A partition relation for triples using a model of Todorčević*

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Received 22 September 1989
Revised 29 June 1990

Abstract

Todorčević has shown that there is a ccc extension $\mathcal{M}$ in which $\text{MA}_{\omega_1} + 2^\omega = \omega_2$ holds and also in which the partition relation $\omega_1 \rightarrow (\omega_1, \alpha)^3$ holds for every dencernable ordinal $\alpha$. We show that the partition relation for triples

$$\omega_1 \rightarrow (\omega 2 + 1, 4)^3$$

holds in the model $\mathcal{M}$, and hence by absoluteness this is a theorem in ZFC.

1. Introduction

For an ordinal $\gamma$, a positive integer $r$ and linear order types $\varphi, \psi_i$ ($i < \gamma$), the partition relation

$$\varphi \rightarrow (\psi_i)^r_{i < \gamma}$$

(1.1)

means that whenever $(S, <)$ is an ordered set of order type $\text{tp}(S) = \gamma$ and \{K$i$: $i < \gamma$\} is a partition of $[S]^r = \{X \subseteq S: |X| = r\}$, then there are $i < \gamma$ and $T \subseteq S$ such that $[T]^r \subseteq K_i$ and $\text{tp}(T) = \psi_i$. In the case when $\gamma = 2$ we write (1.1) as $\varphi \rightarrow (\psi_1, \psi_2)^r$, and the negation of this is expressed by replacing the arrow $\rightarrow$ by a nonarrow $\nrightarrow$.

Very few partition relations of this kind are known when $r \geq 3$ and the order types are not cardinal numbers. Such relations were discussed in some detail in [5], and we proved in that paper that

$$\varphi \rightarrow (\omega + k, 4)^3$$

(1.2)

* Research supported by NSERC Grant No. A5198 and NSF Grant MCS 830361.
holds for any finite \( k \) and linear type \( \varphi \) which satisfies
\[
\varphi \rightarrow (\omega)^4_{\omega}. \tag{1.3}
\]
It was conjectured in [5] that the more general relation
\[
\varphi \rightarrow (\alpha, m)^3
\]
holds for all countable ordinals \( \alpha \) and finite \( m \), but we were unable to extend our method of proof of (1.2) to establish either of the next simplest cases of the conjecture
\[
\omega_1 \rightarrow (\omega 2, 4)^3, \tag{1.4}
\]
or
\[
\omega_1 \rightarrow (\omega + 2, 5)^3. \tag{1.5}
\]
In this paper we use some heavier artillery from [8] in order to prove that
\[
\omega_1 \rightarrow (\omega 2 + 1, 4)^3, \tag{1.6}
\]
which is slightly stronger than (1.4).

Our proof of (1.2) for the case \( \varphi = \omega_1 \) in [5] used the same type of argument employed by Baumgartner and Hajnal in [1]. Since
\[
\omega_1 \rightarrow (\omega + k, 4)^3 \tag{1.7}
\]
is an absolute statement relative to a ccc extension, it is sufficient to prove this under the additional assumption that Martin's Axiom \( \text{MA}_{\omega_1} \) holds. However, we could not prove (1.4) by using the same combinatorial tools. Here we will prove (1.6) using a model of Todorcević. He proved [8] that there is a ccc extension \( \mathcal{M} \) in which \( \text{MA}_{\omega_1} \) holds, \( 2^\omega = \omega_2 \), and also in which the relation
\[
\omega_1 \rightarrow (\omega_1, \alpha)^2 \tag{1.8}
\]
holds for all \( \alpha < \omega_1 \). We prove that (1.6) holds in the model \( \mathcal{M} \) and so, by absoluteness, (1.6) is a theorem of ZFC. Note that (1.8) is independent of the axioms of ZFC since, by an earlier result of Hajnal [3], CH implies that \( \omega_1 \rightarrow (\omega_1, \omega + 2)^2 \). Let us remark that we only use the special case of (1.8) when \( \alpha = \omega 2 + 1 \); we were unable to obtain anything better by using the full strength of (1.8). Also, we should point out that the proof used in [5] to prove (1.7) could, by an argument due to Baumgartner and Hajnal, be adapted to prove (1.2) by using \( \text{MA} \) in place of \( \text{MA}_{\omega_1} \). This argument does not allow us to extend in the same way our proof of (1.6) to the more general relation \( \varphi \rightarrow (\omega 2 + 1, 4)^3 \) for an order type satisfying (1.3). Thus, for example, whether or not the relation
\[
\lambda \rightarrow (\omega 2, 4)^3
\]
holds is still open, where \( \lambda \) is the order type of the reals.
2. Notation and preliminary lemmas

We use the standard notation $[X]^{<\omega}$ to denote the set of all finite subsets of $X$. If $(S, \leq)$ is a linearly ordered set and $x \in S$, then $S(\geq x) = \{ y \in S : y \geq x \}$. Also, if $X, Y$ are subsets of $S$, then we write $X \prec Y$ if $x \prec y$ holds for all $x \in X$ and $y \in Y$. If $\{ K_i : i < \gamma \}$ is any partition of $[S]^\gamma$, then we write

$$\psi \in \text{homog}(K_i)$$

if there is a subset $T \subseteq S$ such that $\text{tp}(T) = \psi$, and $T$ is homogeneous for the class $K_i$, i.e. if $[T]^\gamma \subseteq K_i$. For finite $r, s$ and sets $A, B$, we denote by $[A]^r \otimes [B]^s$ the set of all subsets $X \subseteq A \cup B$ such that $|A \cap X| = r$ and $|B \cap X| = s$.

We need the following easily proved consequence of Ramsey’s Theorem.

**Lemma 2.1.** If $r_i, k_i$ are finite and $f_i : [\omega]^\omega \rightarrow k_i (i < \omega_1)$, then there is a uniform ultrafilter $\mathcal{U}$ on $\omega$ which contains an $f_i$-homogeneous set for each $i < \omega_1$.

**Proof.** Let $\alpha < \omega_1$ and suppose that we have already constructed $f_\beta$-homogeneous sets $U_\beta$ for $\beta < \alpha$ so that the intersection of any finite number of these is infinite. Then there is an infinite set $X$ such that $X \setminus U_\beta$ is finite for each $\beta < \alpha$, and by Ramsey’s Theorem [6] there is an $f_\alpha$-homogeneous set $U_\alpha \subseteq X$. The sets $U_\alpha (\alpha < \omega_1)$ generate an ultrafilter $\mathcal{U}$. \( \square \)

We also need the following special case of Solovay’s Lemma (see e.g. [4, p. 287]).

**Lemma 2.2.** Assume $\text{MA}_{\omega_1}$. If the sets $A_i \subseteq [\omega]^\omega (i < \omega_1)$ have the property that the intersection of any finite number of them is infinite, then there is an infinite set $X \subseteq \omega$ such that $X \setminus A_i$ is finite for all $i < \omega_1$.

3. A proof of (1.6)

We will use the same convention that was used in [5]; the letters $A$ and $B$ (possibly with suffixes or superfixes or primed) will always denote subsets of $\omega_1$ which have respectively order types $\omega$ and $\omega_1$ under the induced ordering.

As already observed in Section 1, it will be enough to prove that (1.6) holds in the model $\mathcal{M}$, i.e. we may and do assume that $\text{MA}_{\omega_1}$ holds and also that (1.8) holds with $\alpha = \omega_2 + 1$. Let $K_0 \cup K_1$ be any partition of $[\omega_1]^3$. We have to show that either

$$\omega_2 + 1 \in \text{homog}(K_0) \quad (3.1)$$

or

$$4 \in \text{homog}(K_1) \quad (3.2)$$
Lemma 3.1. Let \( A < B \) and assume that:

(i) \([A]^2 \otimes [B]^1 \subseteq K_0\),

(ii) the set \( \{ a \in A : \{ a \} \cup s \in K_1 \} \) is finite for each \( s \in [B]^2 \), and

(iii) \((\forall x \in A \cup B)(\forall B_1 \subseteq B)(\exists B_2 \subseteq B_1)(\forall s \in [B_2]^2)\{ x \} \cup s \in K_0\).

Then either (a) \( \omega \in \text{hom}(K_1) \), or (b) there is \( Z \subseteq A \cup B \) such that \( \text{tp}(A \cap Z) = \omega \), \( \text{tp}(B \cap Z) = \omega + 1 \) and \([Z]^3 \subseteq K_0 \) (i.e. (3.1) or (3.2) holds).

Proof. We will assume that (a) is false and deduce that (b) holds. Let \( B = \{ b_\alpha : \alpha < \omega_1 \} \), where \( b_0 < b_1 < \cdots \). We claim that there are \( \alpha < \omega_1 \), \( X \in [A]^\omega \) and \( Y \in \{ \{ b_\beta : \beta < \alpha \} \}^\omega \) such that

\[
[X]^1 \otimes [Y \cup \{ b_\alpha \}]^2 \subseteq K_0,
\]

and

\[
[Y]^2 \otimes [\{ b_\alpha \}]^1 \subseteq K_0.
\]

The lemma follows from the claim since, by Ramsey's Theorem and the assumption that (a) is false, we can assume that \( X \) and \( Y \) are both \( K_0 \)-homogeneous. Then by (i), (3.3) and (3.4) the lemma holds with \( Z = X \cup Y \cup \{ b_\alpha \} \).

Let \( \alpha < \omega_1 \) be fixed. We try to construct the sets \( X \), \( Y \) to satisfy (3.3) and (3.4) in \( \omega \) steps as follows: Let \( n < \omega \) and suppose that we have already constructed \( n \)-element sets \( X_n \subseteq A \) and \( Y_n \subseteq \{ b_\beta : \beta < \alpha \} \) so that

\[
[X_n]^1 \otimes [Y_n \cup \{ b_\alpha \}]^2 \subseteq K_0,
\]

and

\[
[Y_n]^2 \otimes [\{ b_\alpha \}]^1 \subseteq K_0
\]

both hold. If possible we now select \( x_n \in A \setminus X_n \) and \( y_n \in \{ b_\beta : \beta < \alpha \} \setminus Y_n \) so that (3.5) and (3.6) remain true with \( X_n \), \( Y_n \) replaced respectively by \( X_n \cup \{ x_n \} \) and \( Y_n \cup \{ y_n \} \). If it is not possible to choose suitable \( x_n \) and \( y_n \), the construction terminates and we define

\[
n_\alpha = n, \quad X^\alpha = X_n, \quad Y^\alpha = Y_n.
\]

If, for some \( \alpha \), this construction continues for infinitely many steps, then our claim is established. So we can assume that \( n_\alpha \), \( X^\alpha \), and \( Y^\alpha \) are defined as above for all \( \alpha < \omega_1 \). Now by Fodor's theorem [2] there is a stationary set \( S \subseteq \omega_1 \), an integer \( n \in \omega \), and fixed \( n \)-element sets \( X_n \) and \( Y_n \), so that (3.7) holds for each \( \alpha \in S \).

By a finite number of applications of the hypothesis (iii), it follows that there is an uncountable set \( T \subseteq S \) such that

\[
[X_n \cup Y_n]^1 \otimes [\{ b_\alpha : \alpha \in T \}]^2 \subseteq K_0.
\]

Choose \( \gamma, \alpha \in T \) with \( \gamma < \alpha \). By the hypothesis (ii), the set

\[
F = \{ x \in A : \{ x \} \cup s \in K_1 \} \quad \text{for some } s \in [Y_n \cup \{ b_\gamma, b_\alpha \}]^2
\]
is finite and so we can choose \( a \in A \setminus (F \cup X_n) \). Since \( \alpha \) belongs to \( S \) it follows that (3.5) and (3.6) both hold, and these also hold with \( b_\gamma \) in place of \( b_\sigma \) since \( \gamma \) belongs to \( S \). From these facts and (3.7) and by our choice of the element \( a \), it is now a simple matter to check that (3.5) and (3.6) both hold with \( X_n \) replaced by \( X_n \cup \{ a \} \) and \( Y_n \) replaced by \( Y_n \cup \{ b_\gamma \} \). But this contradicts the fact that \( n_\alpha = n \), since for \( \alpha \) the above construction could be continued for at least one more step. \( \square \)

The next lemma shows how we use the hypothesis that

\[ \omega_1 \rightarrow (\omega_1, \omega_2 + 1)^2 \quad (3.9) \]

holds in \( M \).

**Lemma 3.2.** Assume that \( \text{MA}_{\omega_1} \) and (3.9) both hold. Let \( A, B \subseteq \omega_1 \) be such that \( A \prec B \) and \( [A]^2 \otimes [B]^1 \subseteq K_0 \). Then either (3.1) or (3.2) holds.

**Proof.** Let \( \mathcal{U} \) be a uniform ultrafilter on \( A \). For \( s \in [B]^2 \) and \( i = 0 \) or \( 1 \), define \( A_i(s) = \{ a \in A : \{ a \} \cup s \in K_i \} \). Then either \( A_0(s) \) or \( A_1(s) \) belongs to \( \mathcal{U} \). Suppose that there is a subset \( Y \subseteq B \) having order type \( \text{tp}(Y) = \omega_2 + 1 \) and such that \( A_i(s) \in \mathcal{U} \) for all \( s \in [Y]^2 \). Then either \( [Y]^3 \subseteq K_0 \) and (3.1) holds, or there is \( t \in [Y]^3 \cap K_1 \). But in this latter case we can choose \( a \in \bigcap \{ A_i(s) : s \in [t]^2 \} \), and then \( \{ \{ a \} \cup t \}^3 \subseteq K_1 \) and (3.2) holds. Therefore, we may assume that, whenever \( Y \subseteq B \) has order type \( \omega_2 + 1 \), then there is some pair \( s \in [Y]^2 \) such that \( A_0(s) \) belongs to \( \mathcal{U} \). By (3.9) it follows that there is \( B' \subseteq B \) such that \( A_0(s) \) belongs to \( \mathcal{U} \) for all \( s \in [B']^2 \). Also, since \( \text{MA}_{\omega_1} \) holds, it follows by Lemma 2.2 that there is an infinite set \( A' \subseteq A \) such that \( A' \setminus A_0(s) \) is finite for all \( s \in [B']^2 \). Thus, replacing \( A \) by \( A' \) and \( B \) by \( B' \), we may assume that condition (ii) of Lemma 3.1 is satisfied.

Fix an element \( x \in A \cup B \) and a subset \( B_1 \subseteq B \) and consider the partition \( [B_1]^2 = L_0 \cup L_1 \) in which \( s \in L_0 \) if and only if \( \{ x \} \cup s \in K_0 \). If there is \( Y \subseteq B_1 \) such that \( Y \) has order type \( \omega_2 + 1 \) and \( [Y]^2 \subseteq L_1 \), then, by the same argument that was used in the preceding paragraph, it follows that either \( [Y]^3 \subseteq K_0 \), or there is \( t \in [Y]^3 \) such that \( \{ \{ x \} \cup t \}^3 \subseteq K_1 \); in either case the lemma follows. Therefore, we can assume that there is no such \( Y \) and so by (3.9) again, it follows that there is \( B_2 \subseteq B_1 \) such that \( \{ x \} \cup s \in K_0 \) for all \( s \in [B_2]^2 \). In other words, the condition (iii) of Lemma 3.1 also holds. But condition (i) of that lemma holds by hypothesis, and so the result follows. \( \square \)

Specker [7] proved that the partition relation

\[ \omega^2 \rightarrow (\omega^2, m)^2 \quad (3.10) \]

holds for any integer \( m \). The next lemma is a strengthening of this under the assumption that \( \text{MA}_{\omega_1} \) holds.
Lemma 3.3. Assume MA$_{\omega_1}$. Let $W_i (i < \omega)$ be pairwise disjoint infinite sets, $W = \bigcup \{W_i : i < \omega\}$ and let $f_\alpha : [W]^2 \to (\alpha < \omega_1)$. Then there is an uncountable set $B \subseteq \omega_1$ such that either (i) there are $l_0 < l_1 < l_2 < \cdots < \omega$ and sets $H_i \in [W_i]^\omega (i < \omega)$ such that $f_\alpha (s) = 0$ for all $\alpha \in B$ and $s \in \bigcup \{[H_i]^1 \otimes [H_j]^1 : i < j < \omega\}$, or (ii) for any positive integer $m$, there are $m$ integers $l_0 < l_1 < \cdots < l_{m-1}$ and $m$ sets $H_i \in [W_i]^m (i < m)$ such that $f_\alpha (s) = 1$ for all $\alpha \in B$ and $s \in \bigcup \{[H_i]^1 \otimes [H_j]^1 : i < j < m\}$.

Remark. Specker's Theorem corresponds to the case when $W_i = \{\xi : \omega i \leq \xi \leq \omega (i + 1)\} (i < \omega)$ and the $f_\alpha (\alpha < \omega_1)$ are all equal, say $f_\alpha = f$. In this case, if (i) holds then, by Ramsey's Theorem, either there is an infinite set $X$ contained in some $H_i$ such that $f(s) = 1$ for all $s \in [X]^2$ or, for each $i$ there is an infinite set $X$ contained in some $H_i$ such that $f(s) = 1$ for all $s \in [X]^2$ or, for each $i$ there is a set $H_i' \in [H_i]^\omega$ such that $f(s) = 0$ for all $s \in [H]^2$, where $H = \bigcup \{H_i : i < \omega\}$ has order type $\omega^2$. On the other hand, if (ii) holds, then there is a set $Y$ of cardinality $m$ such that $|Y \cap H_i| = 1 (i < m)$ and $|Y|^2 \subseteq K_1$. Of course, the relation (3.10) is absolute and so it follows that this holds in ZFC. In fact, the proof of the lemma given below contains a ZFC proof of Specker's Theorem (see Remark (*) in the following proof).

Proof of Lemma 3.3. Without loss of generality we may assume that $W_i = \{(i, j) : i < j < \omega\} (i < \omega)$. For each $\alpha < \omega_1$, define a function $g_\alpha : [\omega]^4 \to 2^3$ as follows: If $s = \{a, b, c, d\}$ where $a < b < c < d < \omega$, then we set $g_\alpha (s) = \langle g_\alpha^0 (s), g_\alpha^1 (s), g_\alpha^2 (s) \rangle$, where

\[
g_\alpha^0 (s) = f_a (\{\langle a, b \rangle, \langle c, d \rangle \}),
\]

\[
g_\alpha^1 (s) = f_a (\{\langle a, c \rangle, \langle b, d \rangle \}),
\]

\[
g_\alpha^2 (s) = f_a (\{\langle a, d \rangle, \langle b, c \rangle \}).
\]

By Lemma 2.1, there is a uniform ultrafilter $\mathcal{U}$ on $\omega$ which contains a $g_\alpha$-homogeneous set for each $\alpha < \omega_1$, i.e. there are $G_\alpha \in \mathcal{U}$ such that $g_\alpha$ is constant on $[G_\alpha]^4$. By MA$_{\omega_1}$ and Lemma 2.2, there is an infinite set $G \subseteq \omega$ such that $G \setminus G_\alpha$ is finite for each $\alpha < \omega_1$. It follows that there is a finite set $F \subseteq \omega$ and an uncountable set $B \subseteq \omega_1$ such that $G' = GF \subseteq G_\alpha$ holds for all $\alpha \in B$. (Remark (*): In the case when the $f_\alpha$ are all equal, then we may set $G = G' = G_\alpha$, and we do not need Lemma 2.2.) Replacing $B$ by an uncountable subset if necessary, we may assume that the value of $g_\alpha$ on $[G']^4$ is constant for all $\alpha \in B$, say $g_\alpha (s) = \langle i_0, i_1, i_2 \rangle$ for all $\alpha \in B$ and $s \in [G']^4$.

Let $L_n (n < \omega)$ be infinite pairwise disjoint subsets of $G'$ and assume that $l_i < l_j$ holds for $i < j < \omega$, where $l_n = \min (L_n)$. Put $H_n = \{\langle l_n, y : y \in L_n \setminus \{l_n\} \} (n < \omega)$.

If $i_0 = i_1 = i_2 = 0$, then (i) holds. For, if $h = \langle l_m, y_m \rangle \in H_m, h' = \langle l_n, y_n \rangle \in H_n$ and $m < n$, then $f_\alpha (\{h, h'\}) = 0$.

If one or more of $i_0, i_1, i_2$ is equal to 1, then (ii) holds. We will verify this for the case when $i_2 = 1$; the other cases are similar. Choose $m^2$ integers $y_i \in L_i \setminus \{l_i\}$.
for \(0 \leq r, s < m\) so that
\[
l_{m-1}^0 < y_{m-1}^0 < \cdots < y_{m-2}^0 < \cdots < y_{m-2}^1 < \cdots < y_0^1 < y_0^1 < \cdots < y_0^{m-1}
\]
Then for \(\alpha \in B, s < m, i < m\) and \(i < j < m\), we have
\[
f\alpha((l_i, y_i^j), (l_j, y_j^j)) = g^2\alpha((l_i, l_j, y_j^j, y_j^i)) = i_2 = 1. \quad \square
\]

We need one additional lemma. This result follows from Lemmas 3.1, 3.2, and 5.2 of [5], but since it is essential for the present argument, and for the convenience of the reader, we give the proof.

**Lemma 3.4.** Assume \(\text{MA}_{\omega_1}\). Suppose that
\[
[A]^2 \otimes [B]^1 \not\subseteq K_0. \tag{3.11}
\]
whenever \(A, B \subseteq \omega_1\). Then there are subsets \(A'_n\) (\(n < \omega\)) and \(B'\) of \(\omega_1\) such that \(A_0 < A'_1 < \cdots < B'\) and
\[
[A'_m]^2 \otimes [A'_m \cup B']^1 \subseteq K_1
\]
holds for all \(m < n < \omega\).

**Proof.** By Lemma 2.1, for any \(A \subseteq \omega_1\), there is a uniform ultrafilter \(\mathcal{U}\) on \(A\) such that, for each \(\rho \in \omega_1\) there are \(i(\rho) < 2\) and \(A_{\rho} \in \mathcal{U}\) such that \([A_{\rho}]^2 \otimes [\{\rho\}]^1 \subseteq K_{i(\rho)}\). By Lemma 2.2 there is an infinite subset \(X \subseteq A\) such that \(X \setminus A_{\rho}\) is finite for all \(\rho < \omega_1\). There is an uncountable set \(B \subseteq \omega_1\) such that both \(X \setminus A_{\rho} = F\) and \(i(\rho) = i\) are constant for all \(\rho \in B\). Thus, if we put \(A' = X \setminus F\), then \([A']^2 \otimes [B]^1 \subseteq K_i\) from the hypothesis (3.11) it follows that \(i = 1\). It follows from this and a simple induction argument that there are subsets \(A_n\) (\(n < \omega\)) such that \(A_0 < A_1 < \cdots \) and \([A'_m]^2 \otimes [A'_m] \subseteq K_1\) holds for \(m < n < \omega\).

By Lemma 2.1 again, there is a uniform ultrafilter \(\mathcal{U}_n\) on \(A_n\) (\(n < \omega\)) such that for each \(\rho < \omega_1\) there are \(i(n, \rho) < 2\) and \(A(n, \rho) \in \mathcal{U}_n\) such that \([A(n, \rho)]^2 \otimes [\{\rho\}]^1 \subseteq K_{i(n, \rho)}\). By the same argument as above, for each \(n\) there are only countably many \(\rho\) such that \(i(n, \rho) = 0\). Hence there is \(B \subseteq \omega_1\) such that \(i(n, \rho) = 1\) for all \(n < \omega\) and all \(\rho \in B\). By Lemma 2.2, there is an infinite set \(X_n \subseteq A_n\) such that \(X_n \setminus A(n, \rho)\) is finite for all \(\rho \in B\).

For \(j \in \omega, \rho \in B\) and \(\sigma \in [B]^{<\omega}\) define
\[
F(j, \rho) = X_j \setminus A(j, \rho), \quad F(j, \sigma) = \bigcup \{F(j, \rho) : \sigma \in s\}
\]
Now consider the set \(P\) of all ordered pairs \((n, s)\) such that \(n \in \omega, s \in [B]^{<\omega}\) and
\[
B(n, s) = \{\rho \in B : F(j, \rho) \subseteq F(j, s)\} \quad \text{for all } j \leq n
\]
is uncountable. We order \(P\) by the rule that \((n, s) \preceq (n_1, s_1)\) if and only if \(n \leq n_1, s \subseteq s_1\) and \(s_1 \setminus s \subseteq B(n, s)\). Note that this implies that
\[
X_j \cap \bigcap \{A(j, \sigma) : \sigma \in s\} = X_j \cap \bigcap \{A(j, \sigma) : \sigma \in s_1\}
\]
holds for all \(j \leq n\).
We claim that $P$ is ccc, i.e. if $Q$ is an uncountable subset of $P$, then there are $q_1, q_2 \in Q$ and $p \in P$ such that $q_1 \leq p$ and $q_2 \leq p$. To see this, for any $p = \langle n, s \rangle \in P$, define $\bar{p} = \langle F_0, F_1, \ldots, F_n \rangle$, where $F_j = F(j, s)$. Since there are only countably many sequences of this kind, it follows that there are distinct elements $q_1 = \langle n, s_1 \rangle$ and $q_2 = \langle n, s_2 \rangle$ in $Q$ which have a common first term and are such that $\bar{q}_1 = \bar{q}_2$. Put $s = s_1 \cup s_2$, $p = \langle n, s \rangle$. Since $F(j, s) = F(j, s_1) = F(j, s_2)$ for $j \leq n$, it follows that $p \in P$ and $q_1 \leq p$ and $q_2 \leq p$.

For $k < \omega$ and $\rho < \omega_1$, the sets

$$\mathcal{D}_k = \{ \langle n, s \rangle \in P : n \geq k \} \quad \text{and} \quad \mathcal{E}_\rho = \{ \langle n, s \rangle \in P : s \setminus \rho \neq \emptyset \}$$

are cofinal in $\langle P, \leq \rangle$ since, for a given element $\langle n, s \rangle \in P$ there is an uncountable set $B' \subseteq B(n, s)$ such that $F(n, \sigma) = F_j$ for all $j \leq m = \max(n, k)$ and for all $\sigma \in B'$, and so $\langle m, s \cup \{ \alpha \} \rangle \in \mathcal{D}_k \cap \mathcal{E}_\rho$ for $\alpha \in B'$ and $\alpha > \rho$. By MA$_{\omega_1}$ there is an ideal $\mathcal{I} \subseteq P$ which has a non-empty intersection with all the $\mathcal{D}_k$ and $\mathcal{E}_\rho$. Since $\mathcal{I}$ has non-empty intersection with all $\mathcal{E}_\rho$, it follows that $B' = \bigcup \{ s : \langle n, s \rangle \in \mathcal{I} \text{ for some } n \}$ is uncountable.

For $k < \omega$ choose any $\langle n, s \rangle \in \mathcal{D}_k \cap \mathcal{I}$ and define

$$A'_k = X_k \cap \bigcap \{ A(k, \sigma) : \sigma \in s \}.$$ 

Note that this definition of $A'_k$ does not depend upon the particular choice of $\langle n, s \rangle$. For, if $\langle m, t \rangle \in \mathcal{D}_k \cap \mathcal{I}$, then there is a common upper bound of these two elements $\langle l, r \rangle \in \mathcal{D}_k \cap \mathcal{I}$, and then

$$X_k \cap \bigcap \{ A(k, \sigma) : \sigma \in s \} = X_k \cap \bigcap \{ A(k, \sigma) : \sigma \in t \} = X_k \cap \bigcap \{ A(k, \sigma) : \sigma \in r \}$$

Now for any $k < \omega$ and $\rho \in B'$, there is some $\langle n, s \rangle \in \mathcal{I}$ such that $n \geq k$ and $\rho \in s$. Since $A'_k \subseteq A(k, \rho)$ and $i(k, \rho) = 1$, it follows that $[A'_k]^2 \otimes \{ \rho \} \subseteq K_1$.  

We now conclude the proof that (1.6) holds in $\mathcal{M}$.

If there are $A, B \subseteq \omega_1$ such that $[A]^2 \otimes [B]^1 \subseteq K_0$, then the result follows immediately from Lemma 3.2. Therefore, we may assume that

$$[A]^2 \otimes [B]^1 \not\subseteq K_0 \quad \text{(3.12)}$$

whenever $A, B \subseteq \omega_1$. Then by Lemma 3.4 there are sets $A_n$ ($n < \omega$) and $B$ such that

$$A_n < A_1 < \cdots < B \quad \text{and such that}$$

$$[A_m]^2 \otimes [A_n \cup B]^1 \subseteq K_1 \quad \text{(3.13)}$$

holds for $m < n < \omega$. By Lemma 3.3 (with $m = 2$), we can assume that either (i) there are infinite subsets $A'_i \subseteq A_i$ ($i < \omega$) such that $(a_m, a_n, b) \in K_0$ whenever $a_m \in A'_m$, $a_n \in A'_n$, $b \in B$ and $m < n$, or (ii) there are $H_0 \in [A_0]^2$, $H_1 \in [A_1]^2$ such that $(h_0, h_1, b) \in K_1$ whenever $h_i \in H_i$ ($i < 2$) and $b \in B$. If (i) holds then we contradict (3.12) by choosing a set $A$ such that $[A \cap A_i]^1 = 1$ for all $i < \omega$. So we may suppose that (ii) holds. But in this case, since (3.13) holds, we have that $[Y]^1 \subseteq K_1$, where $Y$ is a 4-element set such that $H_0 \subseteq Y$, $[Y \cap H_i]^1 = 1$ and $[Y \cap B]^1 = 1$, and so (3.2) holds. $\Box$
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References
