose m^2 integers $y_r^s\in L_r\backslash\{l_r\}$

for $0 \leq r,s < m$ so that

$$l_{m-1} < y^0_{m-1} < \cdots < y^{m-1}_{m-1} < y^0_{m-2} < \cdots < y^{m-1}_{m-2} < \cdots < y^0_0 < y^1_0 < \cdots < y^{m-1}_0$$

Then for $\alpha \in B$, $s < m$, $t < m$ and $i < j < m$, we have

$$f_\alpha(\{\langle l_i, y^s_i\rangle, \langle l_j, y^t_j\rangle\} = g^2_\alpha(\langle l_i, l_j, y^t_j, y^s_i\rangle) = i_2 = 1. \quad \square$$

We need one additional lemma. This result follows from Lemmas 3.1, 3.2, and 5.2 of [5], but since it is essential for the present argument, and for the convenience of the reader, we give the proof.

Lemma 3.4. *Assume* MA_{ω_1}. *Suppose that*

$$[A]^2 \otimes [B]^1 \nsubseteq K_0, \tag{3.11}$$

whenever $A, B \subseteq \omega_1$. *Then there are subsets* A'_n $(n < \omega)$ *and* B' *of* ω_1 *such that* $A'_0 < A'_1 < \cdots < B'$ *and*

$$[A'_m]^2 \otimes [A'_n \cup B']^1 \subseteq K_1$$

holds for all $m < n < \omega$.

Proof. By Lemma 2.1, for any $A \subseteq \omega_1$, there is a uniform ultrafilter $\mathcal{U}$ on A such that, for each $\rho \in \omega_1$ there are $i(\rho) < 2$ and $A_\rho \in \mathcal{U}$ such that $[A_\rho]^2 \otimes [\{\rho\}]^1 \subseteq K_{i(\rho)}$. By Lemma 2.2 there is an infinite subset $X \subseteq A$ such that $X \backslash A_\rho$ is finite for all $\rho < \omega_1$. There is an uncountable set $B \subseteq \omega_1$ such that both $X \backslash A_\rho = F$ and $i(\rho) = i$ are constant for all $\rho \in B$. Thus, if we put $A' = X \backslash F$, then $[A']^2 \otimes [B]^1 \subseteq K_i$. From the hypothesis (3.11) it follows that $i = 1$. It follows from this and a simple induction argument that there are subsets A_n $(n < \omega)$ such that $A_0 < A_1 < \cdots$ and $[A_m]^2 \otimes [A_n]^1 \subseteq K_1$ holds for $m < n < \omega$.

By Lemma 2.1 again, there is a uniform ultrafilter $\mathcal{U}_n$ on A_n $(n < \omega)$ such that for each $\rho < \omega_1$ there are $i(n, \rho) < 2$ and $A(n, \rho) \in \mathcal{U}_n$ such that $[A(n, \rho)]^2 \otimes [\{\rho\}]^1 \subseteq K_{i(n,\rho)}$. By the same argument as above, for each n there are only countably many ρ such that $i(n, \rho) = 0$. Hence there is $B \subseteq \omega_1$ such that $i(n, \rho) = 1$ for all $n < \omega$ and all $\rho \in B$. By Lemma 2.2, there is an infinite set $X_n \subseteq A_n$ such that $X_n \backslash A(n, \rho)$ is finite for all $\rho \in B$.

For $j \in \omega$, $\rho \in B$ and $s \in [B]^{<\omega}$ define

$$F(j, \rho) = X_j \backslash A(j, \rho), \qquad F(j, s) = \bigcup\{F(j, \sigma) : \sigma \in s\}$$

Now consider the set P of all ordered pairs $\langle n, s\rangle$ such that $n \in \omega$, $s \in [B]^{<\omega}$ and

$$B(n, s) = \{\rho \in B : F(j, \rho) \subseteq F(j, s) \quad \text{for all } j \leq n\}$$

is uncountable. We order P by the rule that $\langle n, s\rangle \leq \langle n_1, s_1\rangle$ if and only if $n \leq n_1$, $s \subseteq s_1$ and $s_1 \backslash s \subseteq B(n, s)$. Note that this implies that

$$X_j \cap \bigcap \{A(j, \sigma) : \sigma \in s\} = X_j \cap \bigcap \{A(j, \sigma) : \sigma \in s_1\}$$

holds for all $j \leq n$.

We claim that P is *ccc*, i.e. if Q is an uncountable subset of P, then there are q_1, q_2 in Q and $p \in P$ such that $q_1 \leq p$ and $q_2 \leq p$. To see this, for any $p = \langle n, s \rangle \in P$, define $\bar{p} = \langle F_0, F_1, \ldots, F_n \rangle$, where $F_j = F(j, s)$. Since there are only countably many sequences of this kind, it follows that there are distinct elements $q_1 = \langle n, s_1 \rangle$ and $q_2 = \{n, s_2\rangle$ in Q which have a common first term and are such that $\bar{q}_1 = \bar{q}_2$. Put $s = s_1 \cup s_2$, $p = \langle n, s \rangle$. Since $F(j, s) = F(j, s_1) = F(j, s_2)$ for $j \leq n$, it follows that $p \in P$ and $q_1 \leq p$ and $q_2 \leq p$.

For $k < \omega$ and $\rho < \omega_1$, the sets

$$\mathcal{D}_k = \{\langle n, s \rangle \in P: n \geq k\} \quad \text{and} \quad \mathcal{E}_\rho = \{\langle n, s \rangle \in P: s \backslash \rho \neq \emptyset\}$$

are cofinal in $\langle P, \leq \rangle$ since, for a given element $\langle n, s \rangle \in P$ there is an uncountable set $B' \subseteq B(n, s)$ such that $F(n, \sigma) = F_j$ for all $j \leq m = \max(n, k)$ and for all $\sigma \in B'$, and so $\langle n, s \rangle \leq \langle m, s \cup \{\alpha\} \rangle \in \mathcal{D}_k \cap \mathcal{E}_\rho$ for $\alpha \in B'$ and $\alpha > \rho$. By MA_{ω_1} there is an ideal $\mathcal{T} \subseteq P$ which has a non-empty intersection with all the $\mathcal{D}_k$ and $\mathcal{E}_\rho$. Since $\mathcal{T}$ has non-empty intersection with all $\mathcal{E}_\rho$, it follows that $B' = \bigcup \{s: \langle n, s \rangle \in \mathcal{T}$ for some $n\}$ is uncountable.

For $k < \omega$ choose any $\langle n, s \rangle \in \mathcal{D}_k \cap \mathcal{T}$ and define

$$A'_k = X_k \cap \bigcap \{A(k, \sigma): \sigma \in s\}.$$

Note that this definition of A'_k does not depend upon the particular choice of $\langle n, s \rangle$. For, if $\langle m, t\} \in \mathcal{D}_k \cap \mathcal{T}$, then there is a common upper bound of these two elements $\langle l, r \rangle \in \mathcal{D}_k \cap \mathcal{T}$, and then

$$X_k \cap \bigcap \{A(k, \sigma): \sigma \in s\} = X_k \cap \bigcap \{A(k, \sigma): \sigma \in t\} = X_k \cap \bigcap \{A(k, \sigma): \sigma \in r\}$$

Now for any $k < \omega$ and $\rho \in B'$, there is some $\langle n, s \rangle \in \mathcal{T}$ such that $n \geq k$ and $\rho \in s$. Since $A'_k \subseteq A(k, \rho)$ and $i(k, \rho) = 1$, it follows that $[A'_k]^2 \otimes [\{\rho\}]^1 \subseteq K_1$. □

We now conclude the proof that (1.6) holds in $\mathcal{M}$.

If there are $A, B \subseteq \omega_1$ such that $[A]^2 \otimes [B]^1 \subseteq K_0$, then the result follows immediately from Lemma 3.2. Therefore, we may assume that

$$[A]^2 \otimes [B]^1 \nsubseteq K_0 \tag{3.12}$$

whenever $A, B \subseteq \omega_1$. Then by Lemma 3.4 there are sets A_n $(n < \omega)$ and B such that $A_0 < A_1 < \cdots < B$ and such that

$$[A_m]^2 \otimes [A_n \cup B]^1 \subseteq K_1 \tag{3.13}$$

holds for $m < n < \omega$. By Lemma 3.3 (with $m = 2$), we can assume that either (i) there are infinite subsets $A'_i \subseteq A_i$ $(i < \omega)$ such that $\{a_m, a_n, b\} \in K_0$ whenever $a_m \in A'_m$, $a_n \in A'_n$, $b \in B$ and $m < n$, or (ii) there are $H_0 \in [A_0]^2$, $H_1 \in [A_1]^2$ such that $\{h_0, h_1, b\} \in K_1$ whenever $h_i \in H_i$ $(i < 2)$ and $b \in B$. If (i) holds then we contradict (3.12) by choosing a set A such that $|A \cap A_i| = 1$ for all $i < \omega$. So we may suppose that (ii) holds. But in this case, since (3.13) holds, we have that $[Y]^3 \subseteq K_1$, where Y is a 4-element set such that $H_0 \subseteq Y$, $|Y \cap H_1| = 1$ and $|Y \cap B| = 1$, and so (3.2) holds. □

References

[1] J. Baumgartner and A. Hajnal, A proof (involving Martin's Axiom) of a partition relation, Fund. Math. 78 (1973) 193–203.
[2] G. Fodor, Ein Bermerkung zur Theorie der regressiven Functionen, Acta Sci. Math. 17 (1956) 139–142.
[3] A. Hajnal, A negative partition relation, Proc. Nat. Acad. Sci. USA 68 (1971) 142–144.
[4] K. Kunen, Set Theory: Studies in Logic and the Foundations of Mathematics, Vol. 102 (North-Holland, Amsterdam, 1980).
[5] E.C. Milner and K. Prikry, A partition theorem for triples, Proc. Amer. Math. Soc. 97 (1986) 488–494.
[6] F.P. Ramsey, On a problem in formal logic, Proc. London Math. Soc. (2) 30 (1930) 264–286.
[7] E. Specker, Teilmengen von Mengen mit Relationen, Comment. Math. Helv. 31 (1957) 302–314.
[8] S. Todorčević, Forcing positive partition relations, Trans. Amer. Math. Soc. 280 (1983) 703–720.