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Analyzing Nash-Williams' partition theorem by means of ordinal types

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Abstract

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There is a natural way of assigning ordinal-valued functions to certain Ramsey-type theorems. In particular, they can be regarded as an extension of the classical notion of Ramsey numbers. The purpose of this paper is to obtain an estimate of these functions for the Nash-Williams' partition theorem.

0. Motivation

This paper is a continuation of our earlier work [7]. The motivation for [7] was twofold. We wanted to generalize 'Ramsey numbers' and thus obtain means to quantatively measure various infinite Ramsey type results, and by doing so we wanted to capture metamathematical phenomena such as the unprovability of certain results in specified logical systems. In this section we further explain these two ideas. The rest of the paper is independent of this section.

Let us consider the following two statements. (Our notation is standard and is explained in the next section.)

For every coloring $r:[\omega]^2 \to \{1, 2\}$ there exists an infinite set $A \subseteq \omega$ such that $r \mid [A]^2$ is constant. (0.0)

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For every coloring $r:[\omega]^4 \to \{1,2\}$ there exists an infinite set $B \subseteq \omega$ such that $r \mid [B]^4$ is constant. (0.1)

Is there any reason to believe that (0.1) is 'harder' or 'stronger' than (0.0)? In [7] we defined 'Ramsey numbers' corresponding to these statements and showed that the 'Ramsey number' of (0.0) is smaller than the 'Ramsey number' of (0.1). But what are the 'Ramsey numbers'? To answer this question we first formulate our definition of the notion of a Ramsey number.

Let U be an infinite set. By $U^{<\omega}$ we denote the set of all non-empty finite sequences of elements of U. Let $T \subseteq U^{<\omega}$. We define the *type* of T to be the least ordinal γ such that there exists a mapping $f: T \to \gamma$ with the property that if a, $b \in T$ and a is a strict initial segment of b then $f(a) > f(b)$. The type is undefined if no such ordinal γ exists.

Let $r:[U]^n \to \{1, \ldots, k\}$ be a coloring, and let $l_1, \ldots, l_k$ be natural numbers. We say that a sequence $a \in U^{<\omega}$ is i-monochromatic if $r(M) = i$ for every n-element set M consisting of terms of a. Let A_i be the set of all i-monochromatic sequences ($i = 1, \ldots, k$), and let $A = (A_1, \ldots, A_k)$. If $g = (g_1, \ldots, g_k)$ is a k-tuple of functions $g_i: U^{<\omega} \to \mathrm{Ord}$ ($i = 1, \ldots, k$) we say that $a \in U^{<\omega}$ is (A, g)-bad (written $a \in \mathrm{Bad}(A, g)$) if the following condition is satisfied.

If $b_1, b_2 \in A_i$ and b_1 is a strict initial segment of b_2, then $g_i(b_1) > g_i(b_2)$. (0.3)

Let $g^{l_i}: U^{<\omega} \to \{0, \ldots, l_i - 1\}$ be defined by

$$g^{l_i}(a) = \max(0, l_i - \text{length of } a),$$

and let $\bar{g} = (g^{l_1}, \ldots, g^{l_k})$. Notice that a is $(A, \bar{g})$-bad if and only if every i-monochromatic subsequence of a has length $\leqslant l_i$ for all $i = 1, \ldots, k$. Now it can be verified (see [7]) that the supremum of the types of all $\mathrm{Bad}(A, g)$ (taken over all colorings $r:[U]^n \to \{1, \ldots, k\}$ and all k-tuples of functions $g = (g_1, \ldots, g_k)$ with $g_i: U^{<\omega} \to \{0, \ldots, l_i - 1\}$) is $R(n; l_1, \ldots, l_k) - 1$, where $R(n; l_1, \ldots, l_k)$ stands for the usual Ramsey number. (It follows from Ramsey's theorem that the types are well defined.)

The generalization is now obvious. If $\gamma_1, \ldots, \gamma_k$ are ordinals, we define the *R-function* $\rho_n(\gamma_1, \ldots, \gamma_n)$ as the supremum of the types of $\mathrm{Bad}(A, g)$ over all colorings $r:[U]^n \to \{1, \ldots, k\}$ and all k-tuples of functions $g = (g_1, \ldots, g_k)$ with $g_i: U^{<\omega} \to \gamma_i$. So in particular, if $\gamma_1, \ldots, \gamma_k$ are all finite, then $\rho_n(\gamma_1, \ldots, \gamma_n) = R(n; \gamma_1, \ldots, \gamma_k) - 1$. The following is a corollary of one of the results of [7].

Theorem 0.4.

$$\left.\omega^{\omega^{\omega^{\cdot^{\cdot^{\omega^k}}}}}\right\}(n-2)\ \textit{times} \leqslant \rho_n\underbrace{(\omega, \ldots, \omega)}_{k\ \text{times}} \leqslant \left.\omega^{\omega^{\omega^{\cdot^{\cdot^{\omega^k}}}}}\right\}(n-1)\ \textit{times},$$

In the definition of the R-function we did not use the fact that A_i consisted of

i-monochromatic sequences. All that was needed was that:

(i) if $b \in A_i$ and a is an initial segment of b, then $a \in A_i$, and

(ii) the type of Bad(A, g) is well defined for all A and g.

Therefore we may define (and examine) the R-functions corresponding to other Ramsey type results for which (ii) is satisfied. This was done in [7] for the Erdős–Szekeres Theorem and its generalization and for the Canonical Ramsey Theorem of Erdős and Rado, and for well-partially-ordered sets. In this paper we investigate the Nash-Williams' Partition Theorem. The upper bound is reasonably easy to obtain, but it is the lower bound which makes the analysis of the Nash-Williams' theorem so complicated.

To understand the connection with logic let us consider the following concept. Let Q be a partially ordered set with a partial ordering $\leqslant$. A (finite or infinite) sequence $q_1, q_2, \ldots$ of elements of Q is called *good* if there are indices i, j such that $i < j$ and $q_i \leqslant q_j$, and is called *bad* otherwise. The set Q is called well-partially-ordered (wpo) if every bad sequence of elements of Q is finite. If Q is wpo we define the *type* of Q, denoted by γ_Q, to be the least ordinal γ for which there exists a mapping f from the set of all non-empty bad sequences of elements of Q into γ such that

$$f(q_1, \ldots, q_n) > f(q_1, \ldots, q_{n+1})$$

for every bad sequence $(q_1, \ldots, q_{n+1})$. It is worth noting that if we define B to be the set of all sequences $(q_1, q_2, \ldots, q_n) \in Q^{<\omega}$ with $q_1 \leqslant q_2 \leqslant \cdots \leqslant q_n$, and $\mathbf{0}$ to be the constant mapping which is zero everywhere then a sequence $(q_1, \ldots, q_n)$ is bad if and only if it is $((B), \mathbf{0})$-bad in the sense defined earlier. Therefore the types of well-partially-ordered sets are a special case of our more general concept of an R-function.

Harvey Friedman [2] discovered that by 'miniaturizing' the assertion "Q is wpo" for certain wpo sets Q one can obtain statements of finite mathematics unprovable in relatively strong fragments of second order arithmetic. Here is an example of such a miniaturization.

> [2] For any positive c, there exists a positive integer $n = n(c)$ such that the following holds. If $T_1, T_2, \ldots, T_n$ is a finite sequence of finite trees with $|V(T_i)| \leqslant c \cdot i$ for all $i \leqslant n$, then there exist indices i and j such that $i < j \leqslant n$ and T_i is homeomorphically embeddable into T_j. (0.5)

That (0.5) is true can be easily derived from a theorem of Kruskal [8] which states that the set of finite trees with the partial ordering "to be homeomorphically embeddable into" is well-partially-ordered (see e.g. [15]). What is not so easy is to establish the unprovability part. The way this is usually done is by:

(i) proving that (0.5) implies that a miniaturization of the statement that a specified ordinal γ is well ordered, and

(ii) applying a result of logic that the above statement is unprovable.

The combinatorial content is now extracted in the proof of (i) (see [15] for details). It turns out that there is a connection between the strength of (0.5) and the type of the underlying wpo set. Therefore the type of a wpo set is a combinatorial invariant which has something to say about metamathematics of the well-partially-ordered set.

Thus our second motivation was to define the R-functions in such a way that this connection with logic will be preserved. And indeed, for example, the Parris–Harrington principle [11] can be looked at as a finite miniaturization of Theorem 0.4. Since the lower bound in 0.4 tends to ε_0 as $n \to \infty$, this is in accordance with the results of [11], because ε_0 is the proof-theoretic ordinal of Peano arithmetic.

In this paper we show an analogous result for the Nash-Williams' Partition Theorem, which implies that here the 'critical' ordinal is Γ_0. The ordinal Γ_0 is an ordinal much bigger than ε_0, defined as follows. Let $\varphi_0(\beta) = \omega^\beta$ and for $\alpha > 0$ let

$$\varphi_\alpha(\beta) = \beta\text{th common fixed point of all } \varphi_{\alpha'}(\alpha' < \alpha).$$

Notice that $\varphi_1(0) = \varepsilon_0$. Now Γ_0 is the least ordinal with the property that if $\alpha, \beta < \Gamma_0$ then $\varphi_\alpha(\beta) < \Gamma_0$.

In [3] Friedman, McAloon and Simpson derived from Nash-Williams' theorem a statement of finite mathematics unprobable in a theory called ATR_0, which is much stronger than Peano arithmetic. The proof-theoretic ordinal of ATR_0 is Γ_0 and so the relation of the main theorem of this paper to [3] is the same as the relation of Theorem 0.4 to [11].

Finally, let us say explicitly that we do not derive any unprovability results. We merely concentrate on combinatorial computation of the 'R-functions.'

1. Introduction

Conventions and Notation 1.1. Let Ω be an infinite set. The symbols $\Omega^{<\omega}$, $[\Omega]^{<\omega}$, Ω^ω, Ω^n, $[\Omega]^n$ denote the sets of non-empty finite sequences in Ω, the set of non-empty finite subsets of Ω, the set of infinite sequences in Ω, the Cartesian product of n copies of Ω and the set of subsets of Ω of cardinality n, respectively. If $a \in \Omega^{<\omega}$ then $|a|$ is the length of a. For $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots) \in \Omega^{<\omega} \cup \Omega^\omega$ we write $a \subseteq b$ and say that a is a subsequence of b if there are $j_1 < j_2 < \ldots$ such that $(a_1, a_2, \ldots) = (b_{j_1}, b_{j_2}, \ldots)$ and $a < b$ if $a \neq b$ and there is an n such that $a = (a_1, \ldots, a_n) = (b_1, \ldots, b_n)$. We shall also write $a \leqslant b$ if $a = b$ or $a < b$. For $a \in \Omega^{<\omega}$ we put $\downarrow a = \{b \in \Omega^{<\omega} \mid b \subseteq a\}$. Put, also, $\bar{a} = \{a_1, \ldots, a_n\}$ for $a = (a_1, \ldots, a_n)$. For $T \subseteq \Omega^{<\omega}$, put $\downarrow T = \bigcup \{\downarrow a \mid a \in T\}$, $\bar{T} = \{\bar{a} \mid a \in T\}$. Further, it will be convenient to denote the ith element of an arbitrary sequence $a \in \Omega^{<\omega}$ by a_i. For $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_m) \in \Omega^{<\omega}$, $a \cdot b$ denotes the concatenation $(a_1, \ldots, a_n, b_1, \ldots, b_m)$. An element $x \in \Omega^{<\omega}$ is often identified with the one-element sequence (x). For a function $f : X \to Y$ and for $M \subseteq X$ the restriction of f to M is denoted by $f \mid M$. The image of a mapping f is denoted by $\operatorname{Im} f$. The *disjoint union* of sets X, Y is the set $X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$. As a rule, however, in $X \sqcup Y$, $X \times \{0\}$ is identified with X and $Y \times \{1\}$ is identified with Y. The class of all ordinals is denoted by Ord. For a set $X \subset \mathrm{Ord}$ we put

$$MX = \sup\{\alpha + 1 \mid \alpha \in X\}.$$

For ordinal numbers α, β we define their *natural sum* by $0 \oplus \alpha = \alpha \oplus 0 = \alpha$, and, for α, $\beta > 0$,

$$\alpha \oplus \beta = \min\{\kappa \in \mathrm{Ord} \mid (\forall \alpha' < \alpha)(\forall \beta' < \beta)\kappa > \alpha' \oplus \beta \quad \text{and} \quad \kappa > \alpha \oplus \beta'\}$$

for a set X, $|X|$ denotes the cardinality of X. We define $\alpha + \beta$ as the type of α followed by β and $\alpha\beta$ as the type of $\alpha \times \beta$ with the lexicographical ordering

$$(x, y) < (z, t) \quad \text{if } x < z \quad \text{or} \quad (x = z \text{ and } y < t).$$

Definition 1.2. A subset $S \subseteq [\Omega]^{<\omega}$ is called a *Sperner system* in Ω if

$$(\forall a, b \in S)(\alpha \subseteq b \rightarrow a = b).$$

We now introduce the Nash-Williams' partition theorem, which is the central object of our study.

Theorem 1.3. (The Nash-Williams' Partition Theorem). *For any Sperner system $S \subseteq [\Omega]^{<\omega}$ and any partition $r: S \rightarrow \{1, \ldots, k\}$ there exists an infinite subset $\Omega' \subseteq \Omega$ such that $r \mid S \cap [\Omega']^{<\omega}$ is constant* (see [10]).

1.4. The language of ordinal types in Ramsey theory (see [7])

Definition 1.4.1. A *tree* is a couple $(T, \leqslant)$ where T is a set and $\leqslant$ is a partial ordering on T such that for every $t \in T$ the set $\{t' \in T \mid t' \leqslant t\}$ is a finite linearly ordered set. A tree $(T, \leqslant)$ is said to be rayless if T contains no infinite subset linearly ordered by $\leqslant$. Note that $(\Omega^{<\omega}, \leqslant)$ is a tree. More generally, all subsets of $\Omega^{<\omega}$ will be regarded as trees with this ordering. A *character* on a tree $(T, \leqslant)$ is a function

$$\varphi: T \rightarrow \mathrm{Ord}$$

such that

$$(\forall x, y \in T)(x < y \rightarrow \varphi(x) > \varphi(y)).$$

A tree $(T, \leqslant)$ is rayless if and only if there is a character on $(T, \leqslant)$. In that case, we define the *ordinal type* γ_T by

$$\gamma_T = \min\{\gamma \mid \text{there is a character } \varphi: T \rightarrow \gamma\}.$$

The Main Definition 1.4.2. A *sheaf* (in Ω) is a subset $A \subseteq \Omega^{<\omega}$ such that ($a \in A$ and $b < a) \rightarrow b \in A$. A *k-sheaf* is a k-tuple of sheaves. By abuse of language, we identify 1-sheaves and sheaves. A *k-system* is a set of k-sheaves. A sheaf A is said to have the *Ramsey property* if

$$(\forall a \in \Omega^{\omega})(\exists b \in \Omega^{\omega})(b \subseteq a \text{ and } (\forall c < b)(c \in A)).$$

A k-sheaf $(A_1, \ldots, A_k)$ is said to have the Ramsey property if the sheaf $A_1 \cup \cdots \cup A_k$ does. A k-system $\mathcal{R}$ is said to have the *Ramsey property* if each $A \in \mathcal{R}$ does.

Let $A = (A_1, \ldots, A_k)$ be a k-sheaf. A $(\gamma_1, \ldots, \gamma_k)$*-testing* is a k-tuple $g = (g_1, \ldots, g_k)$ of functions $g_i : \Omega^{<\omega} \rightarrow \gamma_i$. A sequence $a \in \Omega^{<\omega}$ is called (A, g)-*bad* if each g_i is a character on $\downarrow a \cap A_i$. The subtree of $\Omega^{<\omega}$ of all (A, g)-bad sequences will be denoted by $\mathrm{Bad}(A, g)$.

The following is a result from [7].

Proposition 1.4.2.1. *A sheaf A in Ω has the Ramsey property if and only if* $\mathrm{Bad}(A, g)$ *is a rayless tree for all ordinals* $\gamma_1, \ldots, \gamma_k < |\Omega|^+$ *and every* $(\gamma_1, \ldots, \gamma_k)$*-testing g.*

If A has the Ramsey property we define the R-function $\varphi_A : \mathrm{Ord}^k \rightarrow \mathrm{Ord}$ by

$$\varphi_A(\gamma_1, \ldots, \gamma_k) = \sup\{\gamma_{\mathrm{Bad}(A,g)} \mid g \text{ is a } (\gamma_1, \ldots, \gamma_k)\text{-testing}\}.$$

Similarly, for a k-system $\mathcal{S}$ which has the Ramsey property we define

$$\varphi_{\mathcal{S}}(\gamma_1, \ldots, \gamma_k) = \sup\{\varphi_A(\gamma_1, \ldots, \gamma_k) \mid A \in \mathcal{S}\}.$$

1.5 Definition of the ψ-functions

Let $k \geq 1$ be an integer, S be a Sperner system and $r : S \rightarrow \{1, \ldots, k\}$ be a partition. Define a k-sheaf $A^{S,r} = (A_1^{S,r}, \ldots, A_k^{S,r})$ by putting

$$A_i^{S,r} = \{a \in \Omega^{<\omega} \mid \mathrm{Im}(r \mid (\overline{\downarrow a} \cap S)) = \{i\}\}.$$

The *Nash-Williams k-system* $\mathcal{R}_k$ is defined by

$$\mathcal{R}_k = \{A^{S,r} \mid S \text{ is a Sperner system and } r : S \rightarrow \{1, \ldots, k\}\}.$$

It follows from Theorem 1.3 that $\mathcal{R}_k$ has the Ramsey property. We put

$$\psi(\gamma_1, \ldots, \gamma_k) = \varphi_{\mathcal{R}_k}(\gamma_1, \ldots, \gamma_k).$$

Notation 1.6. Define functions:

$\varphi_0(\beta) = \omega^{\beta}$,

$\varphi_{\alpha}(\beta) =$ the βth common fixed–point of all $\varphi_{\alpha'}$, $\alpha' < \alpha$, for $\alpha > 0$,

$g(\alpha) = \sup\{\varphi_{\gamma}(\beta) \mid \gamma \geq 2 \text{ and } (\beta + 1)\omega^{2^{\gamma\omega}} \geq \alpha\}$.

The Main Theorem 1.7. *Let $k \geq 2$ be an integer, let $\gamma_1, \ldots, \gamma_k$ be ordinals with $g(0) \leq \gamma_i < |\Omega|^+$ $(i = 1, \ldots, k)$, and let $\alpha = \min\{\gamma_1, \ldots, \gamma_k\}$. Then*

$$g(\alpha) \leq \psi(\gamma_1, \ldots, \gamma_k) \leq \varphi_\alpha(\gamma_1 \oplus \cdots \oplus \gamma_k).$$

Proof. Follows from Theorem 2.4 and Theorem 3.16 below. □

2. Prelude—the upper bound

2.1. Let S be a Sperner system and let $T \subseteq \Omega^{<\omega}$ be a rayless sheaf with $\downarrow T = T$. We define

$$C_T(S) = \{a \in T \mid \bar{a} \notin S \text{ and there is a } b \in \Omega^{<\omega} \text{ such that } \overline{a \cdot b} \in S \text{ and } a \cdot b \in T\}$$

and $c_T(S) = \gamma_{C_T(S)}$.
We remark that

$$\overline{C_T(S)} \cap S = \emptyset. \tag{2.1.1}$$

Let, for $\alpha \in \text{Ord}$,

$$\psi_\alpha(\gamma_1, \ldots, \gamma_k) = \sup\{\gamma_T \mid T \text{ is rayless}, \downarrow T = T,\ c_T(S) \leq \alpha \text{ and there exists } (A_1, \ldots, A_k) \in R_k \text{ such that } \gamma_{T \cap A_i} \leq \gamma_i\ (i = 1, \ldots, k)\}.$$

Lemma 2.2. *Let $k \geq 1$ be an integer, let $\gamma_1, \ldots, \gamma_k$ be ordinals and let $\alpha = \min\{\gamma_1, \ldots, \gamma_k\}$. Then $\psi(\gamma_1, \ldots, \gamma_k) = \psi_\alpha(\gamma_1, \ldots, \gamma_k)$.*

Proof. We first prove '$\geq$'. Let T be a rayless sheaf with $\downarrow T = T$ and such that $c_T(S) \leq \alpha$ and $\gamma_{T \cap A_i} \leq \gamma_i$ $(i = 1, \ldots, k)$ for some $A = (A_1, \ldots, A_k) \in \mathcal{R}_k$. For $i = 1, \ldots, k$, let $g_i : T \cap A_i \to \gamma_i$ be characters and let $g = (g_1, \ldots, g_k)$. Then $T \subseteq \text{Bad}(A, g)$, and hence $\gamma_T \leq \psi(\gamma_1, \ldots, \gamma_k)$.

To prove '$\leq$', let g be a $(\gamma_1, \ldots, \gamma_k)$-testing and let $A = (A_1, \ldots, A_k) \in \mathcal{R}_k$. By (2.1.1), $C_{\text{Bad}(A,g)}(S) \subset \text{Bad}(A, g) \cap A_i$ for every $i = 1, \ldots, k$, and hence

$$c_{\text{Bad}(A,g)}(S) \leq \gamma_{\text{Bad}(A,g) \cap A_i} \leq \gamma_i\ (i = 1, \ldots, k).$$

Therefore,

$$\gamma_{\text{Bad}(A,g)} \leq \psi_a(\gamma_1, \ldots, \gamma_k). \qquad \square$$

Theorem 2.3. *Let $k \geq 1$ be an integer and let $\gamma_1, \ldots, \gamma_k$ be ordinals. Then*:

(a) $\psi_0(\gamma_1, \ldots, \gamma_k) \leq \gamma_1 \oplus \cdots \oplus \gamma_k$.

(b) *For $\alpha > 0$ we have*

$$\psi_\alpha(\gamma_1, \ldots, \gamma_k) \leq M\{\psi_{\alpha'}(\psi_\alpha(\gamma_1', \ldots, \gamma_k), \ldots, \psi_\alpha(\gamma_1, \ldots, \gamma_k')) \mid \alpha' < \alpha,\ \gamma_i' < \gamma_i\}.$$

Proof. To prove (a), observe that $c_T(S)=0$ implies

$$S \supseteq \{\{x\} \mid (x) \in T\}.$$

Now (a) follows from Theorem 6.1 in [7].

We prove (b) by induction. Let α and $\gamma_1, \ldots, \gamma_k$ be ordinals and assume that (b) holds for all α' and $\gamma_1', \ldots, \gamma_k'$ such that either $\alpha' < \alpha$, or $\alpha' = \alpha$ and $\gamma_1' \leqslant \gamma_1, \ldots, \gamma_k' \leqslant \gamma_k$ and at least one of these inequalities is strict. Let T be a rayless sheaf with $\downarrow T = T$, and let $A^{S,r} = (A_1^{S,r}, \ldots, A_k^{S,r}) \in \mathscr{R}_k$ be such that $\gamma_{T \cap A_i^{S,r}} \leqslant \gamma_i$ $(i = 1, \ldots, k)$ and $c_T(S) \leqslant \alpha$. We must estimate γ_T. Let $x \in \Omega$ and let $T_x = \{a \in \Omega^{<\omega} \mid (x) \cdot a \in T\}$. Then T_x is a rayless sheaf with $\downarrow T_x = T_x$ and such that

$$T_x \subseteq T. \tag{2.3.1}$$

Let

$$S_x = \{m \in [\Omega]^{<\omega} \mid x \notin m \text{ and } m \cup \{x\} \in S\}$$

and for $m \in S_x$, let $r_x(m) = r(m \cup \{x\})$. It is easily seen that

$$c_{T_x}(S_x) < c_T(S). \tag{2.3.2}$$

By (2.3.1),

$$\{xa \mid a \in T_x \cap A_i^{S_x,r_x} \cap A_i^{S,r}\} \subseteq T \cap A_i^{S,r} \quad \text{for } i = 1, \ldots, k,$$
$$T_x \cap A_i^{S_x,r_x} \cap A_j^{S,r} \subseteq T \cap A_j^{S,r} \quad \text{for } i \neq j,\ i, j = 1, \ldots, k.$$

Thus,

$$\gamma_{T_x \cap A_i^{S_x,r_x} \cap A_i^{S,r}} < \gamma_i \quad \text{for } i = 1, \ldots, k,$$
$$\gamma_{T_x \cap A_i^{S_x,r_x} \cap A_j^{S,r}} \leqslant \gamma_j \quad \text{for } i, j = 1, \ldots, k.$$

Since, obviously, $c_{T_x \cap A_i^{S_x,r_x}}(S) \leqslant c_T(S) \leqslant \alpha$, it follows from the induction hypothesis that

$$\gamma_{T_x \cap A_i^{S_x,r_x}} \leqslant \psi_\alpha(\gamma_1, \ldots, \gamma_i', \ldots, \gamma_k)$$

for some $\gamma_i' < \gamma_i$. By this, (2.3.2) and the induction hypothesis,

$$\gamma_{T_x} \leqslant \psi_{\alpha'}(\psi_\alpha(\gamma_1', \gamma_2, \ldots, \gamma_k), \ldots, \psi_\alpha(\gamma_1, \ldots, \gamma_{k-1}, \gamma_k')),$$

where $\alpha' = c_{T_x}(S_x)$. Since x was arbitrary, the result follows. □

Theorem 2.4. *Let $k \geqslant 1$ be an integer, let $\gamma_1, \ldots, \gamma_k < |\Omega|^+$ be ordinals and let $\alpha = \min\{\gamma_1, \ldots, \gamma_k\}$. Then we have*

$$\psi(\gamma_1, \ldots, \gamma_k) \leqslant \varphi_\alpha(\gamma_1 \oplus \cdots \oplus \gamma_k).$$

Proof. Observe that

$$\varphi_0(\beta) \geqslant \beta, \qquad \varphi_\alpha(\beta) \geqslant M\{\varphi_{\alpha'}(\varphi_\alpha(\beta')) \mid \alpha' > \alpha,\ \beta' < \beta\}.$$

Our result now follows from 2.2 and 2.3. □

3. Fugue

3.0. Outline of the proof

Let *tp* X denote the ordinal type of a well-ordered set X. We are going to construct, for all ordinals α, γ, well-ordered sets $T_\gamma(\alpha)$ such that

$$\text{tp } T_0(\alpha) \geq 2^\alpha \quad \text{for all } \alpha \geq 0, \tag{3.0.1}$$

$$\text{tp } T_\delta(\text{tp } T_\gamma(\alpha)) \leq \text{tp } T_\gamma(\alpha) \quad \text{for all } \alpha \geq 0 \text{ and all } \gamma,\ \delta \geq 0 \text{ with } \gamma \geq \delta + \omega, \tag{3.0.2}$$

$$\text{tp } T_\gamma(\alpha) \leq \text{tp } T_\gamma(\beta) \quad \text{for all } \gamma \geq 0 \text{ and } 0 \leq \alpha \leq \beta. \tag{3.0.3}$$

We are going to construct a Sperner system S_γ in $T_\gamma(\alpha)$, a coloring $r_\gamma : S_\gamma \to 0, 1\}$ and an $((\alpha+1)\cdot\omega^{2^\gamma}, (\alpha+1)\cdot\omega^{2^\gamma})$-testing g_γ such that

$$\text{Dec } T_\gamma(\alpha) \subseteq \text{Bad}(A^{S_\gamma, r_\gamma}, g_\gamma) \cup \{\emptyset\}. \tag{3.0.4}$$

(Here Dec X denotes the set of all decreasing sequences in X.)

We claim that

$$\varphi_\gamma(\alpha) \leq \text{tp } T_{\gamma\cdot\omega}(\alpha) \quad \text{for all } \alpha \geq 0 \text{ and all } \gamma \geq 2. \tag{3.0.5}$$

For let, for $\gamma \in \text{Ord}$, $g_\gamma : \text{Ord} \to \text{Ord}$ be the function defined by $g_\gamma(\alpha) = \text{tp } T_{\gamma\cdot\omega}(\alpha)$. From (3.0.1)–(3.0.3) we deduce

$$g_0(\alpha) \geq 2^\alpha \quad \text{for all } \alpha \geq 0, \tag{3.0.6}$$

$$g_\gamma(\alpha) \text{ is a fixed point of } g_\delta \quad \text{for every } \delta < \gamma \text{ and every } \alpha \geq 0, \tag{3.0.7}$$

$$g_\gamma(\beta) < g_\gamma(\alpha) \quad \text{for all } \gamma \geq 0 \text{ and all } 0 \leq \beta < \alpha. \tag{3.0.8}$$

Hence, $g_\gamma(\alpha) \geq \varphi'_\gamma(\alpha)$ for all α, $\gamma \geq 0$ where $\varphi'_0(\alpha) = 2^\alpha$, $\varphi'_\gamma(\alpha)$ is the αth common fixed point of $\varphi_{\gamma'}$, $\gamma' < \gamma$ for $\gamma > 0$.
But $\varphi'_\gamma(\alpha) = \varphi_\gamma(\alpha)$ for all $\alpha \geq 0$ and $\gamma \geq 2$. (3.0.5) follows.

Now condition (3.0.4) implies

$$\gamma_{\text{Dec}\,T_\gamma(\alpha)} \leq \psi((\alpha+1)\cdot\omega^{2^\gamma}, (\alpha+1)\cdot\omega^{2^\gamma}). \tag{3.0.9}$$

(Recall that $T_\gamma(\alpha)$ is regarded as a tree under the relation $\leq$.) Assume, without loss of generality,

$$T_\gamma(\alpha) \subseteq \Omega. \tag{3.0.10}$$

Since obviously $\text{tp } T_\gamma(\alpha) = \gamma_{\text{Dec}\,T_\gamma(\alpha)\setminus\{0\}}$, (3.0.5) and (3.0.9) imply the desired lower bound.

3.1. The language of category theory

The concepts used here can be found in any elementary text-book of category theory, e.g. [9]. We shall assume that the reader knows the concepts of a *category*, *functor* and *natural transformation*. For a category C, let Obj C denote the class of objects of C and let Mor C denote the class of morphisms of C. For x, $y \in \text{Obj } C$, $C(x, y)$ denotes the set of morphisms from x to y in C. In the sequel, we shall mostly use the category Set of sets and mappings and the category $\mathcal{W}$ of linearly ordered sets and strictly monotone mappings.

Let Δ be a partially ordered set (regarded as a category in the usual way, i.e. objects are elements of Δ and, for $x, y \in \Delta$, $\Delta(x, y) = \emptyset$ if $x \nleq y$ and $|\Delta(x, y)| = 1$ otherwise), let C be an arbitrary category and let

$$F : \Delta \to C$$

be a functor. A *colimit* of F is an object $x \in \text{Obj } C$ together with a bunch of morphisms

$$(\varphi_z \in C(F(z), x) \mid z \in \Delta)$$

such that, for $\alpha \in \Delta(y, z)$,

$$\varphi_z \circ F(\alpha) = \varphi_y$$

and whenever there is an object $t \in \text{Obj } C$ together with a bunch of morphisms

$$(\psi_z \in C(F(z), t) \mid z \in \Delta)$$

such that, for $\alpha \in \Delta(y, z)$,

$$\psi_z \circ F(\alpha) = \psi_t$$

then there is a unique $g \in C(x, t)$ such that for each $z \in \text{Obj } \Delta$

$$\psi_z = g \circ \varphi_z .$$

A partially ordered set Δ is called *directed* if

$$(\forall x, y \in \text{Obj } \Delta)(\exists z \in \text{Obj } \Delta(x \leq z \text{ and } y \leq z).$$

A *directed colimit* is a colimit of a functor

$$F : \Delta \to C$$

where Δ is directed. We say that C *has directed colimits* if for each functor $F : \Delta \to C$ where Δ is directed there is a colimit.

Fact 3.1.1. *The categories $\mathcal{W}$, Set have directed colimits.*

Let $G : C_1 \to C_2$ be a functor. We say that G *preserves directed colimits* if for each functor $F : \Delta \to C_1$ where Δ is directed and for each colimit $(x, (\varphi_z \mid z \in \Delta))$ of $F(G(x), (G(\varphi_z) \mid z \in \Delta))$ is a colimit of GF.

Fact 3.1.2. *The forgetful functor $W : \mathcal{W} \to$ Set preserves directed colimits.*

Compositions 3.1.3. In this paper we shall generally denote compositions of morphisms by '$\circ$' and compositions of functors by '$\cdot$'. Let $S: C \to D$, $T: C \to D$ be functors. A *natural transformation* $\varphi: S \to T$ is a system of morphisms $(\varphi_x: Sx \to Tx \mid x \in \operatorname{Obj} C)$ such that for $f \in C(x, y)$ the following diagram commutes:

$$\begin{array}{ccc} Sx & \xrightarrow{Sf} & Sy \\ {\scriptstyle\varphi_x}\downarrow & & \downarrow{\scriptstyle\varphi_y} \\ Tx & \xrightarrow{Tf} & Ty. \end{array}$$

It is common to write φ instead of φ_x, φ_y if there is no danger of confusion. In natural transformations, we have two kinds of compositions: Let first $R: C \to D$, $S: C \to D$, $T: C \to D$ be functors and let $\sigma: R \to S$, $\tau: S \to T$ be natural transformations. The *vertical composition* of σ and τ is the natural transformation $\tau \circ \sigma: R \to T$ given by $(\tau \circ \sigma)_x = \tau_x \circ \sigma_x$.

Let, on the other hand, $S: C \to D$, $T: C \to D$, $S': D \to E$, $T': D \to E$ be functors and let $\tau: S \to T$, $\tau': S' \to T'$ be natural transformations. The *horizontal composition* of τ and τ' is the natural transformation $\tau' \cdot \tau: (S' \cdot S) \to (T' \cdot T)$ given by $(\tau' \cdot \tau)_x = \tau'_{Tx} \circ S'(\tau_x)$. Note that, by naturality, we also have $(\tau' \cdot \tau)_x = T'(\tau_x) \circ \tau'_{Sx}$ (Proof: For any z, $t \in \operatorname{Obj} D$ and any $g \in D(z, t)$, $\tau'_t \circ S'(g) = T'(g) \circ \tau'_z$. Put $z = Sx$, $t = Tx$, $g = \tau_x$.)

Put, in particular, $T' \cdot \tau = (\mathrm{Id}_{T'}) \cdot \tau$, $\tau' \cdot T = \tau' \cdot (\mathrm{Id}_T)$. We see easily that $(T' \cdot \tau)_x = T'(\tau_x)$, $(\tau' \cdot T)_x = \tau'_{Tx}$. Thus, in general, $\tau' \cdot \tau = (T' \cdot \tau) \circ (\tau' \cdot S) = (\tau' \cdot T) \circ (S' \cdot \tau)$.

We would like to warn the reader that some authors use the symbols $\circ, \cdot$ in different meanings.

3.2. The functors $T_\beta: \mathcal{W} \to \mathcal{W}$ and natural transformations $\kappa^\beta: \mathrm{Id} \to T_\beta$, $\kappa^\beta_\alpha: T_\alpha \to T_\beta$, $\alpha < \beta \in \mathrm{Ord}$

3.2.1. We first define T_0 on objects. Let $\alpha \in \operatorname{Obj} \mathcal{W}$. We define $T_0(\alpha)$ as the set $[\alpha]^{<\omega} \sqcup \alpha$ together with the ordering $<$ given by the following conditions:

$$(\forall x \in \alpha)(\forall y \in T_0(\alpha))(x < \{x\} \text{ and } (y \leq x \text{ or } y \geq \{x\})) \tag{3.2.1.1}$$

$$(\forall x, y \in T_0(\alpha) \backslash \alpha)(x < y \text{ iff } (y \backslash x \neq \emptyset \text{ and } (\forall z \in x \backslash y) z < \max(y \backslash x))). \tag{3.2.1.2}$$

It is easily seen that conditions (3.2.1.1) and (3.2.1.2) indeed specify a unique linear ordering on $T_0(\alpha)$, whose restriction to α coincides with the original ordering on α. For α, $\beta \in \operatorname{Obj} \mathcal{W}$ and $\varphi \in \mathcal{W}(\alpha, \beta)$ define $T_0(\varphi)$ by

$$T_0(\varphi)(x) = \varphi(x) \quad \text{for } x \in \alpha,$$

$$T_0(\varphi)(m) = \{\varphi(x) \mid x \in m\} \quad \text{for } m \in [\alpha]^{<\omega}.$$

(It is easy to check that $T_0(\varphi)$ is strictly increasing if φ is.) Now define κ^0, $\mathrm{Id} \to T_0$ by $(\kappa^0)_\alpha(x) = x$ (for $x \in \alpha$).

3.2.2. Now assume T_γ already defined. Put $T_{\gamma+1} = (T_\gamma)^2$ (T^2 stands for $T \cdot T$). Put, further, for $\beta < \gamma$,

$$\kappa^{\gamma+1} = \kappa^\gamma \cdot \kappa^\gamma = (\kappa^\gamma \cdot T_\gamma) \circ \kappa^\gamma,$$
$$\kappa_\gamma^{\gamma+1} = T_\gamma \cdot \kappa^\gamma,$$
$$\kappa_\beta^{\gamma+1} = \kappa_\gamma^{\gamma+1} \circ \kappa_\beta^\gamma.$$

Note that

$$\kappa^{\gamma+1} = \kappa_\gamma^{\gamma+1} \circ \kappa^\gamma. \tag{3.2.2.1}$$

3.2.3. Now assume $T_{\gamma'}$, already defined for all $\gamma' < \gamma$ where γ is a limit ordinal. We define T_γ as the colimit of the (commutative) diagram of functors

$$T_0 \xrightarrow{\kappa_0^1} T_1 \longrightarrow \cdots \longrightarrow T_{\gamma'} \longrightarrow \cdots \longrightarrow T_{\gamma''} \longrightarrow \cdots, \qquad T_{\gamma'} \xrightarrow{\kappa_{\gamma'}^{\gamma''}} T_{\gamma''}$$

Let, also, κ_β^γ be the colimit mappings arising from the system $(\kappa_\beta^{\gamma'} \mid \gamma' < \gamma)$ for $\beta < \gamma$. Now define

$$\kappa^\gamma = \kappa_\beta^\gamma \circ \kappa^\beta.$$

Observe that this definition does not depend on the choice of $\beta < \gamma$.

3.3. The 'square' transformation $\iota_\gamma^\beta : T_\gamma \to T_\beta T_\gamma$, $\beta, \gamma \in$ Ord

We shall define natural transformations $\iota_\gamma^\beta : T_\gamma \to T_\beta T_\gamma$ by transfinite induction on γ in the following way.

3.3.1. Put, for $\gamma \leqslant \beta$, $\iota_\gamma^\beta = \kappa_\gamma^\beta \cdot \kappa^\gamma$.

3.3.2. For $\gamma = \beta + 1$, put $\iota_{\beta+1}^\beta = T_\beta \cdot \kappa_\beta^{\beta+1}$. (We have $\iota_{\beta+1}^\beta : T_{\beta+1} = T_\beta \cdot T_\beta \to T_\beta \cdot T_\beta \cdot T_\beta = T_\beta T_{\beta+1}$).

3.3.3. Let ι_γ^β be defined for $\gamma > \beta$. Put $\iota_{\gamma+1}^\beta = \iota_\gamma^\beta \cdot T_\gamma$.

3.3.4. Now let $\iota_{\gamma'}^\beta$ be already defined for all $\gamma' < \gamma$ where γ is a limit ordinal. First note that, by the naturality of $\iota_{\gamma'}^\beta$, the following diagram commutes:

$$\begin{array}{ccc} T_{\gamma'} & \xrightarrow{T_{\gamma'} \cdot \kappa^\gamma} & T_{\gamma'} \cdot T_{\gamma'} \\ \downarrow{\scriptstyle \iota_{\gamma'}^\beta} & & \downarrow{\scriptstyle \iota_{\gamma'}^\beta \cdot T_{\gamma'}} \\ T_\beta T_{\gamma'} & \xrightarrow{T_\beta T_{\gamma'} \cdot \kappa^\gamma} & T_\beta T_{\gamma'} T_{\gamma'}. \end{array} \tag{3.3.4.1}$$

Changing the notation, we get the commutative diagram (for $\gamma' \geqslant \beta + 1$):

$$\begin{array}{ccc} T_{\gamma'} & \xrightarrow{\kappa_{\gamma'}^{\gamma'+1}} & T_{\gamma'+1} \\ \downarrow{\scriptstyle \iota_{\gamma'}^\beta} & & \downarrow{\scriptstyle \iota_{\gamma'+1}^\beta} \\ T_\beta T_{\gamma'} & \xrightarrow{T_\beta \kappa_{\gamma'}^{\gamma'+1}} & T_\beta T_{\gamma'+1}. \end{array} \tag{3.3.4.2}$$

This allows us to define $\iota^{\beta}_{\gamma}: T_{\gamma} \to T_{\beta} T_{\gamma}$ as the colimit mapping of a (by induction, commutative) diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & T_{\gamma'} & \xrightarrow{\kappa^{\gamma''}_{\gamma'}} & T_{\gamma''} & \longrightarrow & \cdots . \\ & & & \searrow \quad \swarrow & & & \\ & (T_{\beta} \cdot \kappa^{\gamma}_{\gamma'}) \circ \iota^{\beta}_{\gamma'} & & T_{\beta} T_{\gamma} & & (T_{\beta} \cdot \kappa^{\gamma}_{\gamma''}) \circ \iota^{\beta}_{\gamma''}. & \end{array}$$

The reader may feel that the restriction $\gamma' \geqslant \beta + 1$ in Diagram (3.3.4.2) violates the beauty. Indeed, it is not necessary. For $\gamma' = \beta$ we observe that $\iota^{\beta}_{\beta} = \kappa^{\beta+1}_{\beta} = T_{\beta}\kappa^{\beta}$, $\iota^{\beta+1}_{\beta} = T_{\beta}\kappa^{\beta+1}_{\beta} = T^{2}_{\beta}\kappa^{\beta}$. For $\gamma' < \beta$, we observe that

$$\begin{aligned} (T_{\beta}\kappa^{\gamma'+1}_{\gamma'}) \circ \iota^{\beta}_{\gamma'} &= (T_{\beta}\kappa^{\gamma'+1}_{\gamma'}) \circ (\kappa^{\beta}_{\gamma'} \cdot \kappa^{\gamma'}) = \kappa^{\beta}_{\gamma'} \cdot (\kappa^{\gamma'+1}_{\gamma'} \circ \kappa^{\gamma'}) \\ &= \kappa^{\beta}_{\gamma'} \cdot \kappa^{\gamma'+1} = (\kappa^{\beta}_{\gamma'+1} \cdot \kappa^{\gamma'+1}) \circ \kappa^{\gamma'+1}_{\gamma'} = \iota^{\beta}_{\gamma'+1} \circ \kappa^{\gamma'+1}_{\gamma'}. \end{aligned}$$

Summing up our results, we get the following commutative diagram valid generally for $\gamma' < \gamma$:

$$\begin{array}{ccc} T_{\gamma'} & \xrightarrow{\kappa^{\gamma}_{\gamma'}} & T_{\gamma} \\ \downarrow{\scriptstyle \iota^{\beta}_{\gamma'}} & & \downarrow{\scriptstyle \iota^{\beta}_{\gamma}} \\ T_{\beta} T_{\gamma'} & \xrightarrow[T_{\beta}\kappa^{\gamma}_{\gamma'}]{} & T_{\beta} T_{\gamma}. \end{array} \qquad (3.3.4.3)$$

3.4. The total-image transformation tim_{γ}

Denote by W the forgetful functor $W: \mathcal{W} \to \mathrm{Set}$ and by $K: \mathrm{Set} \to \mathrm{Set}$ the functor given by

$$K(m) = [m]^{<\omega},\ K(f) \text{ is the map induced by } f.$$

Note that there is a natural transformation

$$U: K^{2} \to K,\ U(m) = \bigcup m,$$

called the *union*. We also have a natural transformation

$$S: \mathrm{Id} \to K \text{ given by } S(x) = \{x\}$$

called the *singleton*. It holds that

$$U \circ (K \cdot S) = \mathrm{Id}_{K} = U \circ (S \cdot K).$$

We shall now define natural transformations

$$\mathrm{tim}_{\gamma}: W \cdot T_{\gamma} \to K \cdot W$$

to satisfy

$$\mathrm{tim}_{\gamma} \circ (W \cdot K^{\gamma}) = S \cdot W.$$

3.4.1. Define tim_0 by $(\mathrm{tim}_0)_{\alpha}(\beta) = \{\beta\}$ for $\beta \in \alpha$ and $(\mathrm{tim}_0)_{\alpha}(m) = m$ else.

3.4.2. Now assume tim_γ already defined. To define $\mathrm{tim}_{\gamma+1}$, put

$$(\mathrm{tim}_{\gamma+1}: W \cdot T_\gamma \cdot T_\gamma \to K \cdot W) = (U \cdot W) \circ (\mathrm{K} \cdot \mathrm{tim}_\gamma) \circ (\mathrm{tim}_\gamma \cdot T_\gamma).$$

3.4.3. Now assume that $\mathrm{tim}_{\gamma'}$ has been already defined for all $\gamma' < \gamma$ where γ is a limit ordinal. By the naturality of $\mathrm{tim}_{\gamma'}$, we have the following commutative diagrams (the second one arises from the first one by change of notation):

$$\begin{array}{ccc} WT_{\gamma'} & \xrightarrow{WT_{\gamma'}\cdot\kappa^{\gamma'}} & WT_{\gamma'}T_{\gamma'} \\ {\scriptstyle \mathrm{tim}_{\gamma'}}\downarrow & & \downarrow{\scriptstyle \mathrm{tim}_{\gamma'}\cdot T_{\gamma'}} \\ KW & \xrightarrow[KW\cdot\kappa^{\gamma'}]{} & KWT_{\gamma'} \end{array} \qquad \begin{array}{ccc} WT_{\gamma'} & \xrightarrow{W\kappa_{\gamma'}^{\gamma'+1}} & WT_{\gamma'+1} \\ {\scriptstyle \mathrm{tim}_{\gamma'}}\downarrow & & \downarrow{\scriptstyle \mathrm{tim}_{\gamma'}\cdot T_{\gamma'}} \\ KW & \xrightarrow{KW\cdot\kappa^{\gamma'}} & KWT_{\gamma'}. \end{array}$$

Now since

$$UW \circ (K \cdot \mathrm{tim}_{\gamma'}) \circ (KW_\kappa^{\gamma'}) = \mathrm{Id}_{KW},$$

we get the following commutative diagrams:

$$\begin{array}{ccccc} & & WT_{\gamma'} & \xrightarrow{W\kappa_{\gamma'}^{\gamma'+1}} & WT_{\gamma'+1} \\ & \swarrow{\scriptstyle \mathrm{tim}_{\gamma'}} & & & \downarrow{\scriptstyle \mathrm{tim}_{\gamma'}\cdot T_{\gamma'}} \\ KW & \xleftarrow{U\cdot W} & KKW & \xrightarrow{K\,\mathrm{tim}_{\gamma'}} & KWT_{\gamma'}, \end{array}$$

and consequently

$$\begin{array}{ccc} WT_{\gamma'} & \xrightarrow{W\kappa_{\gamma'}^{\gamma'+1}} & WT_{\gamma'+1} \\ & {\scriptstyle \mathrm{tim}_{\gamma'}}\searrow \quad \swarrow{\scriptstyle \mathrm{tim}_{\gamma'+1}} & \\ & KW. & \end{array}$$

This allows us to define tim_γ as the limit mapping of the (by induction, commutative) diagram

$$\begin{array}{cccccc} \longrightarrow & WT_{\gamma'} & \xrightarrow{W\kappa_{\gamma'}^{\gamma''}} & WT_{\gamma''} & \longrightarrow & \cdots \\ & & {\scriptstyle \mathrm{tim}_{\gamma'}}\searrow \quad \swarrow{\scriptstyle \mathrm{tim}_{\gamma''}} & & & \\ & & KW & & & \end{array}$$

(cf. Fact 3.1.2).

3.5. Sheaf functors

We first introduce a functor

$$\mathrm{Dec}: \mathcal{W} \to \mathrm{Set}$$

assigning to each linearly ordered set α the set of all finite strictly decreasing sequences in α (including the empty sequence) and to each morphism $\beta \to \alpha$ the appropriate induced mapping. Observe that Dec preserves directed colimits. For two functors F_1, $F_2: \mathcal{W} \to \text{Set}$ we shall write $F_1 \subseteq F_2$ if there is a natural transformation $\iota: F_1 \to F_2$ such that for each $\alpha \in \text{Obj}\ \mathcal{W}$, ι_α is an inclusion mapping. Now assume that we are given functors $F_1: \mathcal{W} \to \text{Set}$, $F_2: \mathcal{W} \to \mathcal{W}$ such that $F_1 \subseteq \text{Dec}\ F_2$ and, moreover, for each $\alpha \in \text{Obj}\ \mathcal{W}$, $F_1(\alpha) \subseteq \text{Dec}\ F_2(\alpha)$ is a sheaf in Obj $F_2(\alpha)$. Then we call F_1 an *F_2-shear functor* and write

$$F_1 \lhd \text{Dec}\ F_2.$$

Now let A and B be T_γ- sheaf functors. A natural transformation $\Delta: A \to B$ is called *(γ, λ)-regular*, $(\gamma, \lambda \in \text{Ord})$, if the following conditions hold:

If $\alpha \in \mathcal{W}$ and $a, b \in A(\alpha)$ with $a \leqslant b$ then $\Delta a \leqslant \Delta b$.
Also, $(\forall x \in \overline{\Delta a})(\exists y \in \bar{a}) x \in \text{tim}_\gamma(y)$. (3.5.1)

Put, for $\alpha \in \text{Obj}\ \mathcal{W}$ and $a \in A(\alpha)$,
$t(a) = \{a \cdot (x_0, \ldots, x_k) \mid k \geqslant 0 \text{ and } \Delta(a \cdot (x_0, \ldots, x_{k-1})) = \Delta a\}$. (3.5.2)

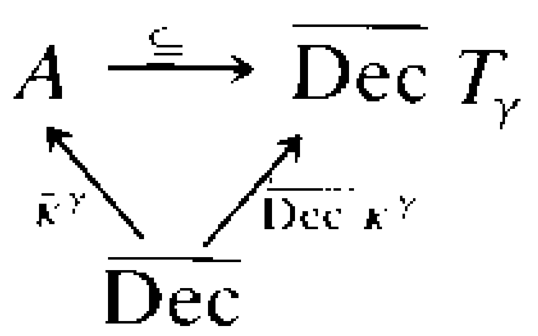
Then, for each $\alpha \in \text{Obj}\ \mathcal{W}$ and $a \in A(v)$,

$$\gamma_{t(a)} \leqslant \lambda.$$

(In particular, the left-hand side exists.)

Let, under the notation of (3.5.2), $a \cdot (y) \leqslant c \in t(a)$ and
$(\Delta a) \cdot (x) \leqslant \Delta c$. Then there is a $z \in \overline{a \cdot (y)}$ such that $x \in \text{tim}_\gamma z$. (3.5.3)

There is a natural transformation $\bar{\kappa}^\gamma: \text{Dec} \to A$ such that the following diagram commutes:



(3.5.4)

$$\Delta \circ \bar{\kappa}^\gamma = \text{Id}. \qquad (3.5.5)$$

3.6. Extension of a $(0, \lambda)$-regular transformation $\Delta_0: A_0 \to$ Dec

Let $A_0 \lhd \text{Dec}\ T_0$ and let

$$\Delta_0: A_0 \to \text{Dec}$$

be a $(0, \lambda)$-regular transformation. We are going to construct T_γ-sheaf functors $A_\gamma \lhd \text{Dec}\ T_\gamma$ and transformations

$$\Delta_\gamma: A_\gamma \to \text{Dec}$$

such that the following diagrams of functors and natural transformations may be completed to commute:

$$\begin{array}{ccccc} & & \mathrm{Dec} & & \\ & \Delta_{\gamma'} \nearrow & & \nwarrow \Delta_{\gamma''} & \\ A_{\gamma'} & & \dashrightarrow & & A_{\gamma''} \\ | & & & & | \\ \mathrm{Dec}\ T_{\gamma'} & & \xrightarrow{\mathrm{Dec}\ \kappa^{\gamma''}_{\gamma'}} & & \mathrm{Dec}\ T_{\gamma''}, \end{array} \tag{3.6.1}$$

where $\gamma' < \gamma'' \in \mathrm{Ord}$;

$$\begin{array}{ccccccc} \mathrm{Dec}\ T_{\gamma'} & \supseteq & A_{\gamma'} & & & & \\ \downarrow i^{\beta}_{\gamma'} & & \downarrow & \searrow & & \Delta_{\gamma'} & \\ \mathrm{Dec}\ T_{\beta}T_{\gamma'} & \supseteq & A_{\beta}T_{\gamma'} & & \searrow & & \searrow \\ & & \downarrow \Delta_{\beta}T_{\gamma'} & & & & \\ & & \mathrm{Dec}\ T_{\gamma'} & \supseteq & A_{\gamma'} & \xrightarrow[\Delta_{\gamma'}]{} & \mathrm{Dec} \end{array} \tag{3.6.2}$$

where $\beta,\ \gamma' \in \mathrm{Ord}$, and $i^{\beta}_{\gamma'} = \mathrm{Dec}\ \iota^{\beta}_{\gamma'}$;

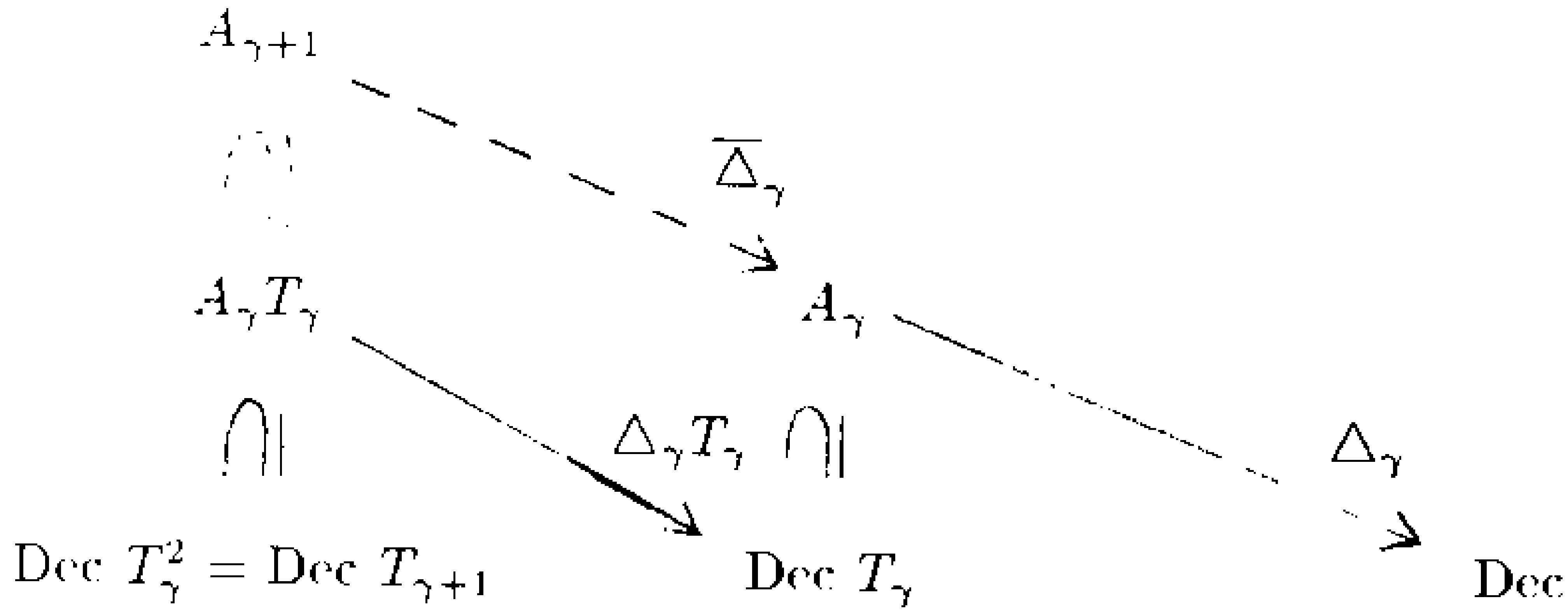
$$\begin{array}{ccccc} \mathrm{Dec} & & & & \\ \mathrm{Dec}\,\kappa^{\gamma'} \downarrow & \searrow & & \mathrm{Id} & \\ \mathrm{Dec}\ T_{\gamma'} & \supseteq & A_{\gamma'} & \xrightarrow[\Delta_{\gamma'}]{} & \mathrm{Dec} \end{array} \tag{3.6.3}$$

where $\gamma' \in \mathrm{Ord}$. Before preceding further, let us show that starting with a $(0, \lambda)$-regular transformation $\Delta_0 : A_0 \to \mathrm{Dec}$, we have (3.6.1)–(3.6.3) for $\gamma' = 0$.

First note that (3.6.1) is trivial. In (3.6.3), the missing map is $\bar{\kappa}^0$ (see (3.5.4) and (3.5.5)). In (3.6.2), the vertical missing map is $A_0\kappa^0$ and the diagonal one is $\bar{\kappa}^0 \circ \Delta_0$. The upper left square is commutative since A_0 is a sheaf functor (note that $\iota^0_0 = T_0\kappa^0$). The right triangle follows from the computation $\Delta_0 \circ (\bar{\kappa}^0 \circ \Delta_0) = (\Delta_0 \circ \bar{\kappa}^0) \circ \Delta_0 = \Delta_0$ (cf. (3.5.5)). The middle triangle is the circumference of the diagram, which is commutative by (3.5.4) and the naturality of Δ_0.

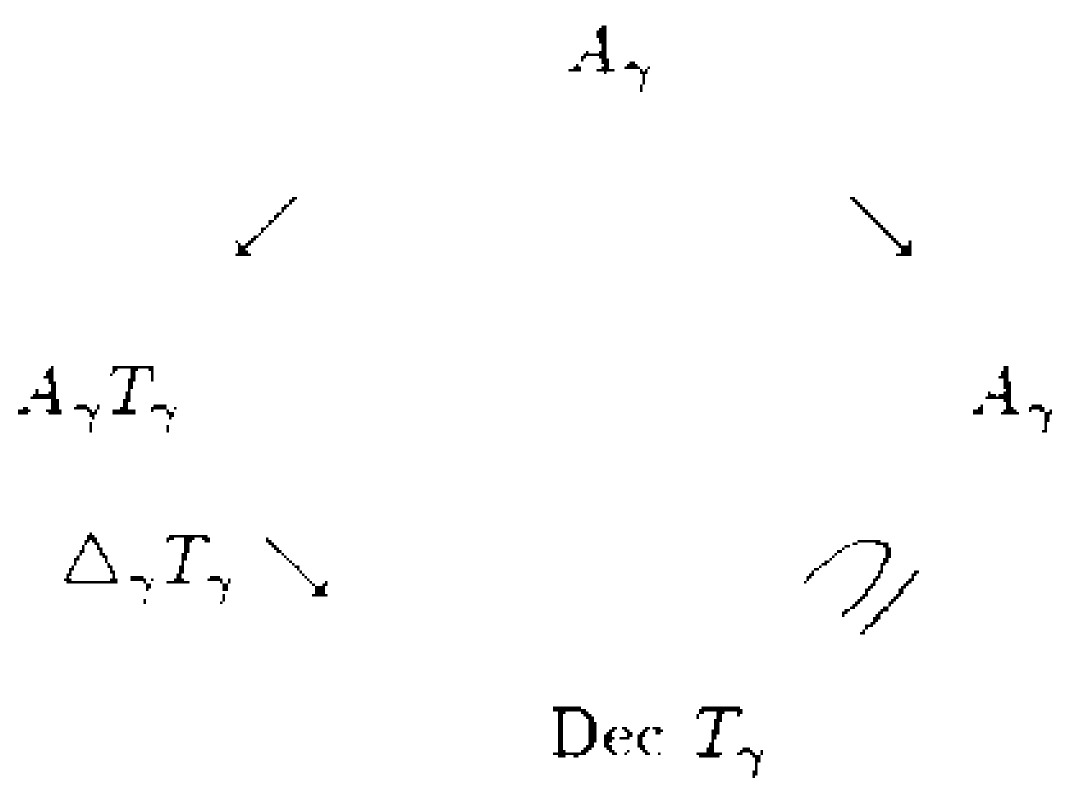
$$\begin{array}{ccccc} A_0 & \xrightarrow{\Delta_0} & \mathrm{Dec} & \xrightarrow{\bar{\kappa}^0} & A_0 \\ \downarrow A_0\kappa^0 & & \downarrow \mathrm{Dec}\ \kappa^0 & \swarrow & \\ A_0T_0 & \xrightarrow[\Delta_0 T_0]{} & \mathrm{Dec}\ T_0 & & \end{array}$$

3.7. The construction



(3.7.1)

Let Δ_γ, A_γ be already defined. Let $A_{\gamma+1}$ be the pullback of Diagram (3.7.1). Let, further, $\Delta_{\gamma+1} = \Delta_\gamma \circ \bar{\Delta}_\gamma$. Now (3.6.1) and (3.6.3) for $\gamma' = \gamma + 1$ can be easily obtained by diagram-chasing (3.7.1) and (3.6.1), (3.6.2), (3.6.3) for $\gamma' \leqslant \gamma$. For example, to get (3.6.1) with $\gamma' = \gamma$, $\gamma'' = \gamma + 1$, consider the part



of Diagram (3.6.2) for $\beta = \gamma' = \gamma$. By the induction hypothesis, this diagram commutes. By pulling-back, we obtain a mapping $A_\gamma \to A_{\gamma+1}$ which satisfies the requisite properties (by 3.3.2 and the remaining parts of Diagram (3.6.2)).

The crucial step in the induction is to prove (3.6.2) for $\gamma' = \gamma + 1$. We distinguish the following two cases.

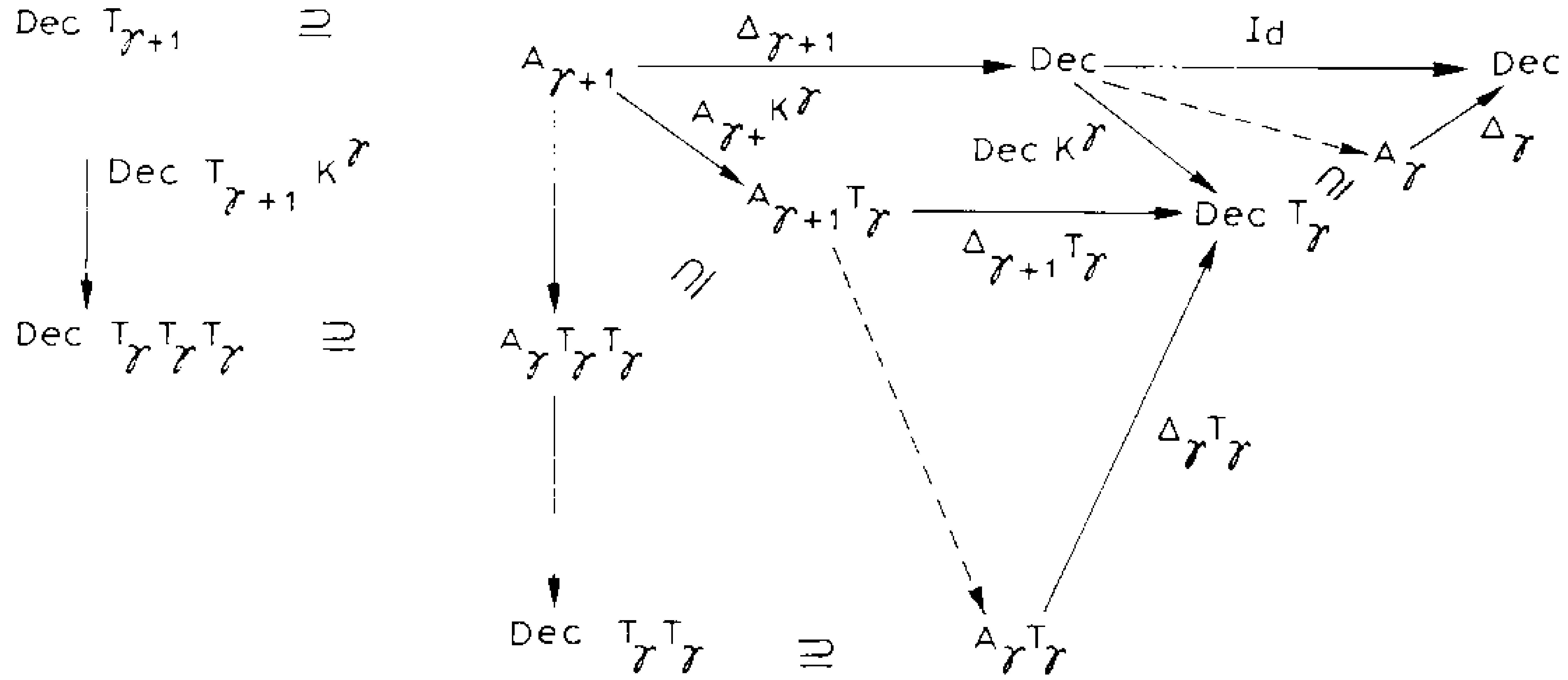
3.7.2 $\beta < \gamma$. By virtue of 3.3.3 and (3.6.2) with $\gamma' = \gamma$, we enjoy the following calculation (valid up to restrictions via inclusion mappings):

$$\Delta_{\gamma+1} \mid A_{\gamma+1} = \Delta_\gamma \circ \bar{\Delta}_\gamma = \Delta_\gamma \circ (\Delta_\gamma \cdot T_\gamma \mid A_{\gamma+1})$$

$$= \Delta_\gamma \circ ((\Delta_\gamma T_\gamma \circ (\Delta_\beta T_\gamma T_\gamma \circ i_\gamma^\beta T_\gamma) \mid A_\gamma T_\gamma) \mid A_{\gamma+1})$$

$$= \Delta_{\gamma+1} \circ \Delta_\beta T_{\gamma+1} \circ i_\gamma^\beta T_\gamma = \Delta_{\gamma+1} \circ \Delta_\beta T_{\gamma+1} \circ i_{\gamma+1}^\beta .$$

3.7.3. $\beta = \gamma$. We have a commutative diagram (see Fig. 1) which proves the



statement by 3.2.2 and 3.3.2. (The lower pentagon is the definition of $\Delta_{\gamma+1}$ composed with T_γ, the upper rhomboid is naturality of $\Delta_{\gamma+1}$ and the upper right triangle is (3.6.3) for $\gamma' = \gamma$.)

In view of (3.6.1), A_γ and Δ_γ for γ limits may be defined by passage to colimits. (3.6.1), (3.6.2) and (3.6.3) for $\gamma' = \gamma$ follow.

The Extension lemma 3.8. *Let* $A_0 \lhd \mathrm{Dec}\ T_0$ *and let* $\Delta_0 : A_0 \to \mathrm{Dec}$ *be a* $(0, \lambda)$ *regular transformation. Then* $A_\gamma \lhd \mathrm{Dec}\ T_\gamma$ *and the transformation* $\Delta_\gamma : A_\gamma \to \mathrm{Dec}$ *is* $(\gamma, \lambda^{2^\gamma})$*-regular.*

Proof. The fact that A_γ is a sheaf functor is proved by an easy induction together with (3.5.1). (3.5.4) and (3.5.5) follow from (3.6.3).

(3.5.3) follows from the definitions of Δ_γ and tim_γ by transfinite induction on γ. For example, the nonlimit step goes as follows: Assume the statement true for some $\gamma \in \mathrm{Ord}$. Now let

$$\alpha \in \mathrm{Obj}\ \mathcal{W}, \qquad a \in A_{\gamma+1}(\alpha), \qquad a \cdot (y) \leqslant c \in t(a), \qquad (\Delta_{\gamma+1} a) \cdot (x) \leqslant \Delta_{\gamma+1} c.$$

The last expression rewrites $(\Delta_\gamma(\Delta_\gamma a)) \cdot (x) \leqslant \Delta_\gamma(\Delta_\gamma c)$. By definition, we conclude that $\Delta_\gamma(c) \in t(\Delta_\gamma(a))$ and by (3.5.1), there is a $\bar{y} \in T_\gamma(\alpha)$ such that

$$\Delta_\gamma(a) \cdot (\bar{y}) \leqslant \Delta_\gamma(c). \tag{3.8.1}$$

(By abuse of notation, we usually drop the indices indicating at what object a natural transformation is applied. Here, Δ_γ stands for $(\Delta_\gamma)_{T_\gamma(\alpha)}$.)

This, by the induction hypothesis, implies that

$$(\exists \bar{z} \in \Delta_\gamma(a) \cdot (\bar{y}))x \in \mathrm{tim}_\gamma\ \bar{z}. \tag{3.8.2}$$

Also by the induction hypothesis and by (3.5.1), however, (3.8.1) implies that

$$(\exists \bar{\bar{z}} \in a \cdot (y))\bar{z} \in \mathrm{tim}_\gamma \bar{\bar{z}}. \tag{3.8.3}$$

By the definition of $\mathrm{tim}_{\gamma+1}$, we conclude that $x \in \mathrm{tim}_{\gamma+1}\ \bar{\bar{z}}$.

We now turn to the proof of (3.5.2). We shall proceed by transfinite induction on γ. For the nonlimit step, we need to realize that $\lambda^{2^{\gamma+1}} = (\lambda^{2^\gamma})^2$ and to perform an easy deliberation essentially analogous to the one demonstrated above in order to prove (3.5.3).

The essential difficulty is in the limit step. Let, thus, γ be a limit ordinal and let for all smaller values of γ (3.5.2) hold true. Our aim is to construct a character

$$\chi : t(a) \to \lambda^{2^\gamma}.$$

For $b \in A_\gamma(\alpha)$, choose a $\gamma(b) < \gamma$ in such a way that

$$b \in (\mathrm{Dec}\ \kappa^\gamma_{\gamma(b)})A_{\gamma(b)}(\alpha).$$

Now put

$$\chi(a \cdot (x_0, \ldots, x_k)) = \begin{cases} \lambda^{2^{\gamma(ax_0)}} & \text{when } k = 0, \\ \chi'(i_\gamma^{\gamma(ax_0)}(ax_0 \cdots x_k)) & \text{when } k > 0 \end{cases}$$

(recall that $i^\beta_\gamma = \mathrm{Dec}\ \iota^\beta_\gamma$) where

$$\chi' : t(i_\gamma^{\gamma(ax_0)}(a)) \to \lambda^{2^{\gamma(ax_0)}}$$

is the character corresponding to the (by the induction hypothesis, $(\gamma(ax_0), \lambda^{2^{\gamma(ax_0)}})$-regular) transformation $\Delta_{\gamma(ax_0)}$ applied at the object $T_\gamma(\alpha)$ of the category $\mathscr{W}$. (This means that we are considering the morphism $\Delta_{\gamma(ax_0)} : A_{\gamma(ax_0)}(T_\gamma(\alpha)) \to \mathrm{Dec}(T_\gamma(\alpha))$.)

It remains to show that χ is correctly defined and namely that

$$c \in t(a) \to i_\gamma^{\gamma(ax_0)}(c) \in t(b)$$

where $b = i_\gamma^{\gamma(ax_0)}(a)$. In the rest of the proof, we shall write γ' instead of $\gamma(ax_0)$. We will show that if, for some

$$i^{\gamma'}_\gamma(c) \in t(b) = i(i^{\gamma'}_\gamma(a)),$$

we have

$$\Delta_{\gamma'}(i^{\gamma'}_\gamma(a)) \neq \Delta_{\gamma'}(i^{\gamma'}_\gamma(c))$$

then we have

$$\Delta_\gamma(a) \neq \Delta_\gamma(c). \tag{3.8.4}$$

In effect, the choice of γ' implies

$$i^{\gamma'}_\gamma(a) \in \mathrm{Im}(\mathrm{Dec}\ T_{\gamma'}\kappa^\gamma : \mathrm{Dec}\ T_{\gamma'}(\alpha) \to \mathrm{Dec}\ T_{\gamma'}T_\gamma(\alpha)).$$

(We have $a \in (\mathrm{Dec}\ \kappa^\gamma_{\gamma'})A_{\gamma'}(\alpha)$ since $A_{\gamma'}$ is a sheaf functor; by Diagram (3.3.4.3), we compute

$$\begin{aligned} i^{\gamma'}_\gamma(a) \in \mathrm{Im}(i^{\gamma'}_\gamma \circ \mathrm{Dec}\ \kappa^\gamma_{\gamma'}) &= \mathrm{Im}(\mathrm{Dec}(\iota^{\gamma'}_\gamma \circ \kappa^\gamma_{\gamma'})) = \mathrm{Im}(\mathrm{Dec}(T_{\gamma'}\kappa^\gamma_{\gamma'} \circ \iota^{\gamma'}_{\gamma'})) \\ &= \mathrm{Im}(\mathrm{Dec}(T_{\gamma'}(\kappa^\gamma_{\gamma'} \circ \kappa^{\gamma'}))) = \mathrm{Im}\ \mathrm{Dec}(T_{\gamma'}k^\gamma).) \end{aligned}$$

Thus, by (3.6.2) and (3.6.3), we have

$$(\Delta_{\gamma'}T_\gamma) \circ i^{\gamma'}_\gamma(a) = (\mathrm{Dec}\ \kappa^\gamma) \circ \Delta_\gamma(a). \tag{3.8.5}$$

In effect, by (3.6.3) we have

$$\Delta_\gamma \circ (\Delta_{\gamma'} T_\gamma) \circ i_\gamma^{\gamma'}(a) = \Delta_\gamma(a). \tag{3.8.6}$$

Putting $i_\gamma^{\gamma'}(a) = \mathrm{Dec}\; T_{\gamma'} \kappa^\gamma(b)$, compute

$$\begin{aligned}(\mathrm{Dec}\;\kappa^\gamma) \circ \Delta_\gamma(a) &= (\mathrm{Dec}\;\kappa^\gamma) \circ \Delta_\gamma \circ (\Delta_{\gamma'} T_\gamma) \circ i_\gamma^{\gamma'}(a) \\ &= (\mathrm{Dec}\;\kappa^\gamma) \circ \Delta_\gamma \circ (\Delta_{\gamma'} \circ \mathrm{Dec}\; T_{\gamma'} \kappa^\gamma(b)) \quad \text{(by naturality)} \\ &= (\mathrm{Dec}\;\kappa^\gamma) \circ \Delta_\gamma \circ \mathrm{Dec}\;\kappa^\gamma \circ \Delta_{\gamma'}(b) \quad \text{(by (3.6.3))} \\ &= (\mathrm{Dec}\;\kappa^\gamma) \circ \Delta_{\gamma'}(b) \quad \text{(by naturality)} \\ &= \Delta_{\gamma'} \circ \mathrm{Dec}\; T_{\gamma'} \kappa^\gamma(b) = (\Delta_{\gamma'} T_\gamma) \circ i_\gamma^{\gamma'}(a).\end{aligned}$$

(Again, by abuse of notation, a general transformation is occasionally identified with its specification to an object.)

Now let $\Delta_{\gamma'}(i_\gamma^{\gamma'}(a))(x) \leqslant \Delta_{\gamma'}(i_\gamma^{\gamma'}(c))$. By (3.5.3), there is a $z \in a \cdot x_0$ with

$$x \in \mathrm{tim}_{\gamma' \iota_\gamma^{\gamma'}(z)}.$$

Again, by Diagram (3.3.4.3), we have $\iota_\gamma^{\gamma'}(z) \in \mathrm{Im}(T_{\gamma'} \cdot \kappa^\gamma)$. Thus, $\mathrm{tim}_{\gamma'} \iota_\gamma^{\gamma'}(z) \subseteq \mathrm{Im}\;\kappa^\gamma$ (by the naturality of $\mathrm{tim}_{\gamma'}$). Thus, for some $\bar{x} \in \alpha$,

$$x = \kappa^\gamma(\bar{x}). \tag{3.8.7}$$

Compute:

$$\begin{aligned}&\Delta_\gamma(c) \quad \text{(by (3.8.6))} \\ &\quad = \Delta_\gamma \circ (\Delta_{\gamma'} T_\gamma) \circ i_\gamma^{\gamma'}(c) \quad \text{(by (3.5.1))} \\ &\quad \geqslant \Delta_\gamma \circ (\Delta_{\gamma'}(i_\gamma^{\gamma'}(a)) \cdot (x)) \quad \text{(by (3.8.5))} \\ &\quad = \Delta_\gamma \circ (((\mathrm{Dec}\;\kappa^\gamma) \circ \Delta_\gamma(a)) \cdot (x)) \quad \text{(by (3.8.7))} \\ &\quad = \Delta_\gamma \circ (((\mathrm{Dec}\;\kappa^\gamma)\Delta_\gamma(a)) \cdot \kappa^\gamma(\bar{x})) \\ &\quad = \Delta_\gamma \circ \mathrm{Dec}\;\kappa^\gamma(\Delta_\gamma(a) \cdot \bar{x}) \quad \text{(by (3.6.3))} \\ &\quad = \Delta_\gamma(a) \cdot \bar{x}.\end{aligned}$$

This concludes the proof of (3.8.4). □

3.9. The extension construction: A combinatorial input

In the sequel, we will be dealing with one particular set of T_γ-sheaf functors $A_\gamma \lhd \mathrm{Dec}\; T_\gamma$ and transformations $\Delta_\gamma : A_\gamma \to \mathrm{Dec}$.

Let $\alpha \in \mathrm{Obj}\;\mathscr{W}$ and $a > b \in T_0(\alpha)$. Then $a \backslash b \neq \emptyset$. Define $\delta(a, b) \in \alpha$ as

$$\begin{array}{ll} \max(a\backslash b) & \text{if } a, b \in [\alpha]^{<\omega}, \\ a & \text{if } a \in \alpha, \\ \max a & \text{if } a \in [\alpha]^{<\omega} \text{ and } b \in \alpha. \end{array}$$

The following observations for $a > b > c \in T_0(\alpha)$ are in order:

$$\delta(a, b) = \delta(b, c) \rightarrow a \in [\alpha]^{<\omega} \quad \text{and} \quad b = \max a, \tag{3.9.1}$$

$$\delta(a, c) = \max(\delta(a, b), \delta(b, c)), \tag{3.9.2}$$

$$b \in \alpha \rightarrow \delta(a, b) \geq b, \tag{3.9.3}$$

$$a \in \alpha \rightarrow \delta(b, c) < a. \tag{3.9.4}$$

For $a = (a_0, \ldots, a_n) \in \operatorname{Dec} T_0(\alpha)$, put $\delta^i = \delta(a_i, a_{i+1})$ and

$$\bar{\delta}(a) = \begin{cases} (\delta^0, \ldots, \delta^{n-1}) & \text{if } a_n \in [\alpha]^{<\omega}, \\ (\delta^0, \ldots, \delta^{n-1}, a_n) & \text{if } a_n \in \alpha. \end{cases}$$

Put, for $\alpha \in \operatorname{Obj} \mathscr{W}$

$$A_0(\alpha) = \{a \in \operatorname{Dec} T_0(\alpha) \mid \bar{\delta}(a) \in \operatorname{Dec}(\alpha)\}, \qquad (\Delta_0)_\alpha = \bar{\delta} \mid A_0(\alpha).$$

To make A_0 a functor, we define for $\varphi \in \mathscr{W}(\alpha, \beta)$

$$A_0(\varphi)((a_0, \ldots, a_n)) = (\varphi(a_0), \ldots, \varphi(a_n)).$$

Fact 3.9.5. *We have* $A_0 \triangleleft \operatorname{Dec} T_0$. *Moreover,* $\Delta_0 : A_0 \rightarrow \operatorname{Dec}$ *is a* $(0, 2)$*-regular transformation.*

Proof. We easily verify (3.5.1) and the first sentence of the statement follows. Also (3.5.2) is obvious. To see (3.5.3), note that $\delta(a, b) \in \operatorname{tim}_0(a)$. Concerning (3.5.4) and (3.5.5), we observe that, for $a = (a_0, \ldots, a_n) \in \operatorname{Dec} \alpha \subseteq \operatorname{Dec} T_0(\alpha)$, $\Delta_0(a) = a$. $\square$

At this point, we consider the sheaf functors $A_\gamma \triangleleft \operatorname{Dec} T_\gamma$ and the $(\gamma, 2^{2^\gamma})$-regular transformations $\Delta_\gamma : A_\gamma \rightarrow \operatorname{Dec}$ defined in 3.6 and 3.7.

3.10. The Transformations $\bar{\delta}_\gamma$, $\gamma \in \mathbf{Ord}$

Let $\operatorname{Seq} : \mathscr{W} \rightarrow \operatorname{Set}$ be the functor assigning to $\alpha \in \operatorname{Obj}(\mathscr{W})$ the set of all finite sequences in α and to $\varphi \in \operatorname{Mor}(\mathscr{W})$ the corresponding induced mapping. From 3.9, we have a natural transformation

$$\bar{\delta} : \operatorname{Dec} T_0 \rightarrow \operatorname{Seq}.$$

We are now going to define natural transformations

$$\bar{\delta}_\gamma : \operatorname{Dec} T_\gamma \rightarrow \operatorname{Seq} T_\gamma$$

in the following manner.

3.10.1. $\bar{\delta}_0 = (\operatorname{Seq} \cdot \kappa^0) \circ \bar{\delta}$.

3.10.2. Let $\bar{\delta}_\gamma$ be already defined. Let $\alpha \in \operatorname{Obj} \mathscr{W}$ and let $a \in \operatorname{Dec} T_{\gamma+1}(\alpha)$.

We define $\bar{\delta}_{\gamma+1}(a)$ to be

$$(\bar{\delta}_\gamma \cdot T_\gamma)(a) \quad \text{if } (\bar{\delta}_\gamma)_{\mathrm{Dec}\, T_\gamma(\alpha)}(a) \notin \mathrm{Im}(\mathrm{Dec}(\kappa^\gamma)_{T_\gamma(\alpha)})$$

and

$$(\mathrm{Seq}\, \kappa_\gamma^{\gamma+1})\bar{\delta}_\gamma(b) \quad \text{if } (\bar{\delta}_\gamma)_{\mathrm{Dec}\, T_\gamma(\alpha)}(a) = \mathrm{Dec}(\kappa^\gamma)_{T_\gamma(\alpha)}(b).$$

3.10.3. As usual, for γ limit we would like to define $\bar{\delta}_\gamma$ by passage to colimits. As usual, this requires a commutative diagram of the form:

$$\begin{array}{ccc} \mathrm{Dec}\, T_\gamma & \xrightarrow{\bar{\delta}_\gamma} & \mathrm{Seq}\, T_\gamma \\ \downarrow{\scriptstyle \mathrm{Dec}\, \kappa_\gamma^{\gamma+1}} & & \downarrow{\scriptstyle \mathrm{Seq}\, \kappa_\gamma^{\gamma+1}} \\ \mathrm{Dec}\, T_{\gamma+1} & \xrightarrow{\bar{\delta}_{\gamma+1}} & \mathrm{Seq}\, T_{\gamma+1}. \end{array} \tag{3.10.3.1}$$

However, using naturality of $\bar{\delta}_\gamma$ and the fact that $\kappa_\gamma^{\gamma+1} = T_\gamma \kappa^\gamma$, we only get a commutative diagram:

$$\begin{array}{ccc} \mathrm{Dec}\, T_\gamma & \xrightarrow{\bar{\delta}_\gamma} & \mathrm{Seq}\, T_\gamma \\ \downarrow{\scriptstyle \overline{\mathrm{Dec}}\, \kappa_\gamma^{\gamma+1}} & & \downarrow{\scriptstyle \overline{\mathrm{Seq}}\, \kappa_\gamma^{\gamma+1}} \\ \mathrm{Dec}\, T_{\gamma+1} & \xrightarrow{\bar{\delta}_\gamma T_\gamma} & \mathrm{Seq}\, T_{\gamma+1}. \end{array} \tag{3.10.3.2}$$

To obtain (3.10.3.1), we need to show that, for $a \in \mathrm{Dec}\, T_\gamma(\alpha)$,

$$(\bar{\delta}_\gamma)_{T_\gamma(\alpha)} \circ \mathrm{Dec}\, \kappa_\gamma^{\gamma+1}(a) = \mathrm{Dec}(\kappa^\gamma)_{T_\gamma(\alpha)}(b) \to (\bar{\delta}_\gamma(b) = \bar{\delta}_\gamma(a)). \tag{3.10.3.3}$$

This cannot be reached without some deeper insight into what the induction has done as yet. Comparing 3.10.2 with 3.7, we obtain the following key results.

$$(\bar{\delta}_\gamma(a) \in \mathrm{Dec}\, T_\gamma(\alpha)) \to (a \in A_\gamma(\alpha)) \text{ and } \bar{\delta}_\gamma(a) = (\mathrm{Dec}\, \kappa^\gamma)\Delta_\gamma(a)), \tag{3.10.4}$$

$$\bar{\delta}_\gamma \circ (\mathrm{Dec}\, \kappa^\gamma) = (\mathrm{seq}\, \kappa^\gamma) \circ (\subseteq : \mathrm{Dec} \to \mathrm{Seq}), \tag{3.10.5}$$

(this is a consequence of (3.10.4) and (3.6.3)) and

$$a \subseteq b \to \bar{\delta}_\gamma(a) \subseteq \bar{\delta}_\gamma(b) \tag{3.10.6}$$

(this follows from (3.9.2) by transfinite induction).

Now to prove (3.10.3.3), note that by the very fact that $(\kappa^\gamma)_{T_\gamma(\alpha)}$ is a $\mathcal{W}$-morphism, we have

$$\mathrm{Im}(\mathrm{Dec}(\kappa^\gamma)_{T_\gamma}(a)) \subseteq \mathrm{Dec}\, T_{\gamma+1}.$$

Thus, the precise of (3.10.3.3) implies

$$(\bar{\delta}_\gamma)_{T_\gamma(\alpha)}(\mathrm{Dec}\, \kappa_\gamma^{\gamma+1}(a)) \in \mathrm{Dec}\, T_{\gamma+1}$$

and hence, by (3.10.4) and the naturality of Δ_γ,

$$\begin{aligned}(\bar{\delta}_\gamma)_{T_\gamma(\alpha)}(\mathrm{Dec}\,\kappa_\gamma^{\gamma+1}(a)) &= (\mathrm{Dec}(\kappa^\gamma)_{T_\gamma(\alpha)})(\Delta_\gamma)_{T_\gamma(\alpha)}(\mathrm{Dec}\,\kappa_\gamma^{\gamma+1}(a)) \\ &= (\mathrm{Dec}(\kappa^\gamma))_{T_\gamma(\alpha)})(\Delta_\gamma)_{T_\gamma(\alpha)}(\mathrm{Dec}\,T_\gamma\kappa^\gamma(a)) \qquad (3.10.7) \\ &= (\mathrm{Dec}(\kappa^\gamma)_{T_{\gamma(\alpha)}}(\mathrm{Dec}\,\kappa^\gamma)\Delta_\gamma(a) = (\mathrm{Dec}\,\kappa^{\gamma+1})\Delta_\gamma(a) \\ &= \mathrm{Dec}(\kappa^\gamma)_{T_\gamma(\alpha)}\circ(\mathrm{Dec}\,\kappa^\gamma)\Delta_\gamma(a).\end{aligned}$$

Since $\mathrm{Dec}(\kappa^\gamma)_\beta$ (as well as $(\kappa^\gamma)_\beta$) is obviously injective for any $\beta \in \mathrm{Obj}\,\mathscr{W}$ (by the mere fact that $(\kappa^\gamma)_\beta$ is a $\mathscr{W}$-morphism), we conclude that

$$b = (\mathrm{Dec}\,\kappa^\gamma)\Delta_\gamma(a) = \bar{\delta}_\gamma(a)$$

(the last equality following from (3.10.4)). Now (3.10.3.3) follows from (3.10.5). The induction is complete.

3.11. A partition

In the sequel, we shall identify $\mathrm{Dec}\,\beta$ with $[\beta]^{<\omega}$ via the obvious bijection $(a_0, \ldots, a_n) \mapsto \{a_0, \ldots, a_n\}$.

Define

$$\begin{aligned}Q_\gamma^1(\alpha) &= \{(q_1, q_2, q_3, q_4) \in \mathrm{Seq}\,T_\gamma(\alpha) \mid q_1 < q_2 \leq q_3 > q_4\},\\ Q_\gamma^2(\alpha) &= \{(q_1, q_2, q_3, q_4) \in \mathrm{Seq}\,T_\gamma(\alpha) \mid q_1 < q_2 > q_3 < q_4\},\\ Q_\gamma^3(\alpha) &= \{(q_1, q_2, q_3, q_4) \in \mathrm{Seq}\,T_\gamma(\alpha) \mid q_1 < q_2 > q_3 \geq q_4\},\\ Q_\gamma^4(\alpha) &= \{(q_1, q_2, q_3, q_4) \in \mathrm{Seq}\,T_\gamma(\alpha) \mid q_1 > q_2 < q_3 > q_4\},\\ Q_\gamma^5(\alpha) &= \{(q_1, q_2, q_3, q_4) \in \mathrm{Seq}\,T_\gamma(\alpha) \mid q_1 > q_2 < q_3 \leq q_4\},\end{aligned}$$

$$Q_\gamma(\alpha) = \bigcup_{i=1}^{5} Q_\gamma^i(\alpha), \qquad \bar{S}_\gamma(\alpha) = \{a \in \mathrm{Dec}\,T_\gamma(\alpha) \mid \bar{\delta}_\gamma(a) \in Q_\gamma(\alpha)\}$$

Observe that $i \neq j \to Q_\gamma^i(\alpha) \cap Q_\gamma^j(\alpha) = \emptyset$.

Next define a mapping $\bar{r}_\gamma : \bar{S}_\gamma(\alpha) \to \{0, 1\}$ by

$$\bar{r}_\gamma(a) = \begin{cases} 0 & \text{if } \bar{\delta}_\gamma(a) \in Q_\gamma^i \text{ with } i \leq 3, \\ 1 & \text{if } \bar{\delta}_\gamma(a) \in Q_\gamma^i \text{ with } i > 3, \end{cases}$$

It is quite clear that $\bar{S}_\gamma(\alpha)$ is usually not a Sperner system. On the other hand, by (3.10.6) we have

$$(a \subseteq b \quad \text{and} \quad a, b \in \bar{S}_\gamma(\alpha)) \to \bar{r}_\gamma(a) = \bar{r}_\gamma(b). \qquad (3.11.1)$$

We shall put

$$S_\gamma(\alpha) = \{a \in \bar{S}_\gamma(\alpha) \mid (\forall b \subsetneqq a) b \notin \bar{S}_\gamma(\alpha)\}, \qquad r_\gamma = \bar{r}_\gamma \mid S_\gamma(\alpha).$$

Now $S_\gamma(\alpha)$ is a Sperner system.

3.12. Some combinatorial properties of $\bar{\delta}$

Choose an $a = (a_0, \ldots, a_l) \in \mathrm{Dec}\, T_0(\alpha)$ and let $\bar{\delta}(a) = (q_0, \ldots, q_k)$ (it can be $k = l$ or $k = l - 1$). We call an $i \in \{0, \ldots, k\}$ *separable* (with respect to a) if one of the following conditions holds:

$$q_i = a_i \quad \text{(in particular, } a_i \in \alpha\text{)}, \tag{3.12.1}$$

$$i < l \text{ and } a_{i+1} \in [\alpha]^{<\omega}. \tag{3.12.2}$$

We shall need the following facts.

3.12.3. *Let $0 \leq i_1 < i_2 < i_3 < i_4 \leq k$. Assume, further, that i_4 is separable. Put $q = (q_i, q_{i_2}, q_{i_3}, q_{i_4})$. Then we have*:

$$q \in Q_0^2(\alpha) \cup Q_0^3(\alpha) \to (\exists b \subseteq a)\bar{r}_0(b) = 0, \tag{3.12.3.1}$$

$$q \in Q_0^4(\alpha) \cup Q_0^5(\alpha) \to (\exists b \subseteq a)\bar{r}_0(b) = 1. \tag{3.12.3.2}$$

Proof. (3.9.2) allows us to construct b explicitly. The following eight cases need to be distinguished.

$$q \in Q_0^2(\alpha) \quad \text{and} \quad a_{i_4} \in [\alpha]^{<\omega}. \tag{3.12.3.3}$$

Then put $b = (a_{i_1}, a_{i_1+1}, a_{i_3}, a_{i_3+1}, a_{i_4+1})$, we have $\bar{\delta}(b) \in Q_0^2(\alpha)$.

$$q \in Q_0^2(\alpha) \quad \text{and} \quad a_{i_4} \in \alpha. \tag{3.12.3.4}$$

Since $a \in \mathrm{Dec}\, T_0(\alpha)$. Since $a \in \mathrm{Dec}\, T_0(\alpha)$, we must have $i_4 - 1 > i_3$ and $q_{i_4-1} > q_{i_4}$. Putting $b = (a_{i_1}, a_{i_1+1}, a_{i_4-1}, a_{i_4})$, we have $\bar{\delta}(b) \in Q_0^1(\alpha) \cup Q_0^3(\alpha)$.

$$q \in Q_0^3(\alpha) \quad \text{and} \quad a_{i_4} \in [\alpha]^{<\omega}. \tag{3.12.3.5}$$

Putting $b = (a_{i_1}, a_{i_1+1}, a_{i_3}, a_{i_3+1}, a_{i_4+1})$, we get $\bar{\delta}(b) \in Q_0^2(\alpha) \cup Q_0^3(\alpha)$.

$$q \in Q_0^3(\alpha) \quad \text{and} \quad a_{i_4} \in \alpha. \tag{3.12.3.6}$$

Putting $b = (a_{i_1}, a_{i_1+1}, a_{i_3}, a_{i_4})$, we get $\bar{\delta}(b) \in Q_0^1(\alpha) \cup Q_0^3(\alpha)$.

$$q \in Q_0^4(\alpha) \quad \text{and} \quad a_{i_4} \in [\alpha]^{<\omega}. \tag{3.12.3.7}$$

Putting $b = (a_{i_1}, a_{i_2}, a_{i_2+1}, a_{i_4}, a_{i_4+1})$, we get $\bar{\delta}(b) \in Q_0^4(\alpha)$.

$$q \in Q_0^4(\alpha) \quad \text{and} \quad a_{i_4} \in \alpha. \tag{3.12.3.8}$$

Putting $b = (a_{i_1}, a_{i_2}, a_{i_2+1}, a_{i_4})$, we get $\bar{\delta}(b) \in Q_0^4(\alpha)$.

$$q \in Q_0^5(\alpha) \quad \text{and} \quad a_{i_4} \quad \in [\alpha]^{<\omega}. \tag{3.12.3.9}$$

Putting $b = (a_{i_1}, a_{i_2}, a_{i_2+1}, a_{i_4}, a_{i_4+1})$, we get $\bar{\delta}(b) \in Q_0^4(\alpha) \cup a_0^5(\alpha)$.

$$q \in Q_0^5(\alpha) \quad \text{and} \quad a_{i_4} \in \alpha. \tag{3.12.3.10}$$

Putting $b = (a_{i_1}, a_{i_2}, a_{i_2+1}, a_{i_4})$, we get $\bar{\delta}(b) \in Q_0^4(\alpha)$. □

3.12.4. *Let $0 \leq i_0 < i_1 < \cdots < i_s \leq k$ be all the separable indices. We have*

$$i_0 \leq 1 \quad \text{and} \quad i_{t+1} \leq i_t + 2. \tag{3.12.4.1}$$

Moreover, we have

$$(i_{t+1} = i_t + 2) \rightarrow (a_{i_{t+1}} \in [\alpha]^{<\omega} \quad \text{and} \quad a_{i_{t+1}} \in \alpha) \tag{3.12.4.2}$$

Proof: This is obvious. □

3.12.5. *Let* $0 \leq t < s$ *and let* $q_{i_t} \geq q_{i_t+1} \geq q_{i_{t+1}}$. *Then we have* $q_{i_t} > q_{i_{t+1}}$.

Proof. We distinguish the following two cases.

$$i_{t+1} = i_t + 1. \tag{3.12.5.1}$$

Then we have $a_{i_t+1}, a_{i_t+2} \in [\alpha]^{<\omega}$. Apply (3.9.1).

$$i_{t+1} = i_t + 2. \tag{3.12.5.2}$$

Then we have $a_{i_t+2} \in \alpha$, $a_{i_t+1} \in [\alpha]^{<\omega}$ (see 3.12.4.2). By (3.9.1), again, we have $q_{i_t} > q_{i_t+1}$. □

3.12.6. *Let* $0 \leq p < t \leq l$ *and let* $q_p \leq q_{p+1} \leq \cdots \leq q_t$. *Then*

$$\{a_p, a_{p+1}, \ldots, a_{t-1}\} \subseteq [\alpha]^{<\omega}. \tag{3.12.6.1}$$

$$q_p < q_{p+1} < \cdots < q_{t-1}, \tag{3.12.6.2}$$

$$\{q_p, q_{p+1}, \ldots, q_t\} \subseteq \operatorname{tim}_0(a_p). \tag{3.12.6.3}$$

Proof. (3.12.6.1) follows from the fact that $a \in \operatorname{Dec} T_0(\alpha)$. (3.12.6.2) is a consequence of (3.9.1). (3.12.6.3) follows from (3.9.2): For $p \leq i \leq t-1$, $q_i = \delta(a_i, a_{i+1}) = \delta(a_p, a_{i+1})$. For $i = t$ we either have $q_t = q_{t-1}$ or we can argue similarly. □

3.13. The sheaf functors $A_\gamma^\epsilon (\epsilon \in \{0, 1\})$ and $(\gamma, \omega^{2^\gamma})$-regular transformations Δ_γ^ϵ

3.13.1. Put

$$A_0^0(\alpha) = \{a \in \operatorname{Dec} T_0(\alpha) \mid \bar{\delta}(a) = (q_0, \ldots, q_k) \text{ and}$$
$$q_0 \leq \cdots \leq q_i > q_{i+1} \geq \cdots \geq q_{k-2} \quad \text{for some } 0 \leq i \leq k-2\},$$
$$A_0^1(\alpha) = \{a \in \operatorname{Dec} T_0(\alpha) \mid \bar{\delta}(a) = (q_0, \ldots, q_k) \text{ and}$$
$$q_0 \geq \cdots \geq q_i < q_{i+1} \leq \cdots \leq q_{k-2} \quad \text{for some } 0 \leq i \leq k-2\}.$$

No doubt, by taking the proper action on mappings, A_0^ϵ become sheaf functors.

3.13.2. The transformation $\Delta_0^0 : A_0^0 \rightarrow \operatorname{Dec}$

For a given $a \in A_0^0(\alpha)$, choose the minimal $i \in \{0, \ldots, k-2\}$ such that $i = k-2$ or $q_i > q_{i+1}$. Let indices $i_1 < \cdots < i_s$ be defined in the following way.

$$i_1 = i. \tag{3.13.2.1}$$

For $1 \leqslant t < s$, $i_{t+1} \in \{i_t + 1, i_t + 2, \ldots, k\}$ is such that
$$q_{i_t} = q_{i_t+1} = \cdots = q_{i_{t+1}-1} > q_{i_{t+1}}. \tag{3.13.2.2}$$

No i_{s+1} satisfying the condition of (3.13.2.2) for $t = s$ exists. (3.13.2.3)

Now define $\Delta_0^0(a)$ as

$$(q_{i_1}, \ldots, q_{i_s}) \quad \text{if } q_{i_s} \in \bar{a}$$

and

$$(q_{i_1}, \ldots, q_{i_{s-1}}) \quad \text{else.}$$

Lemma 3.13.3. *Δ_0^0 is $(0, \omega)$-regular.*

Proof. (3.5.1) is obvious. (3.5.2) follows from (3.12.4), (3.12.5) and (3.12.6). To prove (3.5.3), we have to distinguish the following three possibilities.

$$\Delta_0^0 a = \emptyset. \tag{3.13.3.1}$$

Then our statement follows from 3.12.6.

$$\Delta_0^0 a \neq \emptyset \quad \text{and} \quad q_{i_s} \notin a. \tag{3.13.3.2}$$

Then the statement follows from $q_{i_s} \in \operatorname{tim}_0 a_{i_s}$.

$$\Delta_0^0 a \neq \emptyset \quad \text{and} \quad q_{i_s} \in a. \tag{3.13.3.3}$$

Extending the notation to c compatibly, we have by (3.9.1) $q_{i_{s+1}} = q_{i_s+1} \in \operatorname{tim}_0(c_{i_s+1})$. (3.5.4) and (3.5.5) are obvious. □

3.13.4. The transformation $\Delta_0^1 : A_0^1 \to$ Dec

Let $a \in A_0^1(\alpha)$. Let indices $i_1 < \cdots < i_s$ be defined in the following way.

$$i_1 = 0 \tag{3.13.4.1}$$

For $1 \leqslant t < s$, $i_{t+1} \in \{i_t + 1, i_t + 2, \ldots, k\}$ is such that
$$q_{i_t} = q_{i_t+1} = \cdots = q_{i_{t+1}-1} > q_{i_{t+1}}. \tag{3.13.4.2}$$

No i_{s+1} satisfying the condition of (3.13.4.2) for $t = s$ exists. (3.13.4.3)

Now define $\Delta_0^1(a)$ as

$$(q_{i_1}, \ldots, q_{i_s}) \quad \text{if } q_{i_s} \in \bar{a}$$

and

$$(q_{i_1}, \ldots, q_{i_{s-1}}) \quad \text{else.}$$

Lemma 3.13.5. *Δ_0^1 is $(0, \omega)$-regular.*

Proof. Follows in a similar way as the proof of 3.13.3. Moreover, observe that a case analogous to (3.13.3.1) does not occur. □

3.13.5. Now consider the sheaf functors $A^{\epsilon}_{\gamma} \lhd \text{Dec}\, T_{\gamma}(\epsilon = 0, 1)$ and $(\gamma, \omega^{2\gamma})$-regular transformations $\Delta^{\epsilon}_{\gamma}: A^{\epsilon}_{\gamma} \to \text{Dec}$ constructed by the machinery of 3.6 and 3.7. Observe that by virtue of (3.9.2) and a transfinite induction, we have the following extra property.

Lemma 3.13.6. $(\forall a \in A^{\epsilon}_{\gamma}(\alpha))(\forall c \subseteq \Delta^{\epsilon}_{\gamma}(a)(\exists b \subseteq a)(b \in A_{\gamma}(\alpha)$ *and* $\Delta_{\gamma}(b) = c)$.

We shall next return to the circumstances of Nash-Williams' partition theorem. We shall observe the identification imposed at the beginning of 3.11. Thus, for $\epsilon \in \{0, 1\}$, we have sheaves

$$A^{S_{\gamma}, r_{\gamma}}_{\epsilon} \subseteq \text{Seq}\; T_{\gamma}(\alpha).$$

Put $\bar{A}^{S_{\gamma}, r_{\gamma}}_{\epsilon}(\alpha) = A^{S_{\gamma}, r_{\gamma}}_{\epsilon} \cap \text{Dec}\; T_{\gamma}(\alpha)$. Clearly, by taking the appropriate action on mappings, $\bar{A}^{S_{\gamma}, r_{\gamma}}_{\epsilon}$ turns into a sheaf functor.

The inclusion lemma 3.14. *We have*

$$\bar{A}^{S_{\gamma}, r_{\gamma}}_{\epsilon} \subseteq A^{\epsilon}_{\gamma}.$$

Proof. We shall proceed by a transfinite induction on γ. First we shall handle the case $\gamma = 0$. Let $a \notin A^0_0(\alpha)$. As we can easily check, then there are $0 \leqslant i_1 < i_2 < i_3 \leqslant k-2$ such that

$$q_{i_1} > q_{i_2} < q_{i_3}. \tag{3.14.1}$$

By 3.12.4, there is an $i_4 \in \{k-1, k\}$ such that q_{i_4} is separable. By (3.14.1), $(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) \in Q^4_0(\alpha) \cup Q^5_0(\alpha)$. Hence, by 3.12.3, there is a $b \subseteq a$ with $\bar{r}_0(b) = 1$.

Now let $a \notin A^1_0(\alpha)$. As we may easily check, then there are $0 \leqslant i_1 < i_2 < i_3 \leqslant k-2$ such that

$$q_{i_1} < q_{i_2} > q_{i_3}. \tag{3.14.2}$$

By 3.12.4, there is an $i_4 \in \{k-1, k\}$ such that q_{i_4} is separable. By (3.14.2), $(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) \in Q^2_0(\alpha) \cup Q^3_0(\alpha)$. Hence, by 3.12.3, there is a $b \subseteq a$ with $\bar{r}_0(b) = 0$.

Next we remark that the limit case is trivial, since both sides are obtained by passing to colimits. Thus, the nonlimit step remains.

Let the statement be true for a certain $\gamma \in \text{Ord}$. By definition, we have

$$\bar{A}^{S_{\gamma+1}, r_{\gamma+1}}_{\epsilon}(\alpha) \subseteq \bar{A}^{S_{\gamma}, r_{\gamma}}_{\epsilon}(T_{\gamma}(\alpha)).$$

Thus, by the induction hypothesis,

$$\bar{A}^{S_{\gamma+1}, r_{\gamma+1}}_{\epsilon}(\alpha) \subseteq A^{\epsilon}_{\gamma}(T_{\gamma}(\alpha)).$$

Now we shall prove that for an $a \in \bar{A}^{S_{\gamma+1}, r_{\gamma+1}}_{\epsilon}(\alpha)$ we have

$$\Delta^{\epsilon}_{\gamma}(a) \in \bar{A}^{S_{\gamma}, r_{\gamma}}_{\epsilon}(\alpha). \tag{3.14.3}$$

In effect, let, say, $\epsilon = 0$ (the case of $\epsilon = 1$ is handled similarly). Should (3.14.3) be false, then there is a $c \subseteq \Delta^0_\gamma(a)$ with

$$\bar{\delta}_\gamma(c) \in Q^4_\gamma(\alpha) \cup Q^5_\gamma(\alpha).$$

By 3.13.6, there is a $b \subseteq a$ with

$$\Delta_\gamma(b) = c.$$

By (3.10.4), we obtain

$$\bar{\delta}_\gamma(b) = (\mathrm{Dec}\,\kappa^\gamma)_{T_\gamma(\alpha)}(c).$$

Now by definition 3.10.2 (the second part), we get

$$\begin{aligned}\bar{\delta}_{\gamma+1}(b) &= (\mathrm{Seq}\,\kappa^{\gamma+1}_\gamma)(\bar{\delta}_\gamma(c)) \in (\mathrm{Seq}\,\kappa^{\gamma+1}_\gamma)(Q^4_\gamma(\alpha) \cup Q^5_\gamma(\alpha))\\ &\subseteq Q^4_{\gamma+1}(\alpha) \cup Q^5_{\gamma+1}(\alpha).\end{aligned}$$

We conclude $\bar{r}_{\gamma+1}(b) = 1$ which is a contradiction.

Thus, (3.14.3) is proved. By the induction hypothesis, we conclude that

$$\Delta^\epsilon_\gamma(a) \in A^\epsilon_\gamma(\alpha),$$

implying $a \in A^\epsilon_{\gamma+1}(\alpha)$ directly by the construction 3.7. □

3.15. Some computations

As yet, we did not assume that the argument α of $A^\epsilon_\gamma(\alpha)$, $T_\gamma(\alpha)$, $\bar{A}^{S_\gamma, r_\gamma}_\epsilon(\alpha)$ etc. would be an ordinal. Indeed, it was not even possible to restrict ourselves to working with ordinals. Our construction was based on the categorial behaviour of $\mathscr{W}$, which would substantially change by restricting to ordinals: For example, we would lose directed colimits.

Yet, the case of α being an ordinal is the most interesting one. We shall consider it throughout the rest of the paper. As before, we shall first make the assumption

$$\Omega \supseteq T_\gamma(\alpha). \tag{3.15.1}$$

Note that, by (3.5.2), for any (γ, λ)-regular transformation $\Delta: A \to \mathrm{Dec}$, we have ($A$ being considered a tree by the relation $\preccurlyeq$)

$$\gamma_{A(\alpha)} \leqslant (\alpha + 1) \cdot \lambda + 1. \tag{3.15.2}$$

Indeed, let $\chi: \mathrm{Dec}(\alpha) \to \alpha + 1$ be a character (the summand 1 comes from the empty sequence). Let, for each $b \in A(\alpha)$, $\chi_b: t(b) \to \lambda$ denote the character whose existence is stated by (3.5.2). Let, further, for $a \in A(\alpha)$,

$$b(a) \in A(\alpha)$$

denote the sequence satisfying

$$b(a) \preccurlyeq a, \qquad \Delta b(a) = \Delta a, \qquad c < b(a) \to \Delta c \neq \Delta a.$$

Now define a character $\bar{\chi}: A(\alpha) \to (\alpha+1) \cdot \lambda + 1$ by

$$\bar{\chi}(a) = \begin{cases} (\chi(\Delta a) \cdot \lambda) + \chi_{b(a)}(a) & \text{if } b(a) < a, \\ (\chi(\Delta a) + 1) \cdot \lambda & \text{if } b(a) = a. \end{cases}$$

(3.15.2) is proved.

Now by (3.13) and (3.15.2) and $\emptyset \notin A_\gamma^\epsilon(\alpha)$ we conclude

$$\gamma_{A_\gamma^\epsilon(\alpha)} \leqslant (\alpha+1) \cdot \omega^{2^\gamma}, \quad (\epsilon = 0, 1), \tag{3.15.3}$$

and, by 3.14,

$$\gamma_{A_\epsilon^{S_\gamma, r_\gamma} \cap \mathrm{Dec}\, T_\gamma(\alpha)} \leqslant (\alpha+1) \cdot \omega^{2^\gamma}, \quad (\epsilon = 0, 1). \tag{3.15.4}$$

Thus, there is an $((\alpha+1) \cdot \omega^{2^\gamma}, (\alpha+1) \cdot \omega^{2^\gamma})$-testing $g = (g_0, g_1)$ (recall that g_ϵ is a character on $A^{S_\gamma, r_\gamma} \cap \mathrm{Dec}\, T_\gamma(\alpha)$) such that

$$\mathrm{Dec}\, T_\gamma(\alpha) \subseteq \mathrm{Bad}(A_\epsilon^{S_\gamma, r_\gamma}, g) \cup \{\emptyset\}, \tag{3.15.5}$$

which is (3.0.4). From this, 1.4.2.1 and 1.3 we conclude that

$$\gamma_{\mathrm{Dec}\, T_\gamma(\alpha)} \leqslant \psi((\alpha+1) \cdot \omega^{2^\gamma}, (\alpha+1) \cdot \omega^{2^\gamma}). \tag{3.15.6}$$

Now we show that T_γ satisfies (3.0.1)–(3.0.3). Condition (3.0.1) follows directly from the definition of $T_0(\alpha)$. To prove (3.0.2), we first define a natural transformation

$$\lambda_\beta^n : T_\beta T_{\beta+n} \to T_{\beta+n+1}$$

for $\beta \in \mathrm{Ord}$, $n \in \omega$ by induction on n. For $n = 0$ let $\lambda_\beta^0 : T_\beta T_\beta \to T_{\beta+1}$ be the identity. Is λ_β^n already defined, define λ_β^{n+1} as the composition

$$\begin{array}{ccc} T_\beta T_{\beta+n+1} = T_\beta T_{\beta+n} T_{\beta+n} & \xrightarrow{\lambda_\beta^n T_{\beta+n}} & T_{\beta+n+1} T_{\beta+n} \\ & & \Big\downarrow {\scriptstyle T_{\beta+n+1} \kappa_{\beta+n}^{\beta+n+1}} \\ T_{\beta+n+2} & = & (T_{\beta+n+1})^2. \end{array}$$

For $\gamma = \beta + n$, the naturality of λ_β^n yields a commutative diagram

$$\begin{array}{ccc} T_\beta T_\gamma & \xrightarrow{\lambda_\beta^n} & T_{\gamma+1} \\ {\scriptstyle T_\beta T_\gamma \kappa^\gamma} \Big\downarrow & & \Big\downarrow {\scriptstyle T_{\gamma+1} k^\gamma} \\ T_\beta T_{\gamma+1} & \xrightarrow{\lambda_\beta^n T_\gamma} & T_{\gamma+1} T_\gamma \\ & & \Big\downarrow {\scriptstyle T_{\gamma+1} \kappa_\gamma^{\gamma+1}} \\ & & T_{\gamma+2} \end{array}$$

which is readily rephrased into

$$\begin{array}{ccc} T_\beta T_\gamma & \xrightarrow{\lambda_\beta^\eta} & T_{\gamma+1} \\ {\scriptstyle T_\beta \kappa^\gamma}\downarrow & & \downarrow{\scriptstyle \kappa_{\gamma+1}^{\gamma+2}} \\ T_\beta T_{\gamma+1} & \xrightarrow{\lambda_\beta^{\eta+1}} & T_{\gamma+2}. \end{array}$$

Since T_β obviously preserves directed colimits, we can pass to a colimit transformation

$$T_\beta T_{\beta+\omega} \xrightarrow{\mu_\beta^{\beta+\omega}} T_{\beta+\omega}.$$

Assumed we have already defined a natural transformation

$$\mu_\beta^\gamma : T_\beta T_\gamma \to T_\gamma$$

for some $\gamma \geqslant \beta + \omega$, we also have a natural transformation

$$\mu_\beta^{\gamma+1} = \mu_\beta^\gamma T_\gamma : T_\beta T_{\gamma+1} \to T_{\gamma+1}.$$

The naturality of μ_β^γ yields a commutative diagram

$$\begin{array}{ccc} T_\beta T_\gamma & \xrightarrow{\mu_\beta^\eta} & T_\gamma \\ {\scriptstyle T_\beta \kappa^\gamma}\downarrow & & \downarrow{\scriptstyle \kappa_\gamma^{\gamma+1}} \\ T_\beta T_{\gamma+1} & \xrightarrow{\lambda_\beta^{\gamma+1}} & \mu_{\gamma+2} \end{array}$$

allowing us to pass to colimits and define a transformation

$$\mu_\beta^\gamma : T_\beta T_\gamma \to T_\gamma \quad \text{for any } \gamma \geqslant \beta + \omega. \tag{3.15.7}$$

Recall, however, that morphisms in $\mathscr{W}$ are strictly monotone mappings. Thus, we conclude that

$$\gamma_{\mathrm{Dec}(T_\beta T_\gamma(\alpha))} \leqslant \gamma_{\mathrm{Dec}(T_\gamma(\alpha))} \quad \text{for } \alpha, \beta, \gamma \in \mathrm{Ord},\ \gamma \geqslant \beta + \omega, \tag{3.15.8}$$

which is (3.0.2). To prove (3.0.3), we prove the following.

Lemma 3.15.9. *Let $\beta \subseteq \alpha \in \mathrm{Obj}\, \mathscr{W}$ and let $\rho \in \alpha$. We have*

$$(\forall \sigma \in \beta)\rho > \sigma) \to ((\forall \sigma \in T_\gamma(\beta))\kappa^\gamma(\rho) > \sigma)$$

where $T_\gamma(\beta)$ is considered a part of $T_\gamma(\alpha)$ via the identification $T_\gamma(\subseteq)$.

Proof. An induction on γ. For $\gamma = 0$, the statement is easily verified. On the other hand, for γ limit, we easily get it by passage to colimits. Thus, the non-limit step remains. Let, thus, the statement be true for a certain γ and let $(\forall \sigma \in \beta)\rho > \sigma$. Using the induction hypothesis, we get

$$(\forall \sigma \in T_\gamma(\beta))\kappa^\gamma(\rho) > \sigma$$

and

$$(\forall \sigma \in T_\gamma T_\gamma(\beta))(\kappa^\gamma)_{T_\gamma(\alpha)}(\kappa^\gamma(\rho)) > \sigma.$$

Now

$$(\kappa^\gamma)_{T_\gamma(\alpha)}\kappa^\gamma(\rho) = (\kappa^\gamma \cdot \kappa^\gamma)(\rho) = \kappa^{\gamma+1}(\rho),$$

proving the statement. □

Lemma 3.15.9 is summed up into

$$\gamma_{\mathrm{Dec}T_\gamma(\beta)} < \gamma_{\mathrm{Dec}T_\gamma(\alpha)} \quad \text{for } \beta < \alpha \in \mathrm{Ord}, \tag{3.15.10}$$

which is (3.0.3). This completes the construction outlined in 3.0. We have proved the following theorem.

Theorem 3.16. *For $\gamma \geq 2$, α, $\gamma < |\Omega|^+$, we have*

$$\psi((\alpha+1)\omega^{2^{\gamma\omega}}, (\alpha+1)\omega^{2^{\gamma\omega}}) \geq \varphi_\gamma(\alpha).$$

Remark 3.17. Defining ψ by means of Strong R-functions (see [7]), we could eliminate the cardinality restriction on Ω. Using the methods of [7], one can prove that on arguments $<|\Omega|^+$, the R-functions and the strong R-functions for the Nash-Williams' k-system coincide.

4. Concluding remarks

4.1. Restricting ourselves (say) to two colours, we may define the (classical) Ramsey number $R_n(s_0, s_1)$ as the minimal number K such that for each mapping

$$r: \binom{K}{n} \to 2$$

there is an $X \subseteq K$ and an $i \in 2$ such that

$$|X| = s_i \quad \text{and} \quad r^{-1}(i) \supseteq \binom{X}{n}.$$

The existence of the numbers $R_n(s_0, s_1)$ was first proved by Ramsey [12]. The first attempt to find an explicit upper bound was in Skolem's paper [18]. Roughly, Ramsey's and Skolem's methods lead to the following recursive upper bound:

$$R_n(s_0, s_1) \leq R_{n-1}(R_n(s_0-1, s_1), R_n(s_0, s_1-1)). \tag{4.1.1}$$

In other words, we obtain an upper estimate by a function $A_n(s_0, s_1)$ satisfying

$$A_1(s_0, s_1) = s_0 + s_1 - 1, \tag{4.1.2}$$

$$A_n(s_0, s_1) = A_{n-1}(A_n(s_0-1, s_1), A_n(s_0, s_1-1)). \tag{4.1.3}$$

This is an example of an Ackermann function (function growing faster than any primitively recursive function).

Later, Erdős and Rado found a better upper bound (see [1]). This upper bound can be expressed as a 'tower function' (iterated exponentiation (see [4, Chapter 4].

The results in our Section 2 indicate that the ordinal numbers $\psi_\alpha(\gamma_0, \gamma_1)$ may be regarded as generalizations of the numbers $R_n(s_0, s_1)$, where n, s_0, s_1 become ordinals. (More exactly, $\psi_\alpha(\gamma_0, \gamma_1)$ corresponds to $R_n(s_0+1, s_1+1)-1$.) Indeed, Theorem 2.2 is a precise analogy of the estimate (4.1.1). Also, the function $\varphi_\alpha(\gamma_0 \oplus \gamma_1)$ is an analogy of the Ackermann function $A_n(s_0, s_1)$. Indeed, we have, e.g.,

$$\varphi_{\alpha+1}(\gamma) = \sup\{\varphi_\alpha^n(\varphi_{\alpha+1}(\gamma')) \mid n \in \omega,\ \gamma' > \gamma\}$$

which is an analogy of (4.1.3).

What is interesting and surprising is that an analogy of Erdős and Rado's stronger upper bound [1] does not hold. Instead, we have shown that asymptotically, in the range of countable ordinals, the Ramsey–Skolem's upper bound is quite good.

We would also like to remark that in order to obtain our lower bound, we did use the classical Erdős-Hajnal's stepping-up lemma (see [4, Chapter 4]) as the first step.

4.2. Finally, we would like to make a short remark on the use of category theory in this paper. As the reader might have noticed, we did not use any actual results in category theory except a few elementary observations about the categories $\mathscr{W}$, Set. What is then the purpose of introducing all that 'abstract nonsense'?

We feel that it is the same as in the majority of other fields of mathematics: category theory is a very convenient language.

Recall that in order to make any achievement at all (a hyper-exponential lower bound), we needed to construct $T_\omega(\alpha)$. While $T_n(\alpha)$ can be still plausibly described as $(T_0)^{2^n}(\alpha)$, to define $T_\omega(\alpha)$, we need to identify $T_n(\alpha)$ with a part of $T_{n+1}(\alpha)$. The problem is that the identification can be made in many equally natural ways. The two coming up immediately are via $T_n\kappa^n$ and $\kappa^n \cdot T_n$. Note that they actually are different: for example, $T_1(\alpha)$ consists of α, two copies of $[\alpha]^{<\omega}$ and one copy of $[[\alpha]^{<\omega}]^{<\omega}$. The two mappings $T_0\kappa^0$, $\kappa^0 T_0$ both send α to α, but make a difference in the matter as to which of its copies to send $[\alpha]^{<\omega}$. Naturally, this has an impact on the ordering $\cdots$. Of course, we could define $T_\gamma(\alpha)$ in one blow as an appropriate set of sequences and introduce some peculiar ordering. Although we could carry out the whole proof this way, we would be introducing a lot more information than what we are actually using.

Thus, in the language of a computer programmer, we are using category theory as a *data structure* to keep exactly the amount of information we are using each time. Of course, as a computer programmer would understand, this requires some extra work on *updating* our data structure even through the steps where it does not seem necessary. This is exactly the meaning of our auxillary diagram-chasing lemmas.

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