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A survey on graphs with polynomial growth

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Abstract

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In this paper we give an overview on connected locally finite transitive graphs with polynomial growth. We present results concerning the following topics:
Automorphism groups of graphs with polynomial growth.
Groups and graphs with linear growth.
S-transitivity.
Covering graphs.
Automorphism groups as topological groups.

1. Introduction

The concept of growth was introduced to study finitely generated infinite groups. It first appeared in a paper of Adelson-Velsky and Shreider [1] which was published in 1957. The *growth function* $f_G(n)$ of a group G with respect to a finite generating set H is given by $f_G(0) = 1$ and

$$f_G(n) = |\{g \in G \mid g = h_1 \cdots h_n,\ h_i \in H \cup H^{-1} \cup \{e\}\}|$$

for $n \geq 1$. We say that a group G has *exponential growth* if there is a constant $c > 1$ such that $f_G(n) \geq c^n$ holds for all $n \in \mathbb{N}$. Otherwise G has *non-exponential growth.* In particular, G has *polynomial growth* if there are constants c and d such that $f_G(n) \leq cn^d$ for all $n \in \mathbb{N}$. Furthermore, we say that groups of non-exponential growth which grow faster than any polynomial have *intermediate growth.* Clearly these properties do not depend on the generating set.

Milnor [23] and Wolf [45] carried out the first investigations of growth conditions in groups. In [45] Wolf proved the important result that almost nilpotent groups have polynomial growth (A group G *almost* has property P if a normal subgroup of finite index of G has property P.) For groups G with polynomial growth Milnor [25] conjectured that there always exist constants c_1, c_2 and an integer d such that

$$c_1 n^d \leqslant f_G(n) \leqslant c_2 n^d. \tag{1}$$

He also conjectured that the class of groups with polynomial growth coincides with the class of almost nilpotent groups. In 1972 Bass [3] proved Milnor's first conjecture, i.e. (1), for almost nilpotent groups. The second conjecture was settled by Gromov [14]. This deep result of Gromov [14], together with the above mentioned result of Wolf, is crucial for almost everything we carry out in this paper. For further references we therefore state it as the following theorem.

Theorem 1.1 (Gromov [14], Wolf [45]). *A finitely generated group has polynomial growth if and only if it is almost nilpotent.*

This result also implies the validity of Milnor's first conjecture. This means that (1) always holds for groups with polynomial growth. We call the thus well-defined integer d the growth degree d_G of G.

Wolf also conjectured that a group has exponential growth if its growth function dominates any polynomial, i.e. that there exist no groups with *intermediate* growth. This conjecture holds for solvable groups (cf. [45, 24]). In general it is false, as was first shown by Grigorchuk (cf. [12–13]). He found torsion-free groups as well as p-groups with intermediate growth. The growth functions of one class of groups with intermediate growth constructed by Grigorchuk satisfy e.g. the condition

$$2^{\sqrt{n}} < f_G(n) < 2^{n^{\log_{32} 31}}$$

for sufficiently large n. Further examples of groups with intermediate growth were given by Fabrykowski and Gupta [8].

For additional information about growth conditions in groups we refer to [39, Chapter 12].

2. Growth of graphs and preliminary results

The *growth function* of a locally finite graph X with respect to a vertex $x \in V(X)$ is given by

$$f_X(x, n) = |\{y \in V(X) \mid d(x, y) \leqslant n,\ 0 \leqslant n\}|,$$

where $d(x, y)$ denotes the distance between x and y. For transitive graphs the growth function obviously does not depend on a particular vertex x, hence we denote it by $f_X(n)$. Of course we can then define *polynomial, intermediate* and *exponential growth* analogously to the above definitions for groups. Even in the case of nontransitive graphs these properties do not depend on a particular vertex

x. Only the constants in the bounds for the growth function might be different if we consider the growth function with respect to different vertices. We also mention that we can identify the growth function of a group G, with respect to a generating set H, $e \notin H$, and that of its *Cayley graph* $C(G, H)$, where $C(G, H)$ is defined on G with the edge-set

$$E(C(G, H)) = \{(g, gh) \mid g \in G, h \in H \cup H^{-1}\}.$$

It is easily seen that G acts on $C(G, H)$ by left multiplication. This action is *regular*, i.e. transitive and fixed point free. Conversely a graph X is a Cayley graph of a group G if G acts regularly on X. This result is due to Sabidussi [29]. If the action of G on X is *semiregular*, i.e. fixed point free, one can contract X to a Cayley graph of G, as has been shown by Babai [2]. Both of these results are repeatedly used in the proofs of the results presented here.

By the above it is clear that the most natural examples of graphs satisfying special growth conditions are Cayley graphs of groups with polynomial, intermediate and exponential growth. In fact, one might be tempted to consider results about group growth as results about Cayley graphs. We note that the graphs in Fig. 1 have linear, quadratic and exponential growths, respectively.

The first question, concerning growth functions of locally finite transitive graphs with polynomial growth which immediately arises is, if a relation like (1) also holds for graphs which are not Cayley graphs. The answer to this question, as well as to other problems concerning graphs with polynomial growth, was given by Trofimov [38]. To state the result we need several definitions.

If a group G acts transitively on a graph X, then an *imprimitivity system* of G on X is a partition τ of $V(X)$ into subsets called *blocks*, such that every element of G is permutes the blocks of τ. Among imprimitivity systems we include the partition of $V(X)$ into singletons and into $V(X)$ itself. These are the so called *trivial imprimitivity systems*. The *quotient graph* X_τ is defined as follows: $V(X_\tau)$ is

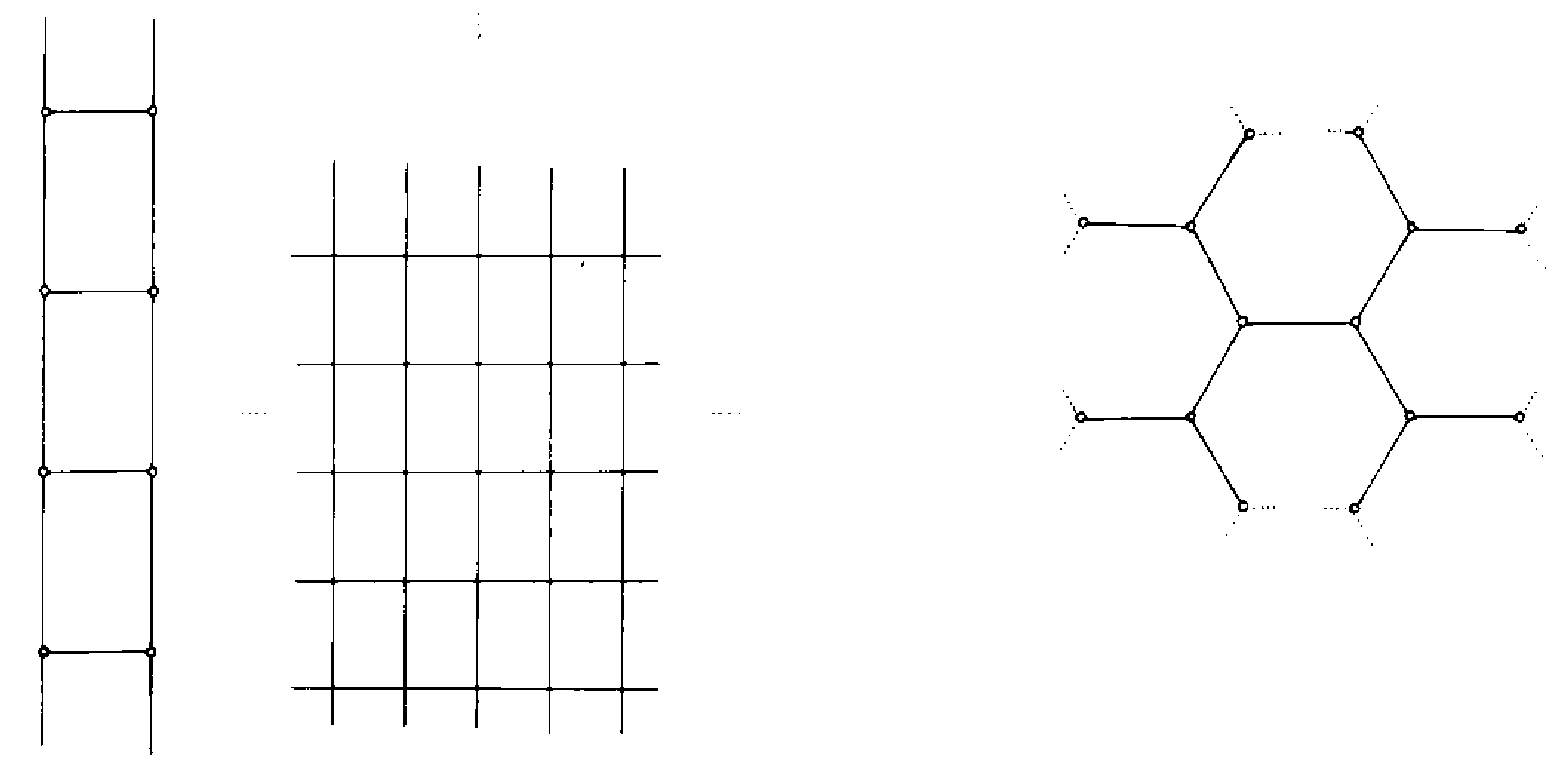

Fig. 1.

the set of blocks and two vertices $v_\tau, w_\tau \in V(X_\tau)$ are adjacent in X_τ if and only if $(v, w) \in E(X)$ for at least two vertices $v \in v_\tau$, $w \in w_\tau$. By G_τ we denote the group acting on X_τ which is induced by G. It is a homomorphic image of G and $G_\tau \leqslant \mathrm{AUT}(X_\tau)$.

Theorem 2.1 (Trofimov [38]). *The following assertions are equivalent for an infinite transitive connected locally finite graph X:*

(i) *X has polynomial growth.*

(ii) *There exists an imprimitivity system τ of $\mathrm{AUT}(X)$ on $V(X)$ with finite blocks such that $\mathrm{AUT}(X_\tau)$ is a finitely generated almost nilpotent group and the stabilizer in $\mathrm{AUT}(X_\tau)$ of a vertex of the graph X_τ is finite.*

Theorem 2.1 combined with the following result of Sabidussi [30] shows that transitive graphs with polynomial growth are closely related to Cayley graphs of almost nilpotent groups.

First a definition. If X is a graph and n a cardinal, then the graph nX is defined on the vertex-set $V(nX) = V(X) \times N$, where N is a set of cardinality n, and

$$E(nX) = \{[(x, \mu), (y, \nu)] \mid [x, y] \in E(X),\ \mu, \nu \in N\}.$$

Theorem 2.2 (Sabidussi [30]). *Let X be a connected transitive graph, G a transitive subgroup of $\mathrm{AUT}(X)$ and let n be the cardinaltiy of the stabilizer in G of a vertex of X. Then nX is a Cayley graph of G.*

Now, let X be a connected transitive graph with polynomial growth. Then X and X_τ (as given by Theorem 2.1) obviously have the same growth degree. Also X_τ and nX_τ, where n is the cardinality of the stabilizer of a vertex of X_τ in $\mathrm{AUT}(X_\tau)$, have the same growth degree. Since, by Theorem 2.2, nX_τ is a Cayley graph of the almost nilpotent group $\mathrm{AUT}(X_\tau)$, the results mentioned in Section 1 imply that the growth degree of nX_τ is a well defined integer d. Hence, also the growth degree of X_τ is the same integer d and since the blocks of τ are finite, the same also holds for the growth degree of X.

Although these remarks show that the automorphism groups of graphs with polynomial growth are in some sense closely related to groups with polynomial growth, there are still many open questions. It is, for example, possible that the automorphism group of a connected locally finite graph X with polynomial growth is uncountable. A simple example being given by Fig. 2. The uncoun-

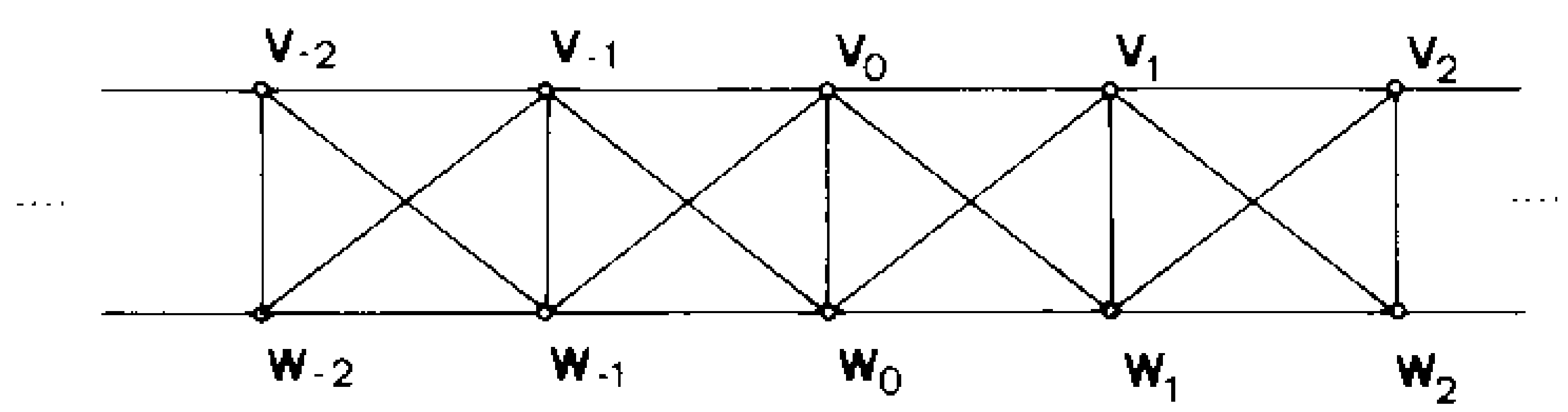

Fig. 2.

tability of the automorphism group of this graph is immediate. In general we have the following criterion by Halin.

Theorem 2.3 (Halin [15]). *The automorphism group of a locally finite connected graph X is uncountable if and only if for every finite subset $F \subset V(X)$ there exists a nontrivial automorphism that fixes F pointwise.*

The graph of Fig. 2 has another interesting property: Let a be the automorphism which permutes v_0 and w_0 but fixes all other vertices of X. By b we denote the automorphism which acts with the orbits $(\ldots, v_{-1}, v_0, v_1, \ldots)$ and $(\ldots, w_{-1}, w_0, w_1, \ldots)$ on X. We now show that the group G generated by a and b is metabelian (solvable of length two) and has exponential growth.

Let $G' = [G, G]$ be the commutator subgroup of G and let $g, h \in G'$. It is obvious that every element of G' leaves invariant the sets

$$\ldots, V_{-1} = \{v_{-1}, w_{-1}\}, \qquad V_0 = \{v_0, w_0\}, \qquad V_1 = \{v_1, w_1\}, \ldots$$

The action of g and h on those sets is induced by the action of a on V_0. Hence the restrictions of g and h to V_i commute for all V_i, $i \in \mathbb{Z}$. So G' is abelian and $[G', G'] = \{e\}$.

Let $g_j = b^j a b^{-j}$ for $j = 0, \ldots, n$, $n \in \mathbb{N}$. We now consider the automorphisms

$$h(\varepsilon_0, \ldots, \varepsilon_n) = g_0^{\varepsilon_0} \cdots g_n^{\varepsilon_n}$$

where $\varepsilon_j \in \{0, 1\}$. Clearly $h(\varepsilon_0, \ldots, \varepsilon_n) = h(\delta_0, \ldots, \delta_n)$ if and only if $\varepsilon_0 = \delta_0, \ldots, \varepsilon_n = \delta_n$. So there are 2^{n+1} different h's. Since the length of those automorphisms with respect to the generators a and b is less or equal to $(2n+1)(n+1)$ we obtain

$$f_G((2n+1)(n+1)) \geq 2^{n+1}$$

for $n = 0, 1, \ldots$. This implies that G does not have polynomial growth. Since G is solvable it thus has exponential growth.

This leads to the following problems:

(1) Groups with exponential growth can act transitively on graphs with polynomial growth. Does this also hold for groups with intermediate growth?

(2) Is it possible to give necessary and sufficient algebraic conditions such that the automorphism groups of graphs with polynomial growth are uncountable?

(3) Is it true that uncountable automorphism groups of graphs with polynomial growth always contain finitely generated solvable subgroups with exponential growth?

Before we present some answers to these questions, we continue with a characterization of growth conditions in the language of ends of graphs.

Two one-way infinite paths P and Q are *equivalent* in X, in symbols $\sim_x$, if there is a third path R which meets both of them infinitely often. The equivalence classes with respect to $\sim_x$ are called *ends*. Obviously the automorphisms of X also act on the set of ends of X. This definition of ends is due to Halin [16], but we emphasize that the concept of ends first appears in papers of Hopf [17] and Freudenthal [9]. They introduced the concept of ends to study discrete groups. For us an end of a group G is an end of a Cayley graph of G. We note that the number of ends of a group (and a transitive graph in general) is either 0 (if the group is finite), 1, 2 or ∞. This number does not depend on the generating set although the number of disjoint one-way infinite paths in each end depends on it if this number is finite!

The fact that a transitive infinite locally finite graph has either 1, 2 or infinitely many ends immediately follows from results in [15, 21]. This, together with another result of Halin [16], readily implies that graphs with infinitely many ends have exponential growth. Also a transitive infinite locally finite graph has linear growth if and only if it has two ends (cf. [20]). (For a detailed description of those graphs see [20, 22].) If a graph has polynomial but nonlinear growth it thus has one end, but the converse is not true: All Cayley graphs of groups with intermediate growth clearly have one end and there also exist graphs with exponential growth and only one end.

Nevertheless, this relation between graphs with polynomial growth and graphs with one or two ends allows the application of many results which where formulated using the concept of ends rather than that of growth. Results of this kind can be found in [15–16, 20–22, 34, 40–41].

3. Automorphism groups of graphs with polynomial growth

An automorphism g of a graph X is called *bounded* if there is a constant k, depending upon g, such that

$$d(x, g(x)) \leqslant k \quad \text{for every } x \in V(X).$$

This concept is closely related to the concept of FC-groups. (A group G is an *FC-group* if the conjugacy class of every $g \in G$ is finite. For the theory of FC-groups we refer to [35].) For, let X be a Cayley graph of a group G such that every $g \in G$ acts as bounded automorphism on X. Then $d(x, gx) \leqslant k$ for every $x \in V(X) = G$ which is equivalent with $d(e, x^{-1}gx) \leqslant k$. But this only holds if $\{x^{-1}gx \mid x \in G\}$ is finite, i.e. if G is an FC-group.

We can then apply one of the results about FC-groups ([27, Theorem 5.1]) to prove the following useful theorem. We recall that a group G is *locally finite* if every finitely generated subgroup of G is finite.

Theorem 3.1 (Godsil, Imrich, Seifter, Watkins and Woess [10]). *Let X be a transitive connected locally finite graph with polynomial growth, and let $B(X)$ be the group of bounded automorphisms of X. Then the set $B_0(X)$ of elements of finite order in $B(X)$ forms a normal subgroup of* AUT(X). *It is locally finite, periodic and acts with finite orbits on $V(X)$.*

This result together with the following nice theorem of Rosset [28], enables us to solve the first problem posed in Section 2.

Theorem 3.2 (Rosset [28]). *If a finitely generated group G has non-exponential growth and H is a normal subgroup of G such that G/H is solvable, then H is finitely generated.*

Theorem 3.3 (Seifter [33]). *Finitely generated groups with intermediate growth cannot act transitively on connected locally finite graphs with polynomial growth.*

Proof. Suppose a finitely generated group G with intermediate growth acts transitively on a connected locally finite graph X with polynomial growth. By τ we denote the imprimitivity system of $\mathrm{AUT}(X)$ on X which is given by Theorem 2.1. Then a homomorphic image G_τ of G also acts transitively on X_τ. By φ we denote the homomorphism from G onto G_τ. From the remarks following Theorem 2.2 we also know that G_τ has the same growth degree as X_τ. Hence $\ker \varphi$ must be infinite for otherwise G also has polynomial growth of the same degree as G_τ. Theorem 1.1 now implies that G_τ contains a nilpotent normal subgroup N_τ of finite index. Since φ is a homomorphism, the subgroup N of G which is generated by all $g \in G$ with $\varphi(g) \in N_\tau$ also has finite index in G. Hence N also is a finitely generated group with intermediate growth. Since $\ker \varphi$ also is a normal subgroup of N, Theorem 3.2 implies that $\ker \varphi$ is finitely generated. But $\ker \varphi \leqslant B_0(X)$ also holds. Hence $\ker \varphi$ must be locally finite by Theorem 3.1, a contradiction. □

This result leads to the following necessary and sufficient algebraic condition for the uncountability of automorphism groups of graphs with polynomial growth.

Corollary 3.4 (Seifter [33]). *The automorphism group of a connected locally finite transitive graph X with polynomial growth is uncountable if and only if it contains a finitely generated subgroup with exponential growth which acts transitively on X.*

If a group with exponential growth acts transitively on X then it is clear that $\ker \varphi$ must be infinite, where φ is defined as in the proof of Theorem 3.3. Then Theorem 2.3 immediately implies that the automorphism group of X is uncountable. The 'only if' part of the proof is a little bit more complicated. We only mention that it mainly depends on Theorem 3.3. For the complete proof we refer to [33].

We recall (see also Section 1) that finitely generated solvable groups have either polynomial or exponential growth. Theorem 3.3 shows that the same holds for finitely generated groups which act transitively on graphs with polynomial growth. Hence it is quite natural to ask if every finitely generated group G which acts transitively on a graph with polynomial growth, contains a solvable subgroup with the same growth properties. It clearly holds if G has polynomial growth and the example given in Section 2 indicates that this might also be true if G has exponential growth. But this turned out to be a quite difficult question. We only have the following partial answer which is a generalization of the example given by Fig. 2.

Theorem 3.5 (Seifter [32]). *Every finitely generated group with exponential growth which acts transitively on a locally finite connected graph X with linear growth contains a finitely generated metabelian subgroup with exponential growth which acts with finitely many orbits on X.*

For graphs with polynomial but nonlinear growth we still do not know if a similar result holds but we think that this should be the case. Hence the following.

Conjecture 3.6. Suppose the finitely generated group G with exponential growth acts transitively on a graph X with polynomial growth. Then G contains a finitely generated solvable subgroup with exponential growth which acts with finitely many orbits on X.

Graphs with linear growth are almost always easier to handle than graphs with higher growth degree. The reason, roughly spoken, is the fact that graphs with linear growth are the only transitive graphs with polynomial growth which have two ends. In the next paragraph we present a rather detailed analysis of groups and graphs with linear growth.

First we prove another interesting property of graphs with polynomial growth. We also give the proof of this result since the construction of suitable imprimitivity systems is one of the main techniques used for proving the results presented in this paper.

Theorem 3.7 (Godsil, Imrich, Seifter, Watkins and Woess [10]). *Let X be a locally finite connected transitive graph with polynomial growth. Then the action of* $\mathrm{AUT}(X)$ *on X is imprimitive.*

Proof. If the blocks of the imprimitivity system τ given by Theorem 2.1 contain more than one vertex, the assertion obviously holds. So we can assume that τ is trivial. By Theorem 2.1 this means that $\mathrm{AUT}(X)$ is almost nilpotent.

If we can find a nontrivial normal subgroup K of $\mathrm{AUT}(X)$ which does not act transitively on X, then the orbits of K on X give rise to a nontrivial imprimitivity system of $\mathrm{AUT}(X)$ on X.

By Theorem 3.1 $B_0(X)$ is a normal subgroup of $\mathrm{AUT}(X)$ which acts with finite orbits on X. Hence if $B_0(X) \neq \{e\}$ we already have a nontrivial imprimitivity system for $\mathrm{AUT}(X)$. Also $B(X)$ is normal in $\mathrm{AUT}(X)$. If $B(X) \neq \{e\}$ does not act transitively on X we again have a nontrivial imprimitivity system. Since $B(X) \neq \{e\}$ always holds, we now have to consider the case that $B(X)$ acts transitively on X.

It follows from [37], Theorem 1, that $B(X) \cong \mathbb{Z}^d$ if $B_0(X) = \{e\}$ and $B(X)$ acts transitively on X, where d denotes the growth degree of X. Then $G = (2\mathbb{Z})^d$ is a subgroup of finite index in $B(X)$ which does not act transitively on X. Since $B(X)$ has the same growth degree as X, it also has the same growth degree as $\mathrm{AUT}(X)$ (cf. the remarks following Theorem 2.2). So $B(X)$ has finite index in $\mathrm{AUT}(X)$ which implies that G also has finite index in $\mathrm{AUT}(X)$. Furthermore, the intersection of all conjugates $gGg^{-1}, g \in \mathrm{AUT}(X)$, is a normal subgroup N of finite index of $\mathrm{AUT}(X)$. Clearly $N \leq G$ also does not act transitively on X, which completes the proof. $\square$

4. Groups and graphs with linear growth

In [43] Wilkie and van den Dries proved the following interesting relation between the first difference function of the growth function of a group G and the index of a subgroup $\cong \mathbb{Z}$ in G.

Theorem 4.1 (Wilkie, van den Dries [43]). *Suppose G is a finitely generated infinite group, $k > 0$, and $f_G(k) - f_G(k-1) \leq k$. Set $c = f(k) - f(k-1)$. Then G contains a subgroup $\cong \mathbb{Z}$ of index $\leq c^4/2$.*

This result shows that a local condition like $f_G(k) - f_G(k-1) \leq k$ implies that G satisfies the global condition of linear growth. But as Wilkie and van den Dries themselves suggested, the bound for the index of a subgroup $\cong \mathbb{Z}$ in G is far from being sharp. This bound can be improved as follows.

Theorem 4.2 (Imrich, Seifter [19]). *Let G satisfy the assumptions of Theorem* 4.1. *Then G contains a subgroup $\cong \mathbb{Z}$ of index $\leq c$. This bound is sharp.*

The proof is based on the result of Wilkie and van den Dries [43] and the characterization of groups with two ends (cf. [34, 4.A.6.5]). By this characterization groups with two ends are exactly those with a homomorphism with finite kernel onto $\mathbb{Z}$ or onto the free produce $\mathbb{Z}_2 * \mathbb{Z}_2$. This implies that $H \cup H^2$ always contains an element of infinite order if H is a finite generating set of a group G with two ends. Based on these facts Theorem 4.2 was proved by a detailed discussion of properties of Cayley graphs with two ends.

To show that the bound is sharp we note that $\langle ab \rangle \cong \mathbb{Z}$ has index 2 in $\langle a, b \mid a^2 = b^2 = e \rangle = \mathbb{Z}_2 * \mathbb{Z}_2$. In this case $c = 2$ holds. Furthermore, let A be any finite group and let G denote the direct product of A by $\langle a, b \mid a^2 = b^2 = e \rangle$. If we set $H = A \cup \{a, b\} \backslash e$ then H generates G and $c = 2|A|$ obviously holds. Also $\langle ab \rangle$ has index $2|A|$ in G.

Theorem 4.2 provides a rather detailed characterization of groups with linear growth. We now consider graphs with linear growth. The example given in Section 2, as well as Theorem 3.5, show that groups with exponential growth can act transitively on graphs with linear growth. On the other hand the remarks following Theorem 2.2 imply that a group with polynomial growth must have linear growth if it acts transitively on a graph with linear growth. Hence the question arises how close the connections between the growth functions of graphs with linear growth and finitely generated groups which have polynomial growth and act transitively on them must be. More precisely: Let X be a graph such that $f_X(n) \leqslant c_X n$ holds for some constant c_X. If a finitely generated group G with polynomial growth acts transitively on X, Theorem 2.1 implies that $f_G(n) \leqslant c_G n$ also holds for some constant c_G. If G acts regularly on X we know (cf. [29]) that X is a Cayley graph of G. Hence there is a generating set H of G such that $f_G(n) \leqslant c_X n$. But what happens if G does not act regularly on X? Is it still possible to find a generating set H of G and a function $h(c_X)$ such that the growth function of G is bounded by $h(c_X)n$?

Since we are now interested in the constant in the bound of the growth function, which clearly depends on the generating set, we write $f_G(H, n)$ for the growth function of a group. We also mention that a *translation* of a graph X is an automorphism which acts without finite orbits on X (sometimes (see e.g. [15]) translations are called automorphisms of type 2). This question can be answered as follows.

Theorem 4.3 (Seifter [31]). *Let the finitely generated group G with polynomial growth act transitively on X with $f_X(n) \leqslant c_X n$ for all $n \geqslant 1$. Then there exists a minimal integer k and a translation g which is central in $B(X) \cap G$ and $d(v, g(v)) \leqslant k$ for all $v \in V(X)$. Furthermore there is a generating set H of G such that*

$$f_G(H, n) < 2c_X(\lfloor c_X \rfloor !)^k n.$$

If, in addition, $\mathrm{AUT}(X)$ is a countable group the following characterization of $\mathrm{AUT}(X)$ can be given.

Theorem 4.4 (Seifter [31]). *Let X be a graph with $f_X(n) \leqslant c_X n$, $n \geqslant 1$, and countable automorphism group. Then there exists a finite generating set H of* $\mathrm{AUT}(X)$ *such that*

$$f_{\mathrm{AUT}(X)}(H, n) \leqslant 2c_X \left\lfloor \frac{c_X}{2} - 1 \right\rfloor ! \, n$$

holds for all $n \geqslant 1$.

Outline of the proofs of Theorems 4.3 and 4.4: One has to determine the cardinality of the stabilizer of a vertex of X in the considered group. Then an application of Theorem 2.2 immediately leads to these results.

Also, the following information about the structure of graphs with linear growth is available.

Proposition 4.5 (Imrich, Seifter [20]). *A transitive connected locally finite graph X has linear growth if and only if it has two ends. Then it is spanned by finitely many 2-way infinite paths and there is a translation which leaves these paths invariant.*

Even if a graph with linear growth is not transitive very much is known about its structure and automorphism group. We refer to [15, 22] but emphasize that the concept of growth is not used in these papers. In [22] graphs with linear growth are called *strips*, in [15] they are simply graphs with two ends and a translation acting on them.

Finally we emphasize that such detailed results cannot be proved for graphs with polynomial growth in general. Even groups with polynomial but nonlinear growth cannot be characterized as in Theorems 4.1 and 4.2. For, let G be the direct product of $\mathbb{Z}^d = \{a_1, \ldots, a_d \mid a_i a_j = a_j a_i \text{ for all } i, j \in \{1, \ldots, d\}, d \geq 2\rangle$, by the group $\mathbb{Z}_3 = \langle b \mid b^3 = e\rangle$. Then we have:

$$f_G(0) = 1, \qquad f_G(1) = 2d + 3, \qquad f_G(2) = 2d^2 + 6d + 3.$$

Hence, if d is sufficiently large, $f_G(2) - f_G(1) \leq 2^{d^*}$ always holds for some $d^* < d$. But this does not imply that G has growth degree d^*.

5. *S*-transitivity

A sequence $(v_0, \ldots, v_s)$ of $s+1$ vertices is called an *s-arc* if for each i, (v_{i-1}, v_i) is an edge of X and $v_{i-1} \neq v_{i+1}$. If a group acts transitively on the s-arcs of X we call X *s-transitive*. For completeness we mention that the so far used term 'transitive' in this context means 0-*transitive*.

In 1981 Weiss [42] showed that finite graphs with valency ≥ 3 cannot be 8-transitive. Surprisingly the same holds for graphs with polynomial growth.

Theorem 5.1 (Seifter [33]). *Let X be a connected locally finite s-transitive graph with polynomial growth and valency at least* 3. *Then $s \leq 7$.*

Outline of the proof. Let X be an s-transitive graph with polynomial growth. Then the graph $Y = X_\tau$ (see Theorem 2.1) is also s-transitive. In addition we know that AUT(Y) is a finitely generated almost nilpotent group. It is then always possible to find an imprimitivity system σ of AUT(Y) on Y with infinite blocks such that Y_σ is finite and also s-transitive. With the exception of the case when Y_σ is a cycle we can then apply Weiss's result [42]. Also the case that Y_σ is a cycle causes no difficulties (cf. [33]).

The following construction shows that the bound given in Theorem 5.1 is sharp.

Let Y be a finite graph and let G be a group. In [5, p. 127], a construction of a covering graph of Y with respect to G is given as follows: Each edge (u, v) of Y gives rise to two 1-arcs, (u, v) and (v, u). By $S(Y)$ we denote the set of 1-arcs and by $\varphi: S(Y) \to G$ a mapping such that $\varphi(u, v) = (\varphi(v, u))^{-1}$ for all $(u, v) \in S(Y)$. The covering graph $\tilde{Y} = \tilde{Y}(G, \varphi)$ of Y with respect to G is defined on the vertex-set $V(\tilde{Y}) = G \times V(Y)$ and two vertices $(g_1, u), (g_2, v) \in V(\tilde{Y})$ are joined by an edge if and only if $(u, v) \in S(Y)$ and $g_2 = g_1\varphi(u, v)$.

Let Y be a finite s-transitive graph with valency at least 3 where $0 \leqslant s \leqslant 7$. The fundamental group $\Pi(Y, u)$ of Y at u is a free group with the edges of $E(Y)\backslash E(T)$ as generators, where T is a spanning tree of Y (cf. [18]). We can also regard $\Pi(X, u)$ as a group generated by the closed walks with base point u which are associated with the edges of $E(Y)\backslash E(T)$. For details we again refer to [18].

We now consider the group $A = \Pi(X, u)/[\Pi(X, u), \Pi(X, u)]$. It is a free finitely generated abelian group. The generators $a_1, \ldots, a_\tau, a_1^{-1}, \ldots, a_\tau^{-1}$ of A are the images of the generators $g_1, \ldots, g_\tau, g_1^{-1}, \ldots, g_\tau^{-1}$ of $\Pi(X, u)$ under the homomorphism from $\Pi(X, u)$ onto A. Hence $A \cong \mathbb{Z}^\tau$. Let T be a spanning tree of Y and let $\varphi, \varphi: Y \to A$, be a mapping which maps those 1-arcs of Y which correspond to edges of $E(X)\backslash E(T)$ onto the generators of A and the edges of T onto the unit element of A. It is easy to see that the covering graph $\tilde{Y}(A, \varphi)$, constructed accordingly to the above rules is connected and locally finite. It also has polynomial growth of degree τ.

We can then apply Theorems 3 and 4 of [7] to show that $\tilde{Y}(A, \varphi)$ is s-transitive if Y is s-transitive. (For details we refer to [11, Proposition 2.3].) Hence, Theorem 5.1 gives a sharp bound for graphs with polynomial growth in general, but there are classes of graphs with polynomial growth with stronger restrictions on s.

Theorem 5.2 (Seifter [33]). *Let X be a connected locally finite graph with valence at least three and let $B(X)$ act transitively on X. Then X cannot be* 3*-transitive.*

In Theorem 5.2 one does not have to assume that X has polynomial growth since this follows from [37], Theorem 1, if $B(X)$ acts transitively on X. We also mention that it is possible to find a better bound for s if the considered graphs have linear growth (cf. [33]).

Not only s-transitivity is restricted by properties of the automorphism group, to some extent the converse also holds.

Theorem 5.3 (Seifter [33]). *Let X be a connected locally finite s-transitive graph with polynomial growth, valency at least three and let $s \geqslant 2$. Then* AUT(X) *is a finitely generated almost nilpotent group.*

Finally we wish to pose the following problem: The smallest known 7-transitive finite graph has 728 vertices and valency 4 (cf. [4], see also [5, p. 164]). Hence the rank of its fundamental group is equal to 729. By the above construction, we only know that a 7-transitive graph with polynomial growth of degree 729 exists. But we think that it should be possible to find e.g. 7-transitive graphs of much smaller growth degree.

Problem 5.4. *Let* $2 \leqslant s \leqslant 7$. Determine the minimal integer $d(s)$, depending upon s, such that there exists a connected locally finite s-transitive graph with polynomial growth of degree $d(s)$.

6. Covering graphs

As we have seen in Section 5 graphs with polynomial growth and high transitivity can be constructed as covering graphs of finite s-transitive graphs. As the next result shows, all graphs with polynomial growth are covering graphs of finite graphs.

Let $p, p: X_1 \to X_2$, denote a homomorphism and let $S(v)$, $v \in V(X_1)$, denote the star consisting of v and all edges incident to v. If $p(S(v))$ is isomorphic to $S(v)$, for every $v \in V(X_1)$, we call p *a covering map* and X_1 a covering graph of X_2.

Theorem 6.1 (Godsil, Seifter [11]). *Let X be a connected locally finite s-transitive, $s \geqslant 0$, graph with polynomial growth. Then there exist infinitely many finite graphs $Y_1, Y_2, \ldots$ such that:*

(1) *X is covering graph of every Y_i, $i \geqslant 1$, and*

(2) *each Y_k, $k \geqslant 2$, is covering graph of the graphs $Y_1, \ldots, Y_{k-1}$.*

(3) *If in addition $s \geqslant 2$ holds, then each Y_i also is at least s-transitive.*

We emphasize that Theorem 6.1 does not exclude the possibility that the finite graphs $Y_1, Y_2, \ldots$ are all s-transitive if $s \leqslant 1$. As the following example shows, some of those finite graphs can even be 'more' transitive than their covering graph with polynomial growth.

Let X be the Cayley graph $C(G, H)$ of the group $G \cong \mathbb{Z} \times \mathbb{Z}_5$ with respect to the generating set $H = \{a, c\}$ where $\langle a \rangle \cong \mathbb{Z}$ and $\langle c \rangle \cong \mathbb{Z}_5$. We denote the vertices of G by (a^j, c^i), $j \in \mathbb{Z}$, $0 \leqslant i \leqslant 4$ (see Fig. 3). Obviously every set $T_j = \{(a^j, c^0), \ldots, (a^j, c^4)\}$, $j \in \mathbb{Z}$, separates the two ends of X. Furthermore the T_j are minimal with respect to this property. We now assume that X is s-transitive for some $s \geqslant 1$. Then there is a $g \in \mathrm{AUT}(X)$ which maps the edge $((a, e), (a, c))$ onto the edge $((a, e), (a^2, e))$. But then the set $g(T_0)$ cannot separate the ends of X, a contradiction. Hence X is not 1-transitive.

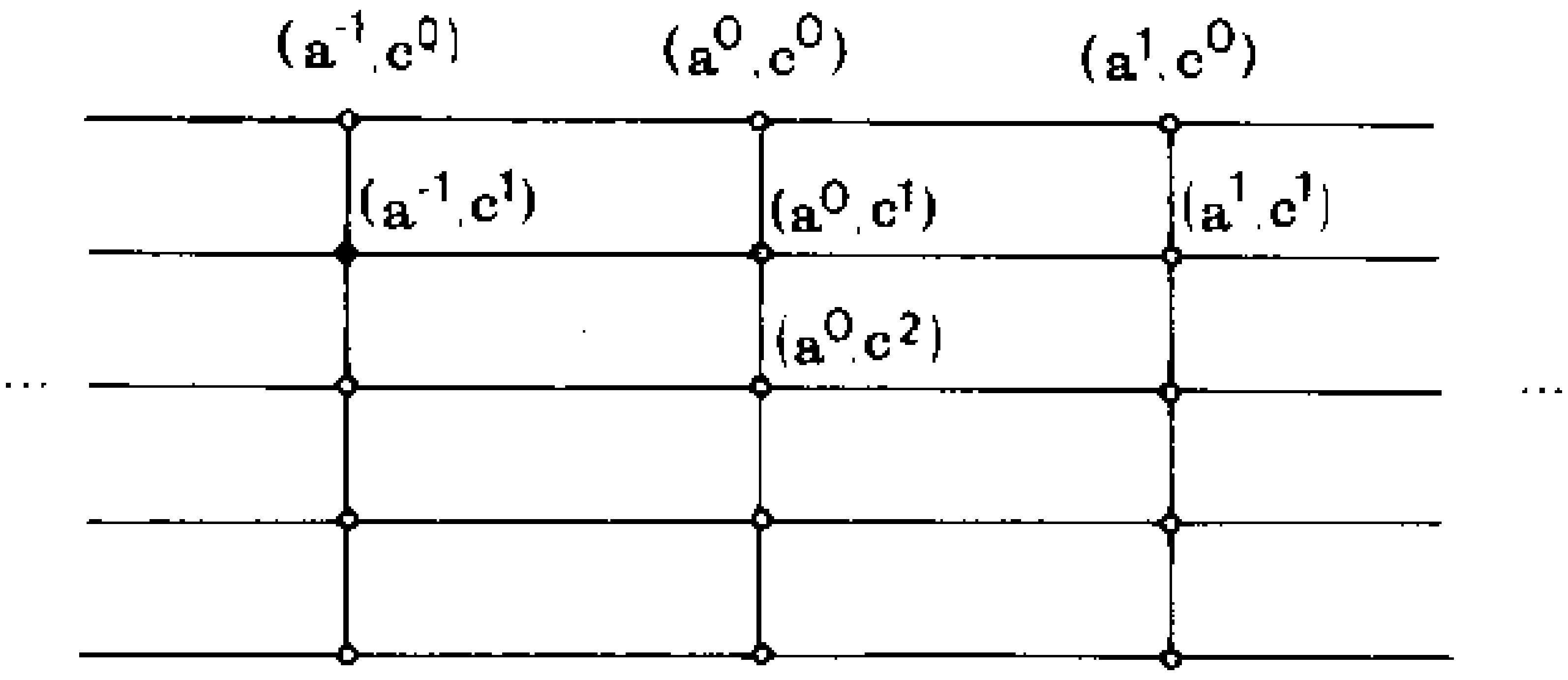

Fig. 3.

Let $\{v_0, \ldots, v_4\}$ denote the vertex set of the complete graph K_5 and let p, $p: X \to K_5$, be a map satisfying:

(a) $\quad p(a^j, c^i) = v_k \quad$ if $j \geqslant 0$ and $i \equiv (k + 2j)(5)$, and,

(b) $\quad p(a^j, c^i) = v_k \quad$ if $j < 0$ and $i \equiv (k + |3_j|)(5)$.

Obviously p is a covering map and K_5 is 2-transitive.

The graphs C_1 and C_2 given by Fig. 4 are Cayley graphs of $\mathbb{Z}_5 \times \mathbb{Z}_5$ and $\mathbb{Z}_3 \times \mathbb{Z}_5$, respectively. Obviously X also covers these graphs. The graph C_1 is clearly 1-transitive but the 2-arc (w_0, w_1, w_2) cannot be mapped onto the 2-arc (w_0, w_1, w_3) since there are two paths of length 2 from w_0 to w_2 but only one path of length 2 from w_0 to w_3. Hence it is not 2-transitive.

The edge $(v_0, v_1) \in E(C_2)$ is contained in the triangle (v_0, v_1, v_2, v_0) but there is no triangle which contains the edge (v_0, v_3). Hence C_2 is not 1-transitive.

If we now choose $Y_i = \mathbb{Z}_5 \times \mathbb{Z}_i$ for suitable i we again obtain an infinite sequence of finite graphs which all satisfy condition three. The reason is that the automorphism group of X is countable, which need not hold for s-transitive graphs with $s \leqslant 1$ in general. For $s \geqslant 2$ this always holds by Theorem 5.3.

As the above example shows, it can happen that some of the Y_i have higher transitivity than their covering graphs with polynomial growth. Of course this is impossible if X is 7-transitive. In this case the result of Weiss [42] implies that all Y_i are also 7-transitive. It would be interesting, but also rather difficult, to establish conditions such that the Y_i have the same transitivity as X if $s < 7$.

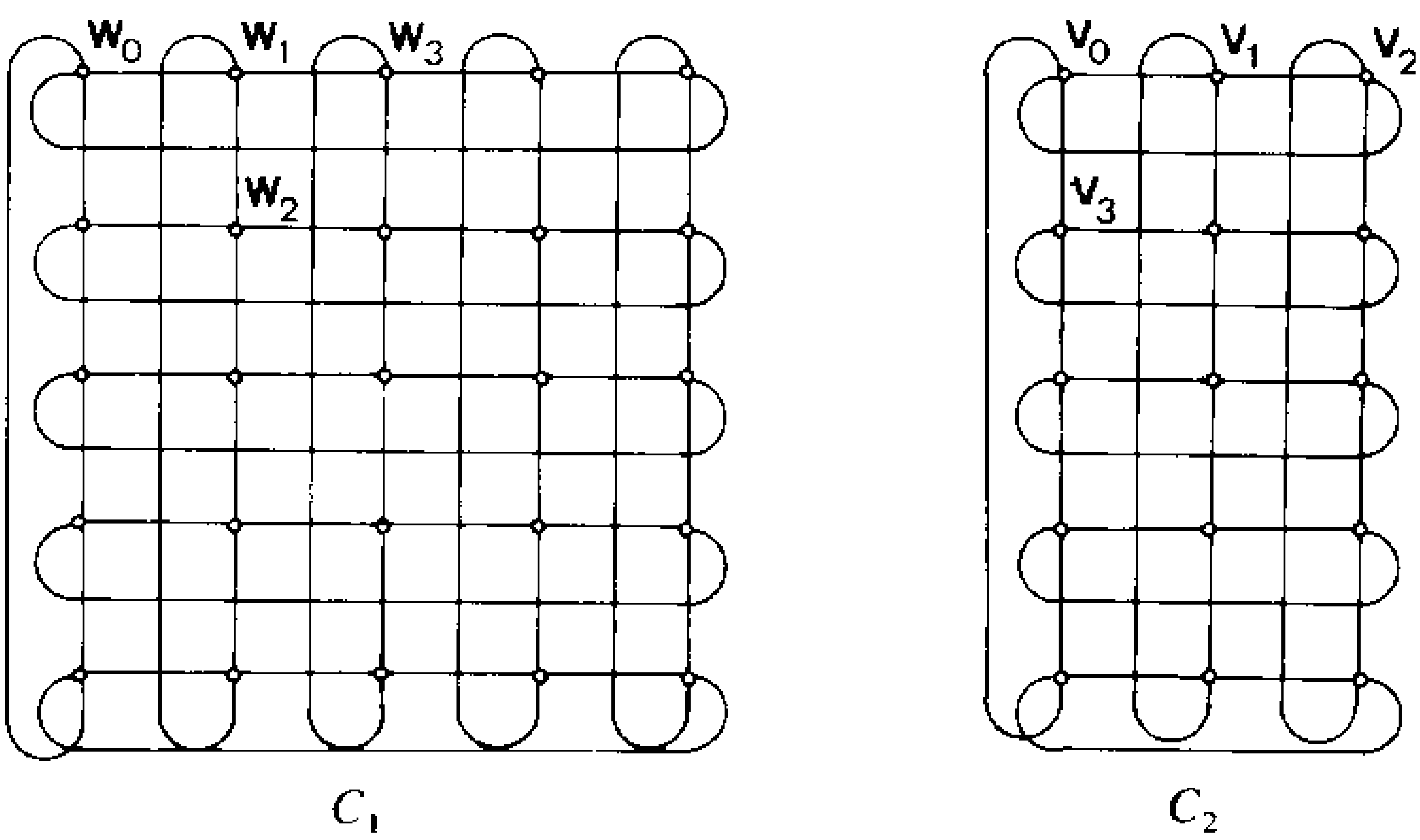

Fig. 4.

Problem 6.2. Find other assumptions than the countability of AUT(X) such that Theorem 6.1, (3), also holds for $s \leq 1$.

Problem 6.3. Let X be a locally finite connected s-transitive graph with polynomial growth and let $s \leq 6$. Is it always possible to find a sequence of finite graphs which satisfy all assertions of Theorem 6.1 and in addition have the property that they are all exactly s-transitive?

7. Automorphism groups as topological groups

In this paragraph we present parts of [36]. In that paper not only graphs with polynomial growth, but infinite connected locally finite graphs in general are considered. Since we assume that this topological approach might also be useful for the investigation of graphs with polynomial growth, we include some of the results in [36] in this survey.

Let X be a connected locally finite graph and let $g_1, g_2 \in \mathrm{AUT}(X)$, $x \in V(X)$. We set:

(a) $\rho_x(g_1, g_2) = 0$ if $g_1 = g_2$,
(b) $\rho_x(g_1, g_2) = n$ if $d(x, g_1^{-1}g_2(x)) = n \in \mathbb{N}$, and,
(c) $\rho_x(g_1, g_2) = 2^{-n-1}$ if the set $\{y \in V(X) \mid d(x, y) \leq n\}$ is fixed by $g_1^{-1}g_2$ and $g_1^{-1}g_2$ does not fix all vertices at distance $n+1$ from x, where $n \geq 0$.

This left-invariant metric defines a topology which is used in [36]. A neighbourhood basis of this topology consists of the pointwise stabilizers of finite subsets of $V(X)$.

Simple compactness arguments then immediately lead to a reformulation (and extension) of Halin's 'uncountability-theorem' ([15, Theorem 6]; see also Theorem 2.3 of this paper).

Theorem 7.1 (Trofimov [36]). *If G is a closed subgroup of* AUT(X), *then either the stabilizer G_x of a vertex of X in G is finite or G_x, and hence G, is uncountable.*

In view of Theorem 7.1 the next result supplies a useful test for the finiteness of the stabilizer of a vertex.

Let $P = (v_0, v_1, \ldots)$ be a one-way infinite path of X. We call P a *geodesic* if $d(v_i, v_j) = |i - j|$ holds for all $v_i, v_j \in V(P)$. The Buseman function α_P, $\alpha_P : V(X) \to \mathbb{Z}$, is defined by

$$\alpha_P(v) = \lim_{i \to \infty} (d(v, v_i) - i), \qquad v \in V(X).$$

Furthermore let $B(P) = \{v \in V(X) \mid \alpha_p(v) \leq 0\}$.

Theorem 7.2 (Trofimov [36]). *Let G act transitively on the connected locally finite graph X. If $\bar{G}$ denotes the closure of G in* AUT(X) *the following assertions are equivalent*:

(1) *The stabilizer of a vertex of X in G is infinite.*
(2) *There is a $g \in \bar{G}$ and a geodesic P in X such that g fixes $B(P)$ pointwise.*

For further combinatorial theorems based on this topological approach we refer to [36]. Also in [44] some interesting results are shown, using a topological approach. Since geodesics seem to play some role concerning properties of the automorphism group we want to refer to [41], where many conditions for the existence of geodesics in graphs are proved.

8. Final remarks

We mention that graphs with polynomial growth also play some role with respect to spectra of infinite graphs. Concerning this topic we only refer to "A survey on spectra of infinite graphs" by Mohar and Woess [26].

One of the reasons for the above considerations was the existence of transitive graphs that are not Cayley graphs. In the case of graphs of polynomial growth it turned out that they are closely related to Cayley graphs. In the case of graphs with nonpolynomial growth it is an open problem whether and how closely they are related to Cayley graphs. Even in the case of graphs with infinitely many ends (and hence exponential growth), which have been investigated thoroughly by Dicks and Dunwoody [6] such a close relationship as in the case of polynomial growth has not been established.

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