Some results on ends and automorphisms of graphs

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Abstract

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Automorphisms σ of a connected graph X satisfying $\sigma(F) \neq F$ for all finite non-empty subsets F of $V(X)$ are of particular interest in the theory of ends. Elementary properties of those automorphisms are studied and linked to the concept of a strip.

1. Introduction and notation

The central concepts in this paper are the notion of an *end* and the notion of a *translation* (type 2 automorphisms in [6]) of a graph X. We will distinguish *proper* translations (hyperbolic automorphisms in [16]) and nonproper translations (parabolic automorphisms in [16]).

First we relate some basic results of [6] to the concept of a *strip*, which was studied in [10]. Then characterizations of proper translations are supplied.

While the results in Section 2 are valid for arbitrary infinite graphs, the emphasis in Section 3 is on the difference between locally finite graphs and the general case.

The proofs are all elementary, and we do not elaborate on the topological aspects of the theory of ends (cf. [8, 11–12, 16]). Neither do we elaborate on the links of this theory to group-theoretical aspects, such as the growth of finitely generated groups. Here we refer the reader to [3, 13–16].

Given a graph X we denote for any subgraph H of X, by $X - H$ and $X - V(H)$ the maximum subgraph of X with vertex set $V(X) - V(H)$, and by ∂H the set of all vertices in $X - H$, which have at least one neighbor in H.

If R is a ray (that is, a one-way infinite path) in X and F a finite subset of $V(X)$, there is a unique component C of $X - F$, which contains all but finitely many vertices of R; we shall say that C *covers* R.

It is easy to see that the following relation $\sim$ is an equivalence relation on the set of rays in X: $R \sim R'$ if for each finite $F \subseteq V(X)$, the rays R and R' are covered by the same component C of $X - F$.

The classes with respect to $\sim$ are called *ends* in X (see [2] and [4]). Since for any finite $F \subseteq V(H)$ all rays in an end l in X are covered by the same component C of $X - F$, we may say that C *covers* l. For distinct ends l and l' in X there exists a finite set $F \subseteq V(X)$ such that l and l' are covered by distinct components of $X - F$. In that event we say that F *separates* l and l'. Clearly distinct ends of X are disjoint subsets of the set of rays in X.

Any element σ of the automorphism group $\operatorname{Aut} X$ of X maps any end l in X onto the end $\sigma(l) = \{\sigma(R): R \in (l)\}$. Clearly σ acts as a permutation of the set $L(X)$ of ends in X. This action of $\operatorname{Aut} X$ on $L(X)$ defines a canonical group homomorphism from $\operatorname{Aut} X$ into the group of permutations of $L(X)$.

An automorphism of X is called a *translation* of X (automorphism of type 2 in [6]), if $\sigma(F) \neq F$ for every finite non-empty subset of $V(X)$.

The following preliminary result is a useful tool in the later proofs.

Lemma 1.1. *For any translation σ of a graph X and any finite set $F \subseteq V(X)$ there exists a positive integer n such that $\sigma^i(F) \cap \sigma^j(F) = \emptyset$, whenever $|j - i| \geq n$.*

It suffices to show, that there is an integer $n > 0$ such that $\sigma^i(F) \cap F = \emptyset$, whenever $i \geq n$. That assertion follows immediately from the fact that for every $x \in V(X)$ the elements $\sigma^i(x)$ $(i \in \mathbb{Z})$ are all distinct.

2. Fixed ends and strips

Given $\sigma \in \operatorname{Aut} X$ we call a set Σ of subgraphs of X *σ-invariant* if $\sigma(Y) \in \Sigma$ for all $Y \in \Sigma$; if in addition for all $Y \in \Sigma$ there is a nonzero integer m such that $\sigma^m(Y) = Y$, we call Σ *σ-periodic*. A σ-invariant end of X will be called a *fixed end* of σ.

By definition a *strip* is a connected locally finite graph with exactly two ends, whose automorphism group contains a translation.

Theorem 2.1. *Let σ be a translation of the connected graph X, and let H be a finite connected subgraph of X such that $\sigma(H) \cap H \neq \emptyset$. Then $S = \bigcup_{i \in \mathbb{Z}} \sigma^i(H)$ is a strip, and $\sigma \mid S$ fixes both ends of S. Each of these ends is a subset of some fixed end of σ. Conversely if some end l of X contains no ray of S then $\sigma^i(l) \neq \sigma^j(l)$ whenever $i \neq j$.*

Proof. By Lemma 1.1 we can determine a positive integer n such that $\sigma^i(H) \cap \sigma^j(H) = \emptyset$ whenever $|j - i| \geq n$. Suppose $x \in \sigma^j(H)$ and $y \in \sigma^i(H)$ are adjacent in S. Then xy is an edge of some $\sigma^k(H)$. From $x \in \sigma^i(H) \cap \sigma^k(H)$ and $y \in \sigma^j(H) \cap \sigma^k(H)$ we deduce $|i - k| \leq n - 1$ and $|j - k| \leq n - 1$, and hence $|i - j| \leq 2n - 2$. Therefore S is locally finite. Moreover, for $\bar{H} = H \cup \sigma(H) \cup \cdots \cup$

$\sigma^{n-2}(H)$ there exist distinct components C_0^+ and C_0^- of $S-\bar{H}$ such that C_0^+ contains all $\sigma^i(H)$ $(i \geq 2n-2)$ and C_0^- contains all $\sigma^i(H)$ $(i \leq -n)$. Since both C_0^+ and C_0^- contain respectively rays R^+ and R^-, S has at least two ends. For any finite set $F \subseteq V(S)$, there exist integers $p < q$ such that $F \cap \sigma^i(H) = \emptyset$ whenever $i \leq p$ or $i \geq q$. Hence $S - F$ has at most two infinite components. So far we have shown that S has exactly two ends; one end l_0^+ of S contains R^+ and the other l_0^- contains R^-.

Clearly $\sigma_0 = \sigma \mid S$ is a translation of S, and l_0^+ and l_0^- are subsets of certain ends l^+ and l^- of X respectively, possibly $l^+ = l^-$. Since R^+ meets infinitely many $\sigma^i(H)$ $(i \geq 0)$, so does $\sigma(R^+)$ and so C_0^+ covers $\sigma_0(l_0^+)$. Since C_0^+ does not cover l_0^- we must have $\sigma_0(l_0^+) = l_0^+$. From $\sigma(l^+) \cap l^+ \supseteq \sigma_0(l_0^+) \cap l_0^+ = l_0^+$ we infer $\sigma(l^+) = l^+$. Similarly l_0^- is a fixed end of σ_0 and $\sigma(l^-) = l^-$.

Finally let l be an end of X other than l^+ and l^-. We choose a finite set $F \subseteq V(X)$ which separates in X the ends l, l^+ and also separates l, l^-. Then we find a finite connected subgraph H_1 of X such that $V(H_1) \supseteq F \cup V(H)$. As shown above, $S_1 = \bigcup_{i\in\mathbb{Z}} \sigma^i(H_1)$ is a strip with ends $l_1^+ \subseteq l^+$ and $l_1^- \subseteq l^-$ By Lemma 1.1 there is an integer $n_1 \geq n$ such that $\sigma^i(H_1) \cap \sigma^j(H_1) = \emptyset$ whenever $|i-j| \geq n_1$.

We choose $R \in l$ and show first that R meets only finitely many $\sigma^i(H_1)$ $(i \in \mathbb{Z})$. Otherwise the component of $X - H_1$ which covers l would contain all but finitely many $\sigma^i(H_1)$ $(i \geq 0)$ or all but finitely many $\sigma^i(H_1)$ $(i \leq 0)$. Since in X the set $V(H_1)$ separates l from l^+ and l^- we arrive at a contradiction.

Hence all but finitely many vertices of R are contained in some component C of $X - S_1$. Since $V(H_1)$ separates in X the ends l from l^+ and l^- the set ∂C can intersect only finitely many $\sigma^i(H_1)$ $(i \in \mathbb{Z})$. In particular ∂C is finite and thus C covers l. Let i be any nonzero integer. For all $j \in \mathbb{Z}$ clearly $\sigma^{ij}(C)$ is another component of $X - S_1$ and $\partial\sigma^{ij}(C) = \sigma^{ij}(\partial C)$. If $\sigma^i(C) \cap C \neq \emptyset$, that is $\sigma^i(C) = C$ then for all $j \in \mathbb{Z}$, $\partial C = \partial\sigma^{ij}(C) = \sigma^{ij}(\partial C)$, which by Lemma 1.1 is impossible.

By the preceding arguments $\sigma^i(C) \cap C = \emptyset$ whenever $i \neq 0$. Therefore $\sigma^i(C) \cap \sigma^j(C) = \emptyset$ whenever $i \neq j$. Since for all $k \in \mathbb{Z}$, $\sigma^k(C)$ covers $\sigma^k(l)$ we have shown that $\sigma^i(l) \neq \sigma^j(l)$ whenever $i \neq j$. □

From Theorem 2.1 we can regain the following results from [6].

Corollary 2.2. *Let σ be a translation of the connected graph X. Then σ has at least one and at most two fixed ends. If $m \in \mathbb{Z}/\{0\}$ then σ^m is a translation, and σ and σ^m have the same fixed ends.*

Proof. Since X is connected we can choose a vertex $x \in X$ and a path H in X joining x to $\sigma(x)$. Hence by Theorem 2.1, σ has at least one and at most two fixed ends.

For any integer $m \neq 0$ clearly σ^m is a translation of X and fixes every fixed end of σ. If σ has only one fixed end l^+, then again by Theorem 2.1, σ^m has no fixed end other than l^+. If σ has two fixed ends l^+ and l^- then they are fixed by σ^m. As already shown a translation has no more than two fixed ends. □.

We remark that Halin's proof for the 'at least'-part is more direct and probably simpler. We also should note that Halin's proof for the 'at most'-part works in general, although he imposed locally finiteness in his statement of this fact.

A translation σ of a connected graph will be called *proper* (*hyperbolic* in [16]) if σ fixes two ends.

Let σ be a translation of the connected graph X. If H is a finite connected subgraph of X such that $\sigma(H) \cap H \neq \emptyset$ then, by Theorem 2.1, some fixed end l_σ^+ of σ contains all rays in $\bigcup_{i \geq 0} \sigma^i(H)$. Note that l_σ^+ is the same end for every choice of H. Indeed, if H_1 and H_2 are two finite subgraphs of X one can find a finite connected subgraph $H \supseteq H_1 \cup H_2$. Then all rays in $\bigcup_{i \geq 0} \sigma^i(H)$ belong to the same fixed end of σ. This fixed end l_0^+ of σ is called the *direction* of σ [6]. If σ has two fixed ends, the fixed end l_σ^- of σ, which is distinct from l_0^+, contains all rays in $\bigcup_{i \leq 0} \sigma^i(H)$. Here H is as in Theorem 2.1 and separates l^-, l^+. In that case $l_\sigma^- = l_{\sigma^{-1}}^+$ and $l_{\sigma^{-1}}^- = l_\sigma^+$.

Theorem 2.3. *Let σ be a proper translation of the connected graph X and H a finite connected subgraph of X such that $\sigma(H) \cap H \neq \emptyset$ and H separates the fixed ends of σ. Then*

$$V\Big(\bigcup_{i \in \mathbb{Z}} \sigma^i(H)\Big)$$

induces a strip $\bar{S} \subseteq X$. Further there exists a finite subgraph $\bar{H}$ of $\bar{S}$ such that $V(\bar{H})$ separates in X the fixed ends of σ and for all components C of $X - \bar{S}$

(i) $\partial C \subseteq \sigma^k(\bar{H})$ *for some* $k \in \mathbb{Z}$ *and*

(ii) $\sigma^i(C) \cap \sigma^j(C) = \emptyset$ *whenever* $i \neq j$.

Proof. Let $S = \bigcup(\sigma^i(H): i \in \mathbb{Z})$. Let n be as provided by Lemma 1.1 for $F = V(H)$. We first show that the graph $\bar{S}$ which is induced by $V(S)$ in X is a strip. Since $V(H)$ separates l_σ^+ and l_σ^-, these ends are covered by distinct components of $X - H$.

One of these components, say C^+, contains all $\sigma^i(H)$ $(i \geq n)$ and another, say C^-, contains all $\sigma^i(H)$ $(i \leq -n)$ as subgraphs.

In particular $\sigma^{-n}(H)$ has no neighbor in $\bigcup_{i \geq n} \sigma^i(H)$. By applying appropriate powers of σ we obtain that $x \in \sigma^i(H)$ and $y \in \sigma^j(H)$ cannot be adjacent in X unless $|j - i| \leq 2n - 1$.

Therefore $\bar{S}$ is locally finite. Since $V(H)$ separates l^+ and l^- in X, $\bar{S}$ has exactly two ends $l_l^+ \subseteq l_\sigma^+$ and $l_1^- \subseteq l_\sigma^-$. Therefore $\bar{S}$ is a strip.

Set $\bar{H} = H \cup \sigma(H) \cup \cdots \cup \sigma^{2n-1}(H)$. We now consider a component C of $X - S = X - \bar{S}$. For any $i \in \mathbb{Z}$, $\sigma^i(C)$ is a component of $X - \bar{S}$ and $\partial\sigma^i(C) = \sigma^i(\partial C)$. If ∂C intersects $\sigma^j(H)$ then $\sigma^{-j-n}(\partial C)$ intersects $\sigma^{-n}(H)$ and hence does not intersect $\bigcup_{i \geq n} \sigma^i(H)$. Therefore ∂C does not intersect $\bigcup_{i \geq j+2n} \sigma^i(H)$. From this we infer that ∂C is finite and, choosing j as the minimum integer satisfying $\sigma^j(H) \cap \partial C \neq \emptyset$, $\partial C \subseteq \sigma^j(H) \cup \cdots \cup \sigma^{j+2n-1} = \sigma^j(\bar{H})$.

If $\sigma^i(C)=\sigma^j(C)$ for some distinct integers i and j, clearly $\sigma^{k(j-i)}(\partial C)=\partial C$ for all $k\in\mathbb{Z}$. This is not possible, since $\partial C\neq\emptyset$ and σ is a proper translation. □.

Corollary 2.4. *For $1\leq i\leq m$ let l_i be a fixed end of some proper translation σ_i of the connected graph X. There exists a finite subset F of $V(X)$, such that two distinct ends l and l' of X can be separated by $\tau(F)$ for some $\tau\in\operatorname{Aut}X$, whenever l or l' is in $\{\varphi(l_i): 1\leq i\leq m,\ \varphi\in\operatorname{Aut}X\}$.*

Proof. Clearly there is a finite set $F_0\subseteq V(X)$, which separates the elements of $\{l: l \text{ fixed end of some } \sigma_i,\ 1\leq i\leq m\}$ pairwise. We determine a connected finite subgraph H of X such that $\sigma_i(H)\cap H\neq\emptyset$ for $1\leq i\leq m$ and $V(H)\supseteq F_0$.

By Theorem 2.3 the set $V(\bigcup(\sigma_i^j(H): j\in\mathbb{Z}))$ induces a strip $\bar{S}_i$ $(1\leq i\leq m)$. Choose a subgraph $\bar{H}_i$ which satisfies (i) and (ii) of Theorem 2.3 with respect to σ_i. We claim that $F=V(\bar{H}_1\cup\cdots\cup\bar{H}_m)=F$ has the desired property.

Consider distinct ends l and l' of X, say $l=\varphi(l_1)$, for some $\varphi\in\operatorname{Aut}X$. As $\varphi^{-1}(l')\neq l_1$, the end $\varphi^{-1}(l')$ is the second fixed end of $\bar{S}_1$ or covered by a component C of $X-\bar{S}_1$. In either case, by Theorem 2.3, there is an integer k such that $\sigma_1^k(\bar{H}_1)$ separates l_1 and $\varphi^{-1}(l')$. Now $\varphi\sigma_1^k(\bar{H}_1)$ separates $\varphi(l_1)=l$ and $\varphi\varphi^{-1}(l')=l'$. As $V(\bar{H}_1)\subseteq F$, clearly $\varphi\sigma_1^k(F)$ separates l and l'. □

We state another result of Halin [6] as a corollary.

Corollary 2.5. *Let σ be a proper translation of the connected graph X, and let the finite subset F of $V(X)$ separate l_σ^+ and l_σ^-. Then any $l\in\{l_\sigma^+, l_\sigma^-\}$ contains at most $|F|$ pairwise disjoint rays.*

Proof. We choose a finite connected subgraph H of X such that $\sigma(H)\cap H\neq\emptyset$ and $V(H)\supseteq F$. Let $R_1,\ldots,R_t$ be pairwise disjoint rays in l_σ^+. Using Lemma 1.1 we determine a positive integer n such that $\sigma^i(H)\cap\sigma^j(H)=\emptyset$ whenever $|j-i|\geq n$.

Let C^+ and C^- be the components of $X-F$ which respectively cover l_σ^+ and l_σ^-. Note that $\bigcup_{i\leq -n}\sigma^i(H)\subseteq C^-$ and $\bigcup_{i\geq n}\sigma^i(H)\subseteq C^+$. In particular $H\subseteq\sigma^n(C^-)$ and consequently $H\cap\sigma^n(C^+)=\emptyset$. Thus $\sigma^n(C^+\cup H)$ spans a connected subgraph of $X-F$. From $\sigma^n(H)\subseteq C^+$ we deduce $\sigma^n(C^+\cup H)\subseteq C^+$ which of course implies $\sigma^n(C^+\cup\partial C^+)\subseteq C^+$. The last inequality will be used in the proof of Theorem 2.7.

For each positive integer i, $\sigma^{ni}(C^+)$ covers l_σ^+ and hence intersects every $R\in l_\sigma^+$. Choose $x\in\partial C^+$. One easily verifies that any vertex in $\sigma^{in}(C^+)$ has distance at least $i+1$ from x $(i\geq 0)$. Therefore $\bigcap_{i\geq n}\sigma^{ni}(C^+)=\emptyset$ and hence there is a positive integer i such that the terminal vertices of $R_1, R_2,\ldots,R_t$ are outside $\sigma^{ni}(C^+)$. For such an integer i each $R\in\{R_1,R_2,\ldots,R_t\}$ intersects $\partial\sigma^{ni}(C^+)$. Now

$$\partial\sigma^{ni}(C^+)=\sigma^{ni}(\partial C^+)\subseteq\sigma^{ni}(F)$$

hence $t\leq|\sigma^{ni}(F)|=|F|$.

Since $l_\sigma^-=l_{\sigma^{-1}}^+$ the proof of the corollary is complete. □

The proof of Theorem 7 in [6] yields the following slightly stronger version of that result. For completeness we sketch the proof.

Proposition 2.6. *Let σ be a translation of the connected graph X. There exists a non-empty σ-periodic set of pairwise disjoint double rays in X.*

Proof. Among all paths P' such that $\sigma^m(P') \cap P' \neq \emptyset$ for some $m > 0$ choose a minimal one and call it P. Then $\sigma^m(P) \cap P \neq \emptyset$ for some $m > 0$, i.e. $\sigma^m(x) \in P$ for some $x \in P$. By the minimality of P we have $P = P[x, \sigma^m(x)]$. Moreover, $\sigma^i(P) \cap \sigma^j(P) = \emptyset$ whenever $m \neq |j - i|$, and $\sigma^{(j-1)m+i}(P) \cap \sigma^{jm+i}(P) = \sigma^{jm+i}(x)$.

Setting $Q_i = \bigcup_{j \in \mathbb{Z}} \sigma^{jm+i}(P)$ for all $i \in \mathbb{Z}$ we see that $\{Q_0, Q_1, \ldots, Q_{m-1}\}$ is a σ-invariant set of pairwise disjoint double rays. □

In the next result we establish a necessary and sufficient condition for an automorphism to be a proper translation.

Theorem 2.7. *An automorphism σ of a connected graph X is a proper translation of X if and only if there exists an induced connected subgraph C and a positive integer n such that $0 < |\partial C| < \infty$ and $\sigma^n(C \cup \partial C) \subseteq C$.*

Proof. If σ is a proper translation, we choose a finite subset F of $V(X)$ which separates l_σ^+ and l_σ^-. With n, H and C^+ as constructed in the proof of Corollary 2.5 we have $\sigma^n(C^+ \cup H) \subseteq C^+$ and $\partial C^+ \subseteq F \subseteq H$.

Now assume $\sigma^n(C \cup \partial C) \subseteq C$ for some positive integer n and some induced connected subgraph C of X with infinite ∂C. Abbreviate $\tau = \sigma^n$. If $i > 0$ then clearly $d(x, y) \geq i$ whenever $x \in \partial C$ and $y \in \tau^i(\partial C)$, hence $d(\partial C, \tau^i(\partial C)) \geq i$.

For any finite non-empty $F \subseteq V(X)$ and any positive integer i we have

$$d(\partial C, \tau^i(F)) \geq d(\partial C, \tau^i(\partial C)) - d(\tau^i(F), \tau^i(\partial C)).$$

From $d(\tau^i(F), \tau^i(\partial C)) = d(F, \partial C)$ we infer $d(\tau^i(F), \partial C) \geq i - d(F, \partial C)$. Therefore we cannot have $\tau(F) = F$. Thus we have shown that τ is a translation. Finally we choose some $x \in \partial C$ and some path H in X joining x to $\tau(x)$. Abbreviate $n = |V(H)| - 1$.

For any $i \in \mathbb{Z}$ we have $\tau^i(x) \in \tau^i(\partial C) \subseteq \tau^{i-1}(C) - \tau^i(C)$.

From $\tau^{i+1+m}(x) \in \tau^i(C)$ and $d(\tau^{i+m+1}(x), \partial\tau^i(C)) \geq m + 1$ we deduce that the path $\tau^{i+m}(H)$ of length m is a subgraph of $\tau^i(C)$.

From $\tau^{i-m}(x) \notin \tau^i(C)$ and $d(\tau^{i-m}(x), \partial\tau^i(C)) \geq m$ we similarly obtain $\tau^{i-m}(C) \supseteq \tau^{i-m-1}(H) \cap \tau^i(C) = \emptyset$. Therefore $\bigcup_{i \geq m} \tau^i(H) \subseteq C$ and C covers all rays of $l_{\tau/S}^+$, where S denotes the strip $\bigcup_{i \in \mathbb{Z}} \tau^i(H)$. Furthermore, $C \cap \bigcup_{i \leq -m-1} \tau^i(H) = \emptyset$ and so C does not cover any ray in $l_{\tau/S}^-$. Hence $l_\tau^+ \neq l_\tau^-$. By Corollary 2.2 also σ is proper. □

3. Nonproper translations and groups of automorphisms

Taking the viewpoint that an end l of X is a 'limit' vertex or a fictitious vertex, we may have the situation that ends l corresponds to real vertices. We call the vertex v of X a *main vertex* of l if for each finite $F \subseteq V(X) - \{v\}$ the component of $X - F$ which contains v covers l (see [7]). It is easy to see that v is a main vertex of l if and only if for each (equivalently some) $R \in l$ there exists an infinite system of openly disjoint paths joining v to distinct elements on R. Given a graph X_0, we call each graph of the form

$$X_0 \cup \bigcup(P_n \colon n = 1, 2, \ldots)$$

a *v-extension* of X_0 if $P_n = P_n[v, w_n]$ $(n = 1, 2, \ldots)$ are openly disjoint paths with distinct endvertices w_n in X_0 and $P_n(v, w_n) \cap X_0 = \emptyset$ $(n = 1, 2, \ldots)$. Note that v may be an element of $V(X_0)$. A union of v_n-extensions $\tilde{X}_n$ of X_0 $(n = 1, 2, \ldots)$ is called an *infinite* extension of X_0 if all v_n $(n = 1, 2, \ldots)$ are distinct and the graphs $\tilde{X}_n - V(X_0)$ $(n = 1, 2, \ldots)$ are pairwise disjoint.

If l is a fixed end of σ and x is a main vertex of l, then clearly $\sigma^i(x)$ is a main vertex of $\sigma^i(l) = l$ for all $i \in \mathbb{Z}$. Therefore Theorem 2.3 yields that the fixed ends of a proper translation in the connected graph X cannot have main vertices.

The following lemma is a useful tool for the construction of v-extensions.

Lemma 3.1. *Let $R_1, R_2, \ldots, R_k$ be rays in the end l of X, and $\bar{F} = V(R_1 \cup \cdots \cup R_k)$. Each component C of $X - \bar{F}$ with infinite ∂C contains a ray of l or a main vertex of l.*

Proof. Let C be a component of $X - \bar{F}$ with infinite ∂C. Without loss of generality we may assume that $V(R_1) \cap \partial C$ is infinite.

We first construct paths $P_n = P_n[v_n, w_n]$ such that $v_n \in V(C)$, $w_n \in V(R_1) \cap \partial C$ and $P_n[v_n, w_n) \subseteq C$ $(n = 1, 2, \ldots)$.

We start by choosing $v_1 \in V(C)$, $w_1 \in V(R_1) \cap \partial C$ and a path P_1 in X, which joins v_1 to w_1 and has inner vertices in C.

Suppose $P_1 = P_1[v_1, w_1], \ldots, P_n = P_n[v_n, w_n]$ have been constructed. We determine a vertex

$$w_{n+1} \in V(R_1) \cap \partial C - \{w_1, \ldots, w_n\}$$

and a path P_{n+1} joining w_{n+1} to a vertex in $V(P_1 \cup \cdots \cup P_n) \cap V(C)$. We can choose $P_{n+1} = P_{n+1}[v_{n+1}, w_{n+1}]$ such that

$$P_{n+1}[v_{n+1}, w_{n+1}) \subseteq C \quad \text{and} \quad P_{n+1} \cap (P_1 \cup \cdots \cup P_n) = v_{n+1}.$$

By construction $T = \bigcup(P_n \colon n = 1, 2, \ldots)$ is *s* tree. If some $v \in V(T)$ has infinite degree in T, then $v = v_n$ for infinitely many n and hence T contains a v-extension of R_1. Now we assume that T is locally finite and choose a ray R in T. For any $v \in V(R)$ let $n(v)$ be the minimum integer n such that v is on P_n. Clearly there is a system $v_1', v_2', \ldots$ of vertices on R such that $n(v_1') < n(v_2') < \cdots$. By construction the paths $P_{n(v_i')}$ $(i = 1, 2, \ldots)$ are pairwise disjoint. Hence R is in l. $\square$

In the following two lemmas we assume that a nonproper translation σ of a connected graph X is given, that Σ is a maximal σ-periodic set of pairwise disjoint double rays in X, and that

$$\bar{F} = \bigcup (V(Q)\colon Q \in \Sigma).$$

Lemma 3.2. *For each component C of $X - \bar{F}$ we have $\sigma^i(C) \cap \sigma^j(C) = \emptyset$ whenever i and j are distinct integers.*

Proof. If $\sigma^i(C) \cap \sigma^j(C) \neq \emptyset$ for some distinct i and j, then $\sigma^i(C) = \sigma^j(C)$ and hence there is a minimum positive integer q such that $\sigma^q(C) = C$. Then $\sigma^q \mid C$ is a translation of C and hence, by Proposition 2.6, there exists a σ^q-invariant set $\{Q_1, \ldots, Q_k\}$ $(k \geq 1)$ of pairwise disjoint double rays in C. Since $C, \sigma(C), \ldots, \sigma^{q-1}(C)$ are pairwise disjoint, the set

$$\Sigma \cup \{\sigma^j(Q_i)\colon 1 \leq i \leq k, 0 \leq j < q\}$$

would be a σ-periodic set of pairwise disjoint double rays which contains Σ as a proper subset. $\square$

Lemma 3.3. *Let $Q \in \Sigma$ and let C be a component of $X - \bar{F}$. If $V(Q) \cap \partial C$ is infinite, then there exist an integer $q > 0$ and an infinite extension X_1 of Q such that $\sigma^q(X_1) = X_1$ and $V(X_1), \ldots, \sigma^{q-1}(V(X_1))$ are pairwise disjoint subsets of $V(Q \cup \cdots \cup \sigma^{q-1}(Q)) \cup (V(X) - \bar{F})$.*

Proof. Let q denote the minimum positive integer such that $\sigma^q(Q) = Q$. Choosing z on Q and setting $H = Q[z, \sigma^q(z)]$ we obtain $Q = \bigcup (\sigma^{iq}(H)\colon i \in \mathbb{Z})$. Assuming that $V(Q) \cap \partial C$ is infinite we can find a vertex x in H such that the set $S = \{i \in \mathbb{Z}\colon \sigma^{iq}(x) \in \partial C\}$ is infinite. Let $S' = \{\sigma^{iq}(x)\colon i \in S\}$.

We first construct a subtree T of X such that S' is the set of endvertices of T and $V(T) \subseteq V(Q \cup C)$. To this end we label $S' = \{w_1, w_2, \ldots\}$ and start the construction by choosing a path T_2 in X, which joins w_1 to w_2 and has all its inner vertices in C. Assuming that a finite tree $T_n \subseteq X$ with $V(T_n) \subseteq V(Q \cup C)$ and set of end vertices $\{w_1, \ldots, w_n\}$ has already been constructed, we add to T_n a minimal path joining w_{n+1} to a vertex of T_n to obtain a tree T_{n+1} with set of endvertices $\{w_1, \ldots, w_{n+1}\}$. Clearly $T = T_2 \cup T_3 \cup \cdots$ has the asserted properties. We distinguish two cases.

Case 1: *Some vertex $v \in V(T)$ has infinite degree in T.*

Since T-$\{v\}$ has infinitely many components we can determine openly disjoint paths $P_n = P_n[v, z_n]$ $(n = 1, 2, \ldots)$ in T such that $z_n \in \{\sigma^{iq}(x)\colon i \in S\}$ $(n = 1, 2, \ldots)$ and all z_n $(n = 1, 2, \ldots)$ are distinct. Then $z_n = \sigma^{i_n \cdot q}(x)$ where $i_n \in S$ $(n =$

$1, 2, \ldots$). Thus $Q' = Q \cup \bigcup (P_n : n = 1, 2, \ldots)$ is a v-extension of Q. Hence for all $j \in \mathbb{Z}$ the graph $\sigma^j(Q')$ is a $\sigma^j(v)$-extension of $\sigma^j(Q)$ such that $\sigma^j(Q') - \sigma^j(Q) \subseteq \sigma^j(C)$. Therefore by Lemma 3.2 the graph

$$X_1 = \bigcup (\sigma^{jq}(Q') : j \in \mathbb{Z})$$

is an infinite extension of Q. Clearly $\sigma^q(X_1) = X_1$, and for every integer r the graph $\sigma^r(X_1)$ is an infinite extension of $\sigma^r(Q)$. Consider integers r_1, r_2 such that $0 \leq r_1 < r_2 < q$. Since $\sigma^{r_1}(Q) \cap \sigma^{r_2}(Q) = \emptyset$,

$$\sigma^{r_1}(X_1 - Q) \subseteq \bigcup (\sigma^{jq+r_1}(C) : j \in \mathbb{Z})$$

and

$$\sigma^{r_2}(X_1 - Q) \subseteq \bigcup (\sigma^{jq+r_2}(C) : j \in \mathbb{Z})$$

we obtain $\sigma^{r_1}(X_1) \cap \sigma^{r_2}(X_1) = \emptyset$.

Case 2: *T is locally finite.*

We construct sequences i_n, j_n and P_n ($n = 1, 2, \ldots$), where P_n is a path in T, which joins $\sigma^{i_n q}(x)$ to $\sigma^{j_n q}(x)$.

We start by choosing i_1 and j_1 in S such that $i_1 < j_1$ and letting P_1 denote the path in T, which joins $\sigma^{i_1 q}(x)$ to $\sigma^{j_1 q}(x)$. Suppose that the path $P_1, P_2, \ldots, P_n$ in T have already been constructed. Since T is locally finite, $T - (P_1 \cup \cdots \cup P_n)$ has only finitely many components, and therefore some component T_{n+1} contains infinitely many elements of S'. Thus we can find i_{n+1} and j_{n+1} in S, such that $j_{n+1} - i_{n+1} \geq n + 1$. Let P_{n+1} denote the path in T_{n+1} joining $\sigma^{i_{n+1} q}(x)$ to $\sigma^{j_{n+1} q}(x)$.

By construction the paths P_n ($n = 1, 2, \ldots$) are pairwise disjoint, further $d(n) = j_n - i_n \geq n$ ($n = 1, 2, \ldots$). Hence we can find an infinite set M of positive integers such that all $d(n)$ ($n \in M$) are distinct and $1 \notin M$. Set $z = \sigma^{i_1 q}(x)$.

For any $n \in M$ the path $P'_n = \sigma^{(i_1 - i_n)q}(P_n)$ joins $\sigma^{i_1 q}(x) = z$ to $\sigma^{(i_1 - i_n + j_n)q}(x) = \sigma^{d(n)q}(z)$ and has inner vertices in $\sigma^{(i_1 - i_n)q}(C)$. Therefore, by Lemma 3.2, the graph

$$Q^* = Q \cup \bigcup (P'_n : n \in M)$$

is a z-extension of Q and $V(Q^*) - V(Q) \subseteq V(X) - \bar{F}$.

Suppose $\sigma^{aq}(P'_n)$ and $\sigma^{bq}(P'_m)$ have a common vertex outside Q, where $n, m \in M$. This means that $\sigma^{(a+i_1-i_n)q}(P_n)$ and $\sigma^{(b+i_1-i_m)q}(P_m)$ have a common vertex w outside Q. But

$$w \in \sigma^{(a+i_1-i_n)q}(C) \cap \sigma^{(b+i_1-i_m)q}(C)$$

hence, by Lemma 3.2, $a + i_1 - i_n = b + i_1 - i_m$. Consequently $P_n = P_m$ hence $n = m$ and $a = b$.

In particular the graph $X_1 = \bigcup (\sigma^{jq}(Q^*) : j \in \mathbb{Z})$ is an infinite extension of Q. Clearly $\sigma^q(X_1) = X_1$.

Now consider integers r_1 and r_2 such that $0 \leq r_1 < r_2 < q$. If $\sigma'_1(X_1)$ and $\sigma'_2(X_1)$ had a common vertex then X_1 and $\sigma^{r_2 - r_1}(X_1)$ would have a common vertex w.

Since $Q \cap \sigma^{r_2-r_1}(Q) = \emptyset$ the vertex w would be in $X - \bar{F}$ and hence be an inner vertex of some P'_n and some $\sigma^{r_2-r_1}(P'_m)$, where $n, m \in M$. Consequently

$$w \in \sigma^{(i_1-i_n)q}(C) \cap \sigma^{r_2-r_1+(i_1-i_m)q}(C)$$

and by Lemma 3.2, $(i_1 - i_n)q = r_2 - r_1 + (i_1 - i_m)q$. Since $0 < r_2 - r_1 < q$ this equality is absurd.

Thus we have shown that $\{X_1, \sigma(X_1), \ldots, \alpha^{q-1}(X)\}$ is a σ-invariant set of pairwise disjoint graphs. □

Theorem 3.4. *Let σ be a nonproper translation of the connected graph X. If Σ is a maximal finite σ-periodic set of pairwise disjoint double rays in X then there exist $q > 0$ and an infinite extension X_1 of some $Q \in \Sigma$ such that $\sigma^q(X_1) = X_1$ and $X_1, \sigma(X_1), \ldots, \sigma^{q-1}(X_1)$ are pairwise disjoint.*

Proof. Let $\Sigma = \{Q_1, \ldots, Q_k\}$. For any subgraph H of X and any subset T of $V(X)$ abbreviate

$$\bar{H} = \bigcup(\sigma^i(H): i \in \mathbb{Z}) \quad \text{and} \quad \bar{T} = \bigcup(\sigma^i(T): i \in \mathbb{Z});$$

clearly $\sigma(\bar{H}) = \bar{H}$ and $\sigma(\bar{T}) = \bar{T}$. For $1 \leq j \leq k$ let q_j denote the minimum positive integer such that $\sigma^{q_j}(Q_j) = Q_j$. Choosing for any j, $1 \leq j \leq k$, a vertex x_j on Q_j and setting

$$H_j = Q_j[x_j, \sigma^{q_j}(x_j)]$$

we obtain $Q_j = \bigcup(\sigma^{iq_j}(H_j): i \in \mathbb{Z})$. Setting

$$F = V(H_1 \cup \cdots \cup H_k)$$

we obtain $\bar{F} = V(Q_1 \cup \cdots \cup Q_k)$.

First we construct a v-extension $Q' = Q \cup \bigcup(P_n: n = 1, 2, \ldots)$ of some $Q \in \Sigma$ such that $V(Q') - V(Q) \subseteq V(X - \bar{F}) \cup \{v\}$.

As σ is nonproper, for any $n = 1, 2, \ldots$ the set $F_n = \bigcup(\sigma^i(F): -n < i < n)$ does not separate in X the ends of Q_1. Hence we can determine a path P'_n in $X - F_n$ having a terminal vertex y_n in $\bigcup(\sigma^i(F): i \leq -n)$ and a terminal vertex z_n in $\bigcup(\sigma^i(F): i \geq n)$ and such that the inner vertices of P'_n, if any, are in $X - \bar{F}$. Choosing i_n and j_n such that $y_n \in \sigma^{i_n}(F)$ and $z_n \in \sigma^{j_n}(F)$ we have $\sigma^{-i_n}(y_n), \sigma^{-j_n}(z_n) \in F$ $(n = 1, 2, \ldots)$. Therefore we can determine an infinite set M_0 of integers such that all $\sigma^{-i_n}(y_n)$ $(n \in M_0)$ are the same vertex $y \in F$ and all $\sigma^{-j_n}(z_n)$ $(n \in M_0)$ are the same vertex $z \in F$.

Let us abbreviate $d(n) = j_n - i_n$ $(n = 1, 2, \ldots)$. As $d(n) \geq 2n$ $(n = 1, 2, \ldots)$ we can determine an infinite subset M_1 of M_0 such that all $d(n)$ $(n \in M_1)$ are distinct. Exploiting the fact that

$$\sigma^{-i_n}(z_n) = \sigma^{d(n)}(z) \quad \text{and} \quad \sigma^{-j_n}(y_n) = \sigma^{d(n)}(y)$$

are elements of $\bar{F} = V(Q_1 \cup \cdots \cup Q_k)$ we can find an infinite set $M_2 \subseteq M_1$ and elements Q_κ, Q_λ in $\{Q_1, \ldots, Q_k\}$, such that all $\sigma^{-d(n)}(y)$ $(n \in M_2)$ are in $V(Q_\kappa)$ and all $\sigma^{d(n)}(z)$ $(n \in M_2)$ are in $V(Q_\lambda)$.

Now observe that for all $n \in M_2$ the path $P_n'' = \sigma^{-i_n}(P_n')$ joins y to $\sigma^{d(n)}(z)$ and has inner vertices, if any, in a unique component C_n' of $X - \bar{F}$. If for some infinite subset M_3 of M_2 the paths P_n'' are openly disjoint then

$$Q_\lambda' = Q_\lambda \cup \bigcup (P_n'' \colon n \in M_3)$$

is a y-extension of Q_λ.

In the remaining case all but finitely many $n \in M_2$ satisfy $|V(P_n)| \geq 3$. Moreover, there is an infinite subset M_3 of M_2 such that $|V(P_n)| \geq 3$ for all $n \in M_3$, and all C_n' $(n \in M_3)$ are the same component C' of $X - \bar{F}$. For all $n \in M_3$ the path $\sigma^{-j_n}(P_n') = \sigma^{i_n - j_n}(P_n'')$ joins $\sigma^{-j_n}(y_n) = \sigma^{d(n)}(y)$ to $\sigma^{-j_n}(z_n) = z$ and has inner vertices in $\sigma^{-d(n)}(C_n') = \sigma^{-d(n)}(C')$. Thus

$$Q_\kappa' = Q_\kappa \cup \bigcup (\sigma^{-d(n)}(P_n'') \colon n \in M_3)$$

is a z-extension of Q_κ.

This completes the construction of the v-extension $Q' = Q \cup \bigcup (P_n \colon n = 1, 2, \ldots)$ of Q, where $Q \in \Sigma$. Let q denote the minimum positive integer such that $\sigma^q(Q) = Q$. In view of Lemma 3.3 we may assume that for any component C of $X - \bar{F}$ the boundary ∂C is finite. Let $P_n = P_n[v, w_n]$ $(n = 1, 2, \ldots)$.

We claim that for every positive integer n the set M_n of all positive integers m, such that $n \neq m$ and $(\bar{P}_n - \bar{F}) \cap (\bar{P}_m - \bar{F}) \neq \emptyset$, is finite. Assume to the contrary that for some positive integer n_0 the set M_{n_0} is infinite. Then for all $n \in M_{n_0}$ the paths P_{n_0} and P_n have inner vertices in respectively C_{n_0} and C_n. Abbreviate $P = P_{n_0}$, $M = M_{n_0}$ and $C = C_{n_0}$. By assumption there exist for each $n \in M$ integers i_n and j_n such that $\sigma^{i_n}(P)$ and $\sigma^{j_n}(P_n)$ have inner vertices in common, consequently $\sigma^{i_n}(C) = \sigma^{j_n}(C_n)$. Note that $v, w_n \in \partial C_n$ for all $n \in M \cup \{n_0\}$. Abbreviating $d(n) = j_n - i_n$ we have $C = \sigma^{d(n)}(C_n)$ and consequently

$$\sigma^{d(n)}(v), \sigma^{d(n)}(w_n) \in \sigma^{d(n)}(\partial C_n) = \partial C$$

for all $n \in M$. But clearly $\{\sigma^{d(n)}(v) \colon n \in M\}$ or $\{\sigma^{d(n)}(w_n) \colon n \in M\}$ is infinite, contrary to our assumption, that ∂C be finite.

Having proved the above claim, we can easily construct an infinite set M of positive integers such that $(\bar{P}_n - \bar{F}) \cap (\bar{P}_m - \bar{F}) = \emptyset$ whenever n and m are distinct elements of M. Set

$$Q^* = Q \cup \bigcup (P_n \colon n \in M).$$

As Q^* is a v-extension of Q, for all $i \in \mathbb{Z}$ the graph $\sigma^i(Q^*)$ is a $\sigma^i(v)$-extension of $\sigma^i(Q)$.

We claim that $X_1 = \bigcup (\sigma^{jq}(Q^*) \colon j \in \mathbb{Z})$ is an infinite extension of Q. Assume to the contrary that $\sigma^{j_1 q}(Q^*) - Q$ and $\sigma^{j_2 q}(Q^*) - Q$ have a common vertex w_0, where $j_1 \neq j_2$, then w_0 is in $X - \bar{F}$ and hence an inner vertex of some $\sigma^{j_1 q}(P_{n_1})$ and of some $\sigma^{j_2 q}(P_{n_2})$, where $n_1, n_2 \in M$. Then $n_1 = n_2$ since otherwise $(\bar{P}_{n_1} - \bar{F}) \cap (\bar{P}_{n_2} - \bar{F}) \neq \emptyset$.

Since $w_0 \in \sigma^{j_1q}(C_{n_1}) \cap \sigma^{j_2q}(C_{n_1})$ we obtain a contradiction to Lemma 3.2.

Clearly $\sigma^q(X_1) = X_1$, and for all $0 \leq r < q$, the graph $\sigma^r(X_1)$ is an infinite extension of $\sigma^r(Q)$. It remains to show that $X_1, \sigma(X_1), \ldots, \sigma^{q-1}(X_1)$ are pairwise disjoint. Suppose $\sigma^{r_1}(X_1)$ and $\sigma^{r_2}(X_1)$ have a common vertex, where $0 \leq r_1 < r_2 < q$. Then X_1 and $\sigma^{r_2-r_1}(X_1)$ have a common vertex w. As

$$Q \cap \sigma^{r_2-r_1}(Q) = \emptyset \quad \text{and} \quad \{\sigma^{jq}(v): j \in \mathbb{Z}\} \cap \{\sigma^{jq+r_2-r_1}(v): j \in \mathbb{Z}\} = \emptyset$$

the vertex w is an inner vertex of some P_{n_1} and of some $\sigma^{r_2-r_1}(P_{n_2})$, where $n_1, n_2 \in M$. By the properties of M we must have $n_1 = n_2$, consequently $C_{n_1} \cap \sigma^{r_2-r_1}(C_{n_1}) \neq \emptyset$ contrary to Lemma 3.2.

Thus $\{X_1, \sigma(X_1), \ldots, \sigma^{q-1}(X_1)\}$ is a σ-invariant set of pairwise disjoint infinite extensions of respectively $Q, \sigma(Q), \ldots, \sigma^{q-1}(Q)$. □

Corollary 3.5. *If σ is a nonproper translation of the connected graph X then the fixed end of σ in X contains infinitely many pairwise disjoint rays in X.*

Proof. Consider a maximal σ-invariant periodic set Σ of pairwise disjoint double rays. If Σ is infinite, clearly the ends of any $Q \in \Sigma$ are subsets of l_σ^+.

If $\Sigma = \{Q_1, \ldots, Q_n\}$ we determine q, Q and X_1 according to the theorem. It is easy to construct infinitely many pairwise disjoint rays in X_1. Clearly those rays are in l_σ^+. □

Theorem 3.6. *Let X be a connected graph with at least two ends, and let G be a subgroup of* Aut *X which has only finitely many orbits on $V(X)$. If G is vertex-transitive, or at least two ends of X have no main vertices, then G contains a proper translation.*

Proof. If G is a vertex transitive, we invoke Theorem 1 in [8], which says that there exist an induced subgraph C of X and an element σ of G satisfying $0 < |\partial C| < \infty$ and $\sigma(C \cup \partial C) \subseteq C$. Then σ is a proper translation by Theorem 2.7.

Now assume that there exist distinct ends l_1 and l_2, neither of which has a main vertex. Clearly there is a finite connected subgraph H of X, which separates l_1 and l_2 and contains additionally at least one element from each orbit of G on $V(X)$. Let C_1 and C_2 denote the components of $X - H$ which cover respectively l_1 and l_2.

We first show that, for $i = 1$ and 2, there exists a vertex x_i in C_i such that $d(x_i, H) > \operatorname{dia} H$, that is $\min(d(x_i, v): v \in V(H)) > \max(d(v, w): v, w \in V(H))$. Assuming the contrary we also may assume that C_1 contains no such vertex x_1. Setting $\operatorname{dia}(H) = k$ we obtain a decomposition

$$V(H) \cup V(C_1) = V_0 \cup V_1 \cup \cdots \cup V_k,$$

where $V_i = \{v \in V(H \cup C_1): d(v, H) = i\}$ $(0 \leq i \leq k)$. For each $v \in V_i$ $(0 < i \leq k)$ we choose a unique edge which is incident with v and some vertex in V_{i-1}.

Clearly all chosen edges define a forest with vertex set $V(C_1 \cup H)$ and each component T of that forest contains a unique vertex $v(T)$ in $V(H)$. Further we determine a ray R in l_1 such that $R \subseteq C_1$. Some component T of the above forest contains infinitely many vertices of R, say the distinct vertices $w_1, w_2, \ldots$ For $n = 1, 2, \ldots$ let P_n denote the path in T which connects $v(T)$ to w_n. As T has finite diameter we can determine a vertex w in T at maximum distance from $v(T)$ in T, such that w is on infinitely many P_n, say on $P_{n_1}, P_{n_2}, \ldots$ $(n_1 < n_2 < \cdots)$. By the choice of w there exists an infinite subsequence of $P_{n_1}[w, w_{n_1}], P_{n_2}[w, w_{n_2}], \ldots$ which defines an infinite set of openly disjoint paths. Therefore w is a main vertex of l_1, a contradiction.

Thus for $i = 1$ and 2 we can find x_i in C_i such that $d(x_i, H) > \operatorname{dia} H$, further an element $y_i \in V(H)$ and an element σ_i in G such that $\sigma_i(y_i) = x_i$. Then for $i = 1$ and 2,

$$x_i \in V(C_i) \cap \sigma_i(H) \quad \text{and} \quad d(x_i, H) > \operatorname{dia} H = \operatorname{dia} \sigma_i(H)$$

imply

$$H \cap \sigma_i(H) = \emptyset.$$

If for $i = 1$ and 2, $H \cap \sigma_i(C_i) = \emptyset$ then $\sigma_i(C_i \cup H)$ induces a connected subgraph of $X - H$ and hence $\sigma_i(C_i \cup H) \subseteq C_i$. In that case σ_1 or σ_2 is a proper translation by Theorem 2.7.

If for $i = 1$ and $i = 2$, $H \cap \sigma_i(C_i) \neq \emptyset$, then $H \subseteq \sigma_1(C_1) \cap \sigma_2(C_2)$. In that case both $\sigma_1(C_2 \cup H)$ and $\sigma_2(C_1 \cup H)$ induce connected subgraphs of $X - H$, hence

$$\sigma_1(C_2 \cup H) \subseteq C_1 \quad \text{and} \quad \sigma_2(C_1 \cup H) \subseteq C_2.$$

Then $\sigma_1\sigma_2(C_1 \cup H) \subseteq \sigma_1(C_2) \subseteq C_1$ and $\sigma_2\sigma_1(C_2 \cup H) \subseteq C_2$ and, by Theorem 2.7, both $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$ are proper translations. □

It is not difficult to construct graphs X, such that Aut X has exactly two orbits on $V(X)$ and exactly two orbits on $L(X)$. Let X_1 and X_2 be two vertex-transitive 2-connected non-isomorphic graphs. For each $v \in V(X_1)$ let $X_2(v)$ be a copy of X_2. Let X be constructed from the disjoint union of X_1 and all $X_2(v)$ $(v \in V(X_1))$ by adding all edges $[v, w]$, where $v \in V(X_1)$ and $w \in V(X_2(v))$. Clearly Aut X has two orbits on $V(X)$. If each of X_1 and X_2 has exactly one end orbit, then Aut X has two end-orbits. If Aut X_1 and Aut X_2 are torsion groups, then Aut X contains no translation. Examples for graphs X_1 for which Aut X_1 is a vertex-transitive torsion group are given in [1]. X_1 can be chosen infinite, connected and locally finite. So in a certain sense Theorem 3.6 is sharp.

Let us call an end l of the graph X a *thick* end of X if l contains an infinite subset of pairwise disjoint rays. If Aut X is a vertex-transitive and $|L(X)| \geq 2$, by Theorem 3.6 the graph X allows a proper translation. But as noted above, fixed ends of proper translations are not thick. Therefore the ends of a vertex-transitive graph X with $|L(X)| \geq 2$ cannot all be thick. On the other hand it is easy to construct end-transitive locally finite graphs X with $|L(X)| = 2^{\omega}$ in which each end is thick.

Let T_d be the tree in which one vertex v_0 has degree $d-1$ and all other vertices have degree d $(d \geqslant 2)$. For $h=0, 1, \ldots$ let $S(h)$ denote the set of vertices in $T=T_d$ at distance h from v_0. Let $p(1)<p(2)<\cdots$ be the sequence of prime numbers. For $n=2, 3, 4, \ldots$ we add to T an edge between $v \in S(p(n))$ and $w \in S(p(n^2))$, whenever v is on the unique path $T[v_0, w]$ joining v_0 and w in T. Clearly the arising graph $X=X_d$ is locally finite. We first show that each end of X contains a (unique) ray in T with initial vertex v_0. Given a ray R in X we construct a sequence $x_1, x_2, \ldots$ in $V(R)$ such that

$$V(R) \cap V(T[v_0, x_m]) = \{x_1, x_2, \ldots, x_m\}$$

and the subray R_m of R with initial vertex x_m contains no element of $T[v_0, x_m)$. Let x_1 be the $\leqslant-$minimum of $V(R_1)$, where $\leqslant$ is the order related to the rooted tree (T_d, v_0) and v_0 is the $\leqslant$-minimum of $V(T)$. Now assume that $x_1, \ldots, x_m$ have been constructed with the above properties. The second vertex y_m of R_m is not on $T[v_0, x_m]$. As y_m and x_m are $\leqslant-$comparable, x_m lies on $T[v_0, y_m]$. Let x_{m+1} be the first vertex on $T(x_m, y_m]$ in $V(R_m)$. Then $x_1, \ldots, x_{m+1}$ has the stipulated properties. Now

$$\bar{R} = \bigcup_{m=1}^{\infty} T[v_0, x_m]$$

is a ray in T. Let $v_0, v_1, \ldots$ be the vertices of $\bar{R}$, where $v_h \in S(h)$. For any positive integer n, which is not a perfect square, we have a ray $\bar{R}_n$ in X with vertices $v_{p(n)}, v_{p(n^2)}, v_{p(n^4)}, \ldots$ Those rays are pairwise disjoint. This shows that each end of X is thick. The edges of T_d are exactly the edges of X_d, which have at least one end vertex of degree d or $d-1$ in $X(d)$. As v_0 is the only vertex of degree $d-1$ in X and in T, we have $\operatorname{Aut} X_d = \operatorname{Aut} T_d$.

However question which groups $G \leqslant \operatorname{Aut} X$ (X connected) contain proper translations is in general unanswered.

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