Discrete Mathematics 95 (1991) 255–270
North-Holland

Universal elements and the complexity of certain classes of infinite graphs

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Received 16 December 1989
Revised 20 March 1990

Abstract

Komjáth, P. and J Pach, Universal elements and the complexity of certain classes of infinite graphs, Discrete Mathematics 95 (1991) 255–270.

A class of graphs has a universal element G_0, if every other element of the class is isomorphic to an induced subgraph of G_0. In Sections 1–4 we give a survey of some recent developments in the theory of universal graphs in the following areas: (1) Graphs universal for isometric embeddings, (2) universal random graphs, (3) universal graphs with forbidden subgraphs, (4) universal graphs with forbidden topological subgraphs. Section 5 is devoted to the problem of deciding how far a class of graphs $\mathcal{G}$ is from having a universal element. We introduce a new measure of the complexity of the class $\mathcal{G}$, denoted by $\mathrm{cp}(\mathcal{G})$. This is defined to be the minimum cardinal κ such that there exist κ elements in $\mathcal{G}$ with the property that any other element of $\mathcal{G}$ can be embedded into at least one of them as an induced subgraph. $\mathcal{G}$ has a universal element if and only if $\mathrm{cp}(\mathcal{G}) = 1$. Among other theorems we prove that (i) the complexity of the class of all countable graphs without $n \geq 2$ independent edges is finite; (ii) for any cardinal κ, $\omega_1 \leq \kappa \leq 2^{\omega}$, it is consistent that the complexity of the class of all locally finite countable graphs is equal to κ. In Section 6 we consider some analogous questions for hypergraphs.

1. Isomorphic embeddings and isometric embeddings

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Given any $x, y \in V(G)$, let $d_G(x, y)$ denote the *distance* of x and y in G, i.e., the length of

* Research supported in part by Hungarian SF grant OTKA-1805 and MSRI, Berkeley, NSF-DMS 8505550.
** Research supported in part by Hungarian SF grant OTKA-1814, NSF grant CCR-8901484 and DIMACS-NSF Science and Technology Center, NSF-STC 88-09648.

Elsevier Science Publishers B.V.

the shortest path connecting them. An *isomorphic embedding* of G into another graph G_0 is an injection $f: V(G) \to V(G_0)$ with the property

$$f(x)f(y) \in E(G_0) \iff xy \in E(G)$$

for every $x, y \in V(G)$. If there exists such an embedding f, then we also say that G is (isomorphic to) an *induced subgraph* of G_0.

An *isometric embedding* of G into another graph G_0 is an injection $f: V(G) \to V(G_0)$ with the property that

$$d_{G_0}(f(x), f(y)) = d_G(x, y)$$

for every $x, y \in V(G)$. We say that H is an *isometric subgraph* of G_0 if $H \subseteq G_0$ and

$$d_H(x, y) = d_{G_0}(x, y)$$

for every $x, y \in V(H)$. Evidently, every isometric embedding is an isomorphic embedding, and every isometric subgraph is an induced subgraph.

Rado [25–26] made the following interesting observation, which raised a number of new questions.

Theorem 1.1 (Rado). *There exists a countable graph G_0 such that every countable graph is isomorphic to an induced subgraph of G_0.*

Proof. Let the vertex set of G_0 be the union of countably many disjoint sets $V_0 \cup V_1 \cup V_2 \cup \cdots$. Let $|V_0| = 1$, and assume that the part of G_0 induced by the set $W = \bigcup_{i<n} V_i$ has already been defined. For every subset $A \subseteq W$, take a different vertex $x_A \notin W$, and join it to all elements in A and to none in $W - A$. Let $V_n = \{x_A : A \subseteq W\}$. It is now clear that, for every countable graph G with $V(G) = \omega = \{0, 1, 2, \ldots\}$, there is an isomorphic embedding $f: V(G) \to V(G_0)$ such that $f(i) \in V_i$ for all i. Later we shall see that the simple idea behind this treelike construction can be developed to establish similar results for many classes of graphs. □

The question whether Theorem 1.1 can be generalized to isometric embeddings was raised by Howorka [17], and was answered by Pach [22] in the affirmative.

Theorem 1.2 (Pach). *There exists a countable graph G_0 such that every countable graph can be isometrically embedded into G_0.*

Proof. It is sufficient to construct a countable graph G^* such that every countable *connected* graph G can be isometrically embedded into G^*. Then the union of countably many vertex-disjoint copies of G^* will meet the requirements for G_0.

Let V_n denote the set of all $n+1$ by $n+1$ symmetric integer matrices $D = (d_{ij})_{i,j=0}^{n}$ for which one can find a connected graph G with $V(G) = \omega$ such that $d_G(i, j) = d_{ij}$ for all $0 \leq i, j \leq n$. Let $V(G^*) = \bigcup_{n<\omega} V_n$. Two vertices (matrices) $D = (d_{ij})_{i,j=0}^{n}$ and $D' = (d'_{ij})_{i,j=0}^{m}$ $(m \leq n)$ will be joined by an edge in G^* if and only if $d_{ij} = d'_{ij}$ for all $0 \leq i, j \leq m$, and $d_{nm} = 1$.

It is not hard to see now that if G is any connected graph on $V(G) = \omega$, then

$$f(n) = (d_G(i, j))_{i,j=0}^{n}, \quad n < \omega,$$

is an isometric embedding of G into G^*. □

A countable graph G is called *isometrically constructible* if one can find a nested sequence of finite isometric subgraphs $G_1 \subset G_2 \subset G_3 \subset \cdots$ of G such that any $x \in V(G)$ is contained in some $V(G_i)$. It would be interesting to find a nice (meaningful) characterization of the isometrically constructible graphs. The only nontrivial result in this direction is due to Pach [22].

Theorem 1.3 (Pach). *Every countable planar graph is isometrically constructible.*

2. Universal graphs and zero-one laws

A completely different approach to Theorem 1.1 that leads to many far-reaching generalizations, was taken by Pósa and Fagin [9] (see also [8]). Roughly speaking, the idea is that a countable random graph almost surely contains every countable graph as an induced subgraph. Furthermore, the countable random graph is (almost surely) unique.

In order to be a little more precise, we need the following definition: A graph G is said to have *property* P_{ij} if for any distinct vertices $x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j \in V(G)$ one can find a vertex x adjacent to all of the x and none of the y. The proof of the following two statements is straightforward.

Lemma 2.1. *Let $0 < p < 1$ be fixed, and let $G_{n,p}$ denote a random graph on n vertices, whose edges are chosen independently with probability p. Then*

$$\lim_{n \to \infty} \text{Prob}[G_{n,p} \textit{ has property } P_{ij}] = 1$$

holds for any $i, j < \omega$.

Lemma 2.2 *Let G' and G'' be two countable graphs satisfying P_{ij} for all $i, j < \omega$. Then G' and G'' are isomorphic.*

Lemma 2.1 yields that the system $\mathcal{T}$ consisting of all statements P_{ij} $(i, j < \omega)$ is a *consistent* theory (every finite subsystem $\mathcal{T}' \subset \mathcal{T}$ has a finite model). Hence, by the Gödel Completeness Theorem, $\mathcal{T}$ has a countable model (which, of course, cannot be finite). Moreover, Lemma 2.2 implies that this countable model G_0 is uniquely determined. On the other hand, G_0 has the very strong property that the embedding of any *finite* subgraph of any countable graph G into G_0 can be extended to an embedding of G into G_0. In particular, any countable graph G is isomorphic to an induced subgraph of G_0.

A similar argument leads to the following zero-one law.

Theorem 2.3 (Fagin). *For every first order statement Q, and for every $0<p<1$,*

$$\lim_{n\to\infty} \mathrm{Prob}[G_{n,p} \text{ satisfies } Q] = 0 \text{ or } 1.$$

Proof. It is easy to see that $\mathcal{T}$ is a *complete* theory, i.e., any statement S is either provable or disprovable from $\mathcal{T}$. For if not, then letting $\mathcal{T}' = \mathcal{T} \wedge S$ and $\mathcal{T}'' = \mathcal{T} \wedge (\neg S)$ we would obtain two consistent theories with two different countable models G' and G'', contradicting Lemma 2.2.

Assume now that Q is a first order statement which can be proved from $\mathcal{T}$. Then there exists a natural number k such that the proof uses only P_{ij} with $i, j<k$. Thus, Prob[$G_{n,p}$ does not satisfy Q] $\leq \sum_{i,j<k}$ Prob[$G_{n,p}$ does not have property P_{ij}], and this sum approaches 0, as n tends to infinity. Similarly, if the negation of Q can be proved from $\mathcal{T}$, then Prob[$G_{n,p}$ satisfies Q] $\to 0$. □

The same plan can be followed when the edge probability p is not a constant but $p=p(n)$ is a function approaching 0. In this case, Shelah and Spencer [29] established the following result, suggesting that it makes certain sense to speak about a 'universal countable random graph with edge probability $p(n)$' (a rather strange looking notion, indeed). Let $p(n) \ll q(n)$ mean that $\lim_{n\to\infty} p(n)/q(n) = 0$.

Theorem 2.4 (Shelah, Spencer). *Assume that $n^{-1-1/k} \ll p(n) \ll n^{-1-1/(k+1)}$ for some integer $k \geq 1$, and Q is a first order statement. Then*

$$\lim_{n\to\infty} \mathrm{Prob}[G_{n,p(n)} \text{ satisfies } Q] = 0 \text{ or } 1.$$

As before, the proof gives a uniquely determined countable model (graph) G_0, which has the interesting feature that, if we wish to decide whether a given first order statement is true in $G_{n,p(n)}$ with probability tending to 1, then it is sufficient to check whether or not it holds for G_0. (However, one must admit that some of the most interesting graph-theoretic properties, like planarity, connectedness, etc., cannot be expressed by first order statements. In this sense, the 'universality' of G_0 is limited).

Shelah and Spencer can also prove that Theorem 2.4 remains valid for $p(n)=n^{-\alpha}$, where α can be any fixed irrational number. Yet the most surprising phenomenon in this field is that the above zero–one law is false for many reasonably smooth edge probability functions $p(n)$.

3. Universal graphs with forbidden subgraphs

A class of graphs $\mathcal{G}$ is said to have a *universal element* (or *universal graph*) G_0 if any other element of the class is isomorphic to an induced subgraph of G_0.

Using this terminology, Theorem 1.1 states that the class of all countable graphs possesses a universal element (or, in other words, there exists a universal countable graph). The idea of the proof given in Section 1 can readily be generalized to the class of all countable graphs containing no K_r, a complete subgraph with r vertices.

Proposition 3.1. *The class of all countable K_r-free graphs has a universal element for every $2 \leq r < \omega$.*

Moreover, if the Generalized Continuum Hypothesis (GCH) is assumed, then Proposition 3.1 remains valid for the class of all K_r-free graphs of any infinite cardinality.

Given an infinite cardinal γ and a family of so-called *forbidden subgraphs* $\mathcal{H} = \{H_1, H_2, \ldots\}$, let $\mathcal{G}_\gamma(\mathcal{H})$ denote the class of all graphs of size γ containing no subgraph isomorphic to any element of $\mathcal{H}$. (Note that, according to this definition, the elements of $\mathcal{G}_\gamma(\mathcal{H})$ avoid $\mathcal{H}$ in the strong sense: they must not contain any H_i even as a not necessarily induced subgraph). In the case when $\mathcal{H} = \{H\}$ consists of a single forbidden subgraph, we shall write $\mathcal{G}_\gamma(H)$ instead of $\mathcal{G}_\gamma(\{H\})$. The elements of $\mathcal{G}_\gamma(H)$ are often called H-free.

The easy proof of the following statement has been well known (in folklore) for a long time.

Proposition 3.2. *There is no universal element in $\mathcal{G}_\omega(K_\omega)$.*

Proof. Assume, for a contradiction, that there is a universal graph $G_0 \in \mathcal{G}_\omega(K_\omega)$. Let G_0' denote the graph obtained from G_0 by adding a new vertex v adjacent to all other points. Clearly, G_0' is also K_ω-free. Thus, by our assumption, there is an isomorphic embedding $f : V(G_0') \to V(G_0)$. However, in this case $\{f(v), f^2(v), f^3(v), \ldots\}$ induces an infinite complete subgraph in G_0, contradiction. □

Diestel, Halin and Vogler [7] generalized this argument to establish the following stronger statement.

Theorem 3.3 (Diestel, Halin, Vogler). *Let $\mathcal{H}$ be a non-empty class of countable graphs, each containing an infinite path. Then there is no universal element in $\mathcal{G}_\omega(\mathcal{H})$.*

Given any (finite or infinite) cardinals α, β, let $K_{\alpha,\beta}$ denote a complete bipartite graph with α resp. β elements in its classes. The $K_{1,\omega}$-free graphs, i.e., graphs without vertices of infinite degree, are often called *locally finite.*

De Bruijn (see [26]) showed that $\mathscr{G}_\omega(K_{1,\omega})$, the class of all countable locally finite graphs, does not possess a universal element. (In Section 5 we will prove a much stronger result, based on the original idea of de Bruijn.) Hajnal and Pach [18] proved that $\mathscr{G}_\omega(K_{2,2})$ has no universal element. On the other hand, using GCH, Rado was able to establish the following result for every regular cardinal $\gamma > \omega$, which was extended by Shelah [27] to every $\gamma > \omega$.

Theorem 3.4 (Rado, Shelah). *The class $\mathscr{G}_\gamma(K_{1,\gamma})$ has a universal element for every $\gamma > \omega$.*

We proved the following general theorem telling exactly in which cases $\mathscr{G}_\gamma(K_{\alpha,\beta})$ has a universal element and in which cases it does not, provided that α is finite [20]. Of course, it contains all the above mentioned results as special cases.

Theorem 3.5 (Komjáth, Pach). *Assume GCH. Let $1 \leqslant \alpha \leqslant \beta \leqslant \gamma$ be cardinals, α finite, γ infinite. Then $\mathscr{G}_\gamma(K_{\alpha,\beta})$ has a universal element if and only if*
(i) *$\gamma > \omega$, or*
(ii) *$\gamma = \omega$, $\alpha = 1$ and $\beta \leqslant 3$.*

The basic technique used for the construction of most universal graphs is a natural (nevertheless, powerful) extension of the (trivial) argument in Section 1 proving Theorem 1.1. To be a little more precise, we need some definitions.

A graph (or a structure) G is called *homogeneous* if any isomorphism between two finite subgraphs (substructures) of G can be extended to an automorphism of G. (Note that e.g. the countable universal graph constructed in Section 2 is clearly homogeneous. A complete list of countable homogeneous graphs is given in [21].) From this definition, we immediately obtain the following.

Claim 3.6. Let G be a homogeneous graph of size γ, and let $\mathscr{H}$ denote the family of all finite graphs not isomorphic to any induced subgraph of G. Then G is a universal element in $\mathscr{G}_\gamma(\mathscr{H})$.

A much stronger version of this claim is proved in [1; Theorem 3.2]. A class $\mathscr{G}$ of finite graphs (or structures) is called an *amalgamation class* if, for every pair $G_1, G_2 \in \mathscr{G}$ with a common induced subgraph (substructure) F, one can find an element $G^* \in \mathscr{G}$ such that $G^* \supseteq G_1 \cup G_2$.

Theorem 3.7 (Fraïssé). *Given any family $\mathscr{H}$ of finite graphs (structures), let $\mathscr{G}_\omega(\mathscr{H})$ be the class of all countable graphs (structures) that do not contain any element of $\mathscr{H}$ as a subgraph (not necessarily induced structure). Then $\mathscr{G}_\omega(\mathscr{H})$ has a homogeneous universal graph (structure) if and only if its finite elements form an amalgamation class.*

For more details about amalgamation classes, see [10]. If we want to show that a certain class of countable graphs possesses a universal element, then we can either use Theorem 3.7 directly, or we can apply it to some larger class of structures (obtained by adding some extra relations, coloring etc.). In the latter case, we obtain the universal graph by deleting these additional relations from the universal element in the larger class.

This technique was used by Komjáth Mekler and Pach [19] to establish the following results. Let C_k and P_k denote a cycle of length k and a path of length k, respectively.

Theorem 3.8 (Komjáth, Mekler, Pach). *For any positive integer k, the class $\mathscr{G}_\omega(\{C_3, C_5, \ldots, C_{2k+1}\})$ of all countable graphs containing no odd cycles of length at most $2k+1$ possesses a universal element.*

Theorem 3.9 (Komjáth, Mekler, Pach). *For any infinite cardinal γ and positive integer k, there is a universal element in the class*

(i) *$\mathscr{G}_\gamma(P_k)$ of all graphs of size γ containing no path of length k;*

(ii) *$\mathscr{G}_\gamma(\{C_k, C_{k+1}, \ldots\})$ of all graphs of size γ containing no cycles of length at least k.*

There are many further interesting related questions for uncountable cardinals. We do not discuss them here, because they lead to difficult model-theoretic problems. We only mention the following result, which is a special case of the existence theorem on saturated and special models (see e.g. [2]).

Theorem 3.10. *Assume GCH. If $\gamma > \omega$ and H is a finite graph, then $\mathscr{G}_\gamma(\mathscr{H})$ contains a universal element.*

The interested reader can find more results on universal graphs of size greater than or equal to 2^ω in [24, 16, 28].

4. Universal graphs with forbidden topological subgraphs

The following theorem of Pach [23] settled a problem of Ulam.

Theorem 4.1 (Pach). *There is no universal element in the class of all countable planar graphs.*

Given a graph H, let $\operatorname{top} H$ denote the family of graphs 'topologically equivalent' to H, i.e., the family of all subdivisions of H. (A *subdivision* of H is a graph arising from H by replacing its edges with independent paths.) Accordingly, for a system of graphs $\mathscr{H} = \{H_i\colon i \in I\}$, let $\operatorname{top} \mathscr{H} = \bigcup_{i\in I} \operatorname{top} H_i$. Using this notation, Theorem 4.1 asserts that $\mathscr{G}_\omega(\operatorname{top}\{K_5, K_{3,3}\})$ does not have a universal element. The idea of the proof given in [23] stems from the observation that the class of finite planar graphs is far from having the amalgamation property discussed in the last section.

Using the same idea, one can easily prove that $\mathcal{G}_\omega(\text{top}\, K_{3,3})$ does not have a universal element either. Diestel [4] extended this result by showing the following.

Theorem 4.2 (Diestel). *Let* $2 \leq \alpha \leq \beta < \omega$, $\beta \geq 4$. *Then* $\mathcal{G}_\omega(\text{top}\, K_{\alpha,\beta})$ *has no universal element.*

A different approach was taken in [7] to prove the non-existence of universal elements in some classes of graphs with forbidden topological subgraphs. In particular, it enables us to give an alternative proof of Theorem 4.1.

To sketch their method we need some definitions. Let G be a countable graph, λ an ordinal. A family $\{G_\kappa\colon \kappa < \lambda\}$ of induced subgraphs of G is said to form a *simplicial decomposition* if

(i) $G = \bigcup_{\kappa<\lambda} G_k$,

(ii) $(\bigcup_{\kappa<\mu} G_\kappa) \cap G_\mu =: S_\mu$ is a complete graph (simplex) for every $0 < \mu < \lambda$,

(iii) no S_μ contains any G_κ, $\kappa \leq \mu$.

A graph is called *prime* if it has no simplicial decomposition into more than one subgraph, which is easily seen to be equivalent to the fact that it has no separating complete subgraph. It is well known (see [6, 11–14]) that every K_ω-free countable graph has a *prime decomposition*, i.e., a simplicial decomposition in which all elements are prime.

An element of $\mathcal{G}_\omega(\text{top}\, \mathcal{H})$ is called *maximal*, if the addition of any edge would result in a graph no longer in the class. The *subdivision base* $B(\mathcal{H})$ is defined as the class of all graphs that occur as a member of a prime decomposition of some maximal element of $\mathcal{G}_\omega(\text{top}\, \mathcal{H})$.

It is not hard to prove the following crucial result showing how these concepts relate to universal graphs. (See e.g. [5]).

Theorem 4.3 (Diestel, Halin, Vogler). *If the subdivision base* $B(\mathcal{H})$ *of some family* $\mathcal{H}$ *of finite graphs is uncountable, then* $\mathcal{G}_\omega(\text{top}\, \mathcal{H})$ *has no universal element.*

From this, one can deduce the following.

Theorem 4.4 (Diestel, Halin, Vogler). *Given* $2 \leq r < \omega$, *the class* $\mathcal{G}_\omega(\text{top}\, K_r)$ *has a universal element if and only if* $r \leq 4$.

It might be interesting to note that Theorem 3.3 immediately implies that $\mathcal{G}_\omega(\text{top}\, K_\omega)$ cannot have a universal graph.

Let $\mathcal{G}$ be a class of graphs. An element G_0 of $\mathcal{G}$ is called *weakly universal*, if every other element G of $\mathcal{G}$ can be embedded into G_0 as a not necessarily induced subgraph. That is, there exists $f : V(G) \to V(G_0)$ such that

$$xy \in E(G) \;\Rightarrow\; f(x)f(y) \in E(G_0).$$

As a matter of fact, in all the above mentioned cases when we were able to prove the non-existence of a universal element in some class of graphs $\mathcal{G}$, it was also true that $\mathcal{G}$ had no weakly universal graph. However, the two notions do not always coincide, as is indicated by the following result of Diestel [4].

Theorem 4.5 (Diestel). $\mathcal{G}_\omega$ (top $K_{2,3}$) *has a weakly universal element, but it does not contain a universal one.*

Very similar results are true for classes of graphs of type $\mathcal{G}_\omega$(hom $\mathcal{H}$), where hom H is defined as the family of all graphs that can be contracted to $\mathcal{H}$.

5. The complexity of a class of graphs

Given a class of graphs $\mathcal{G}$, let cp($\mathcal{G}$), the *complexity of* $\mathcal{G}$, be defined as the smallest cardinal κ such that there exist κ elements in $\mathcal{G}$ with the property that any other element is isomorphic to an induced subgraph of at least one of them. Obviously, $\text{cp}(\mathcal{G}) = 1$ means that $\mathcal{G}$ has a universal element.

For the sake of simplicity, in this section we will be concerned with the complexity of $\mathcal{G}_\omega(H)$, i.e., the class of all countable graphs with one forbidden subgraph. To simplify the notation, let

$$c(H) = \text{cp}(\mathcal{G}_\omega(H)).$$

If H is *connected,* then the union of disjoint H-free graphs is also H-free. This immediately implies that $c(H) \leq \omega$ if and only if $c(H) = 1$, i.e., $\mathcal{G}_\omega(H)$ has a universal graph.

However, if H is *disconnected* then $c(H)$ can be a natural number different from 1. For instance, if H consists of two disjoint edges, then K_3, $K_{1,\omega} \in \mathcal{G}_\omega(H)$, but K_3 has no proper extension belonging to this class. Hence, $c(H) \geq 2$. In fact, $c(H) = 2$, because any H-free graph is an induced subgraph of K_3 or $K_{1,\omega}$.

For any two graphs H_1 and H_2, let $H_1 + H_2$ denote the union of their vertex-disjoint copies. We write nH for $H + H + \cdots + H$, where the number of summands is n.

Theorem 5.1. *For* $2 \leq n < \omega$:

(i) $1 < c(nK_2) < \omega$,

(ii) $1 < c(K_n + K_2) < \omega$.

Proof. (i) To see that $\mathcal{G}_\omega(nK_2)$ does not possess a universal element, it is enough to observe that $(n-1)K_3$, $K_{1,\omega} \in \mathcal{G}_\omega(nK_2)$, but no proper extension of $(n-1)K_3$ belongs to this class.

The upper bound can be proved by induction on n. As we have seen before, it is true for $n = 2$. Assume that we have already proved that there are finitely many elements in $\mathcal{G}_\omega((n-1)K_2)$ such that any other element can be embedded into at least one of them, i.e., $c((n-1)K_2) < \omega$. Since $\mathcal{G}_\omega((n-1)K_2) \subseteq \mathcal{G}_\omega(nK_2)$, it is sufficient to find finitely many elements in $\mathcal{G}_\omega(nK_2)$ which embed every $G \in \mathcal{G}_\omega(nK_2) - \mathcal{G}_\omega((n-1)K_2)$. Fix such a G, and let $X \subseteq V(G)$, $|X| = 2n-2$, denote the vertex set of a maximum system of independent edges in G. We divide the remaining vertices of G into 2^{2n-2} disjoint classes (some of which might be empty) so that two vertices belong to the same class if and only if they are adjacent to the same elements of X. If a class contains at least n vertices, then we extend it to an infinite class by adding ω new vertices connected to X in the same way. Thus, we obtain a new graph $G^* \supseteq G$. Clearly, $G^* \in \mathcal{G}_\omega(nK_2)$, because any set of n independent edges in G^* could use only at most n vertices from the same class, therefore they would already occur in G. On the other hand, the number of different graphs that can be obtained as G^* is obviously finite. There are at most

$$2^{\binom{2n-2}{2}}$$

graphs induced by X, and the number of elements in each of the 2^{2n-2} classes of the remaining vertices of G^* is either an integer between 0 and $n-1$, or ω. Summarizing, we found at most

$$2^{\binom{2n-2}{2}}(n+1)^{2^{2n-2}}$$

elements in $\mathcal{G}_\omega(nK_2)$ such that any $G \in \mathcal{G}_\omega(nK_2) - \mathcal{G}_\omega((n-1)K_2)$ is an induced subgraph of at least one of them.

(ii) Clearly, K_{n+1}, $K_{1,\omega} \in \mathcal{G}_\omega(K_n + K_2)$, and K_{n+1} is a maximal element of this class. Hence, $c(K_n + K_2) > 1$.

As for the upper bound, by Proposition 3.1 it is sufficient to show that there are finitely many elements in $\mathcal{G}_\omega(K_n + K_2)$ such that any $G \in \mathcal{G}_\omega(K_n + K_2) - \mathcal{G}_\omega(K_n)$ is contained in at least one of them as an induced subgraph. Fix such a G and let $X \subseteq V(G)$, $|X| = n$, denote the vertex set of a maximum complete subgraph of G. We can classify the remaining vertices of G, as before, and extend each class of size at least 2 to an infinite class, to obtain a $(K_n + K_2)$-free graph $G^* \supseteq G$. The number of different graphs that can be obtained as G^* is again finite, as required. □

As a matter of fact, the above argument proves the following slightly more general statement.

Theorem 5.2. *Let H be a finite graph with $c(H) < \omega$. Then $1 < c(H + K_2) < \omega$.*

We conjecture that the same theorem holds for any K_r, $r \geq 3$. This would yield that if H is the sum of finitely many disjoint finite complete graphs, then $c(H) < \omega$.

To illustrate the difficulty of this question, we discuss another special case.

Theorem 5.3. $1 < c(2K_3) < \omega$.

Proof. The lower bound follows from the fact that $K_5 \in \mathcal{G}_\omega(2K_3)$ cannot be extended in the class by adding new triangles, hence there is no $2K_3$-free graph containing both K_5 and (say) the graph consisting of three edge-disjoint triangles with a point in common.

By Proposition 3.1, there exists a universal countable K_3-free graph. Let G_0 denote the graph obtained from this by replacing each vertex v be a set of two independent points $\{v_1, v_2\}$, and adding a new vertex which is connected to all v_1. Clearly, G_0 is a universal element in the class of all countable graphs whose triangles can be covered by one point. Thus, it is sufficient to show that there are finitely many elements in $\mathcal{G}_\omega(2K_3)$ such that any $2K_3$-free graph G whose triangles cannot be covered by one vertex is an induced subgraph of at least one of them.

Let us fix such a graph G.

Claim A. *There is a subset $A \subseteq V(G)$, $|A| \leq 7$ such that $|A \cap V(T)| \geq 2$ for every triangle $T \subseteq G$.*

Let v_1, v_2, v_3 be the vertices of a triangle $T_0 \subseteq G$. For every i, pick a triangle T_i (if it exists) such that $V(T_i) \cap \{v_1, v_2, v_3\} = \{v_i\}$. Let A be the union of the vertex sets of these triangles. Using the fact that G is $2K_3$-free, we get $|A| \leq 7$. Let $T \subseteq G$ be any triangle. Assume, in order to obtain a contradiction, that $V(T) \cap A = \{v_i\}$, and choose a triangle $S \subseteq G$ not covered by v_i. If $|V(S) \cap \{v_1, v_2, v_3\}| = 2$ then either S and T_i, or S and T are disjoint triangles. If $V(S) \cap \{v_1, v_2, v_3\} = \{v_j\}$, $j \neq i$, then T_j and T are disjoint. This contradiction proves Claim A.

Claim B. *There is a subset $B \subseteq V(G)$, $B \supseteq A$, $|B| \leq 8$ such that $B \cap V(T_1) \cap V(T_2) \neq \emptyset$ for any two triangles $T_1, T_2 \subseteq G$.*

If $B = A$ does not have the required property, then there are two triangles $T_1, T_2 \subset G$ with $V(T_1) \cap V(T_2) = \{x\}$ for some $x \notin A$. If $B = A \cup \{x\}$ does not satisfy the requirements, then one can choose two triangles $S_1, S_2 \subseteq G$ such that $V(S_1) \cap V(S_2) = \{y\}$ for some $y \notin A \cup \{x\}$. Since there are no two disjoint triangles in G, $(V(T_1) \cup V(T_2)) \cap A = (V(S_1) \cup V(S_2)) \cap A$ is a 4-element subset of A, all of whose points are connected to both x and y. But then T_1 and $(T_2 - \{x\}) \cup \{y\}$ are two disjoint triangles, a contradiction proving Claim B.

We define the *type* of G as the subgraph of G induced by B, and the collection $\mathcal{C}$ of those subsets $C \subseteq B$ for which there exists a vertex in G connected to all elements of C and none of $B - C$. Obviously, there are at most

$$2^{\binom{|B|}{2}} 2^{2^{|B|}}$$

different types.

Claim C. *The family of all graphs with a given type has a universal element.*

This can be shown by the usual treelike construction (see e.g. Theorem 3.7). We have to show only that the *amalgamation* property holds. Assume that G_1' and G_2' are finite induced subgraphs of two elements G_1 and G_2 of the class of type $(B, \mathscr{C})$. Let G_i'' denote the subgraph of G_i induced by $B \cup V(G_i')$, $i = 1, 2$. Then it is easy to see that $G_1'' \cup G_2''$ is $2K_3$-free and it can be extended to a $2K_3$-free graph of type $(B, \mathscr{C})$. This completes the proof of Claim C, and hence Theorem 5.3. □

It is obvious that $c(H) \leqslant 2^\omega$ for any forbidden subgraph H, because the total number of countable graphs is 2^ω. It is known that the following holds.

Theorem 5.4 (Hajnal, Komjáth). $c(K_\omega) = \omega_1$.

We will show here that $c(K_{1,\omega})$ can take any value between ω_1 and 2^ω.

Given two functions $f, g : \omega \to \omega$, we write $f \leqslant^* g$ if $f(n) \leqslant g(n)$ for all but finitely many n. A family of functions $\{f_\gamma\colon \gamma \in \Gamma\}$ is called *dominating* if for every $f : \omega \to \omega$ there exists $\gamma \in \Gamma$ with $f_\gamma \geqslant^* f$. The *domination number d* is defined as the smallest size $|\Gamma|$ of a dominating family. (See e.g. [3].)

As we mentioned in Section 3, $c(K_{1,\omega}) > 1$, i.e., the fact that the class of all countable, locally finite graphs does not have a universal element, was proved by de Bruijn. Now we prove the following stronger result.

Theorem 5.5. *$c(K_{1,\omega})$ is equal to the domination number d.*

Proof. Assume first that, for some $\kappa < d$, there is a system $\{G_\gamma\colon \gamma < \kappa\}$ of countable, locally finite graphs with $V(G_\gamma) = \omega$ and with the property that any other element of $\mathscr{G}_\omega(K_{1,\omega})$ is an induced subgraph of some G_γ. We can obviously assume that each G_γ is connected. Let $f_\gamma(n) < \omega$ denote the number of vertices of G_γ that can be reached for some vertex $i < n$ by a path of length at most n $(n = 1, 2, \ldots)$, and put $f_\gamma(0) = 1$. Since $\kappa < d$, these functions cannot form a dominating system. That is, one can find a function $g : \omega \to \omega$ such that for every $\gamma < \kappa$, $g(n) > f_\gamma(n)$ for infinitely many n. We can obviously suppose that $g(0) = 1$. Let G be a countable, locally finite graph on the vertex set ω such that the number of vertices that can be reached from 0 by a path of length n is at least $g(n)$, for every n. Assume that $h : V(G) \to V(G_\gamma)$ is an embedding of G into some G_γ, as an induced subgraph. However, if $h(0) = k$, then this implies that $f_\gamma(n) \geqslant g(n)$ for all $n > k$, contradicting the definition of G.

Next we prove that $c(K_{1,\omega}) \leqslant d$. It follows immediately from the definitions that we can also find a family of functions $\{f_\gamma\colon \gamma < d\}$ such that for every $g : \omega \to \omega$ there exists a γ with the (stronger) property that $g(n) \leqslant f_\gamma(n)$ for *all* $n < \omega$. The vertex set of any countable, locally finite graph G can be decomposed into disjoint finite parts $V_0 \cup V_1 \cup V_2 \cup \cdots$ such that no two points $x \in V_i$, $y \in V_j$, $j > i+1$, are joined by an edge. Let $g_G(n) = |V_n|$, $n < \omega$. Then there exists $\gamma < d$ such that $g_G(n) \leqslant f_\gamma(n)$ for every n. In this case we shall say that the *type of G* is γ. However, an easy recursive construction shows that there is a countable, locally finite graph G_γ containing every countable, locally finite graph of type γ as an induced subgraph. □

Finally, we note that many earlier results for the non-existence of universal elements in certain classes of graphs (especially those whose proof was based on the idea presented in [23]) can be generalized to stronger statements about the complexity of these classes. For example, the following result partially generalizes Theorem 3.5.

Theorem 5.6. (i) $c(K_{1,r}) = 2^\omega$ *for every* $4 \leqslant r < \omega$,
(ii) $c(K_{r,s}) = 2^\omega$ *for every* $2 \leqslant r \leqslant s < \omega$.

6. The complexity of a class of hypergraphs

The *complexity* of a class of hypergraphs $\mathscr{H}$ can be defined as the smallest cardinal κ such that there exist κ elements $H^*_\alpha \in \mathscr{H}$ $(\alpha < \kappa)$ with the property that any other element of $\mathscr{H}$ can be embedded into some H^*_α as an induced subhypergraph. (An *induced subhypergraph* of H^*_α consists of all hyperedges contained in a given subset of $V(H^*_\alpha)$, the vertex set of H^*_α.)

In order to simplify the exposition, we shall only consider classes of countable, 3-*uniform* hypergraphs, i.e., *triplet systems*, although the results clearly generalize to k-uniform hypergraphs for any $k \geqslant 3$.

Given a triplet system F, let $\mathscr{H}^3_\omega(F)$ denote the class of all countable 3-uniform hypergraphs containing no subhypergraph isomorphic to F. (These hypergraphs are also called *F-free*.) Let $c(F)$ denote the complexity of $\mathscr{H}^3_\omega(F)$.

A triplet system F is said to be *weakly complete*, if for any two elements $x, y \in V(F)$ there is a hyperedge (triplet) $T \in E(F)$ with $x, y \in T$.

Proposition 6.1. *If F is a weakly complete, finite triplet system, then* $c(F) = 1$.

Proof. To prove that $\mathscr{H}^3_\omega(F)$ has a universal element, it is sufficient to show that its finite members form an amalgamation class (see Theorem 3.7). Let H_1 and H_2 be two (not necessarily disjoint) 3-uniform F-free hypergraphs which coincide on $V(H_1) \cap V(H_2)$. If $H_1 \cup H_2$ contains an isomorphic copy of F, then it must intersect both $V(H_1) - V(H_2)$ and $V(H_2) - V(H_1)$. However, no pair of points $x \in V(H_1) - V(H_2)$, $y \in V(H_2) - V(H_1)$ is contained in a triplet of $H_1 \cup H_2$, contradicting the assumption that F is weakly complete. □

Clearly, $c(F) \leqslant 2^\omega$ for any F. Our next assertion shows that this bound can be attained.

Theorem 6.2. *Let F consist of two triplets differing only in one point, i.e., $V(F)=\{a, b, c, d\}$ and $F=\{\{a, b, c\}, \{a, b, d\}\}$. Then $c(F)=2^\omega$.*

Proof. Let us define a 3-uniform hypergraph H, as follows:

$$V(H)=\{x_i\colon 1\leq i\leq 8\}\cup\{y_j\colon 1\leq j<\omega\},$$
$$E(H)=\{\{x_i, y_j, y_{j+1}\}\colon j\equiv i \pmod 8\}.$$

Obviously, H is F-free. Furthermore, if $f_1, f_2\colon V(H)\to V(H^*)$ are two embeddings of H into an F-free hypergraph H^* such that

$$f_1(y_1)=f_2(y_1),\qquad f_1(x_i)=f_2(x_i)\quad 1\leq i\leq 8,$$

then $f_1\equiv f_2$. Indeed, $f_1(y_j)=f_2(y_j)$ follows by induction on j from the fact that H^* is F-free.

For any function $g\colon\omega\to\{0, 1\}$, let H_g be defined as the hypergraph obtained from H by adding the triplets

$$\{y_{32k+1}, y_{32k+9}, y_{32k+17+8g(k)}\}$$

for every $\kappa<\omega$. It is clear that H_g is also F-free.

Assume now, in order to obtain a contradiction, that there is a system of fewer than 2^ω F-free hypergraphs H^*_α such that every H_g can be embedded into at least one of them. Then one can find an α such that H_g can be embedded into H^*_α for 2^ω different g's. Let $f_g\colon V(H_g)\to V(H^*_\alpha)$ denote such an embedding. Since the number of 9-tuples in $V(H)$ is countable, there exist $g_1\neq g_2$ such that

$$f_{g_1}(y_1)=f_{g_2}(y_1),\qquad f_{g_1}(x_i)=f_{g_2}(x_i),\quad 1\leq i\leq 8.$$

This implies that $f_{g_1}\equiv f_{g_2}=:f$. Pick an integer k so that $g_1(k)\neq g_2(k)$. Obviously, both of $\{f(y_{32k+1}), f(y_{32k+9}), f(y_{32k+17})\}$ and $\{f(y_{32k+1}), f(y_{32k+9}), f(y_{32k+25})\}$ are contained in H^*_α, contradicting the assumption that H^*_α is F-free. □

Let K^3_n denote the system of all triplets of an n element set. If F_1 and F_2 are two triplet systems, then F_1+F_2 stands for the system obtained by taking the union of disjoint copies of F_1 and F_2. Similarly, $nF=F+F+\cdots+F$, where n is the number of terms.

Theorem 6.3. *For every $1<n<\omega$, $1<c(nK^3_3)<\omega$.*

Proof. The easy proof of the lower bound is left to the reader. To establish the upper bound, first we show that for any nK^3_3-free triplet system H one can find $A, B\subseteq V(H)$, $A\cap B=\emptyset$, $|A|=a<n$, $|B|\leq(3n-2)(3n-3)$, such that every triplet $T\in E(H)$ either intersects A or has at least two elements in common with B. Furthermore, there are no pairwise disjoint triples $T_1, T_2, \ldots, T_{n-a}\in E(H)$ with $T_i\cap A=\emptyset$ $(1\leq i\leq n-a)$.

Let $A \subseteq V(H)$, $|A| < n$ be a maximal subset satisfying the condition that H has no $n - |A|$ triplets disjoint from each other and from A. Let $T_1, T_2, \ldots, T_k \in E(H)$ be a maximal system of pairwise disjoint triplets in $V(H) - A$. If $|A| = n - 1$, then we are done. If $|A| < n - 1$ then, by the maximality of A, for any $t \in T_1 \cup T_2 \cup \cdots \cup T_k$ there exist $n - |A| - 1$ disjoint triplets $T_j^{(t)}$ $(1 \leqslant j \leqslant n - |A| - 1)$ in $V(H) - (A \cup \{t\})$. Set

$$B = \Big(\bigcup_{1 \leqslant i \leqslant k} T_i\Big) \cup \Big(\bigcup_{t \in \bigcup T_i,\, 1 \leqslant j \leqslant n - |A| - 1} T_j^{(t)}\Big).$$

Then $|B| \leqslant 3(n - |A| - 1) + 9(n - |A| - 1)^2 \leqslant (3n - 2)(3n - 3)$. On the other hand, if a triplet $T \in E(H)$ is disjoint from A then it has to intersect at least one T_i (by the maximality of k). If $|T \cap B| = 1$, then $T \cap B = \{t\}$ for some $t \in \bigcup T_i$, and T is disjoint from all $T_j^{(t)}$ $(1 \leqslant j \leqslant n - |A| - 1)$, contradicting the definition. Thus, A and B satisfy the requirements.

The points and the point pairs of $V(H) - A - B$ can now be classified (colored) according to the types of the triplets containing them. (The *type* of a triplet $T \in E(H)$ is its intersection with $A \cup B$.) This structure (coloured graph) has to omit a finite number of finite substructures (colored subgraphs). Now it is not difficult to build a universal structure of this kind, embedding all H having the same A, B and the same collection of types determined by the triplets of H. □

Theorem 6.4. $c(\omega K_3^3) = \omega$.

Proof. Let H be a countable, 3-uniform hypergraph containing no ωK_3^3. Let $T_1, T_2, \ldots, T_k \in E(H)$ be a non-extendable collection of triplets in H. Then H is $(3k+1)K_3^3$-free, so $c(\omega K_3^3) \leqslant \omega$ follows from Theorem 6.3. The same argument shows that finitely many ωK_3^3-free graphs cannot embed nK_3^3 if n is large, hence $c(\omega K_3^3) \geqslant \omega$. □

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