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Bounded graphs

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Abstract

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A graph G is called bounded if for every mapping $f: V(G) \to \mathbb{N}$ there exists a sequence $(c_n)_{n\in\mathbb{N}}$ of naturals such that for every ray $U = (u_1, u_2, u_3, \ldots)$ of G an n_0 can be found with $f(u_i) < c_i$ for all $i \geqslant n_0$. All locally finite connected graphs are bounded. Four basic types of unbounded graphs are given, and it is conjectured that one of these configurations must be present in every unbounded graph. This conjecture is proved for the graphs with rayless blocks and the graphs which do not contain an infinite system of disjoint rays.

The idea to introduce the class of graphs indicated in the title arose from Rado's study of universal graphs [6], especially de Bruijn's proof of the statement that there is no universal locally finite countable graph (i.e., no locally finite countable graph which contains a copy of every other such graph).

First some terminology has to be developed. Let $(a_i)_{i\in\mathbb{N}}$, $(b_i)_{i\in\mathbb{N}}$ be sequences of natural numbers. We will say that (b_i) *majorizes* (a_i) if there is an n_0 such that $b_i > a_i$ for all $i \geqslant n_0$.

Lemma 1. *If $(a_i^\nu)_{i\in\mathbb{N}}$, $\nu = 1, 2, 3, \ldots,$ is a countable family of sequences of natural numbers, then there is a sequence $(b_i)_{i\in\mathbb{N}}$ which majorizes these sequences $(a_i^\nu)_{i\in\mathbb{N}}$ simultaneously.*

Proof. Let $b_i := 1 + \max(a_i^\nu : \nu \leqslant i)$. □

Now let G be a graph, $U = (u_0, u_1, u_2, \ldots)$ a ray (i.e., a one-way infinite path) in G. Here the u_i are the vertices of U in their natural order, which means that $[u_i, u_{i+1}]$ are the edges of U. Let $f: V(G) \to \mathbb{N}$ be a function and $b = (b_i)_{i\in\mathbb{N}}$ a sequence of natural numbers. We say that b majorizes f on U if b majorizes the sequence $f(u_0), f(u_1), f(u_2), \ldots.$ Further we say that b *ray-majorizes* f (with respect to G, or on G), if b majorizes f on every ray of G.

Now we define the central notion of this article. A graph G is *bounded* if for every function $f: V(G) \to \mathbb{N}$ there is a sequence of natural numbers which ray-majorizes f on G. The problem that will concern us is to try to determine the graphs which are bounded.

It is clear that every subgraph of a bounded graph is again bounded. Moreover, to check whether a connected graph is bounded it suffices to consider rays starting in some fixed vertex.

Lemma 2. *Let G be a connected graph, $f: V(G) \to \mathbb{N}$ a function, and assume that $(b_i)_{i\in\mathbb{N}}$ majorizes f on all those rays which start in some fixed vertex u_0. Then there is a sequence $(c_i)_{i\in\mathbb{N}}$ of natural numbers which ray-majorizes f on G.*

Proof. Without loss of generality we may assume (b_i) to be monotonically increasing; otherwise we replace b_i by $\max(b_1, \ldots, b_i)$. Now let $c_i = b_{2i}$. Let $V = (v_0, v_1, v_2, \ldots)$ be a ray in G. By the connectedness of G there is a ray $U = (u_0, u_1, u_2, \ldots)$ in G such that $U \cap V$ is a ray with the vertices $u_{i_0+\nu} = v_{j_0+\nu}$ $(\nu = 0, 1, 2, \ldots)$. By assumption there is a k such that $f(u_{i_0+\nu}) < b_{i_0+\nu}$ for all $\nu \geq k$. Then for all $\nu \geq \max(k, i_0)$ we have $f(v_{j_0+\nu}) < b_{i_0+\nu} \leq b_{2(j_0+\nu)} = c_{j_0+\nu}$. □

The following lemma is proved similarly.

Lemma 3. *If H is a bounded subgraph of G such that every ray of G has a subray in H, then G itself is bounded.*

As a consequence of Lemmas 1 and 3 we have the following.

Lemma 4. *If $F \subset G$ is finite and $G - F$ has at most countably many components which are all bounded, then G itself is bounded.*

Let us consider some examples of unbounded graphs which seem to be the prototypes for this class of graphs. Since there are no more than 2^ω distinct maps from the vertex set of a ray to $\mathbb{N}$ (ω denotes the smallest infinite cardinal), it is clear that the union of 2^ω disjoint rays is unbounded. We remark that, in the absence of the Continuum Hypothesis, Martin's Axiom (which is independent of ZFC) implies the existence of an uncountable system of disjoint rays forming a bounded graph. To avoid set theoretic intricacies, we shall therefore assume the Continuum Hypothesis in what follows.

Note further that the ω-regular tree T_ω is not bounded. For if any injective mapping $f: V(T_\omega) \to \mathbb{N}$ is given, every finite path can be extended by a vertex of arbitrarily large value under f. Similarly one establishes that every subdivision of T_ω is unbounded. By analogous reasoning one finds that also the *bundle graph* and the *fan graph* (Fig. 1) and their subdivisions are unbounded. (The bundle graph arises from a ray by replacing each edge in 'even position' with a $K_{2,\omega}$; the fan graph is obtained from a ray $(u_0, u_1, u_2, \ldots)$ by 'planting' disjoint rays R_n on u_{2n} and connecting u_{2n-1} to all the vertices of R_n by edges, for $n = 1, 2, \ldots$).

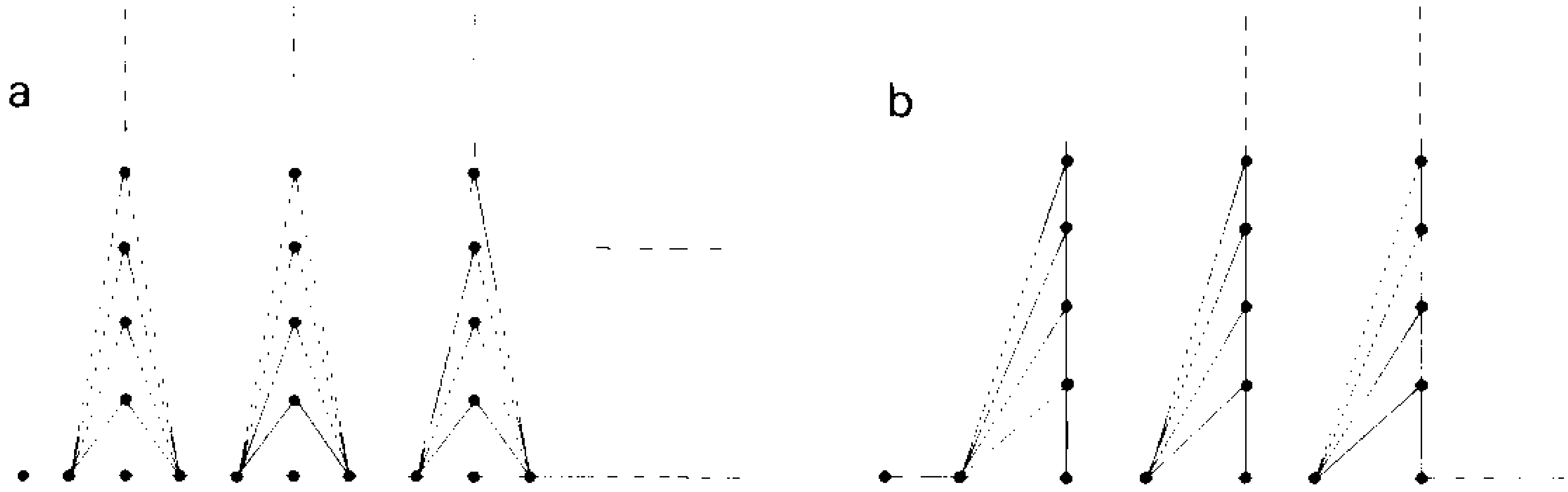

Fig. 1. The bundle graph (a) and the fan graph (b).

Note that the graph of Fig. 2, which looks so similar to the fan graph, is nevertheless bounded (apply Lemma 1 to the vertical rays, choosing the majorizing f_n monotonic).

A large variety of bounded graphs is given by the following theorem.

Theorem 1. *Every locally finite countable graph G is bounded.*

Proof. By Lemma 4 we may assume G to be connected. Let $V(G) = \{v_0, v_1, v_2, \dots\}$ be an enumeration of the vertices of G. Consider an arbitrary function $f: V(G) \to \mathbb{N}$. Let V_i be the set of all $x \in V(G)$ with distance at most i from $\{v_0, \dots, v_i\}$. Clearly every V_i is finite. Now let $b_i := 1 + \max(f(x): x \in V_i)$. If $U = (u_0, u_1, u_2, \dots)$ is a ray in G, then $u_0 = v_k$ for some $k \in \mathbb{N}$. Then for $i \geq k$ every u_i belongs to V_i; hence $f(u_i) < b_i$ for all $i \geq k$. Thus the sequence (b_i) ray-majorizes f on G. □

Now it is easy to show de Bruijn's result [6] that the class of locally finite countable graphs has no universal element. Let G be a graph from this class and choose $f(v)$, for each $v \in V(G)$, as the degree of v in G. By Theorem 1 there is a sequence (b_i) of naturals which ray-majorizes f on G. Let H be any graph having a ray $U = (u_0, u_1, \dots)$ such that the degree of u_i is $\geq b_i$ for each i. (For instance H can be chosen as a locally finite tree.) Then H cannot be isomorphic to a subgraph

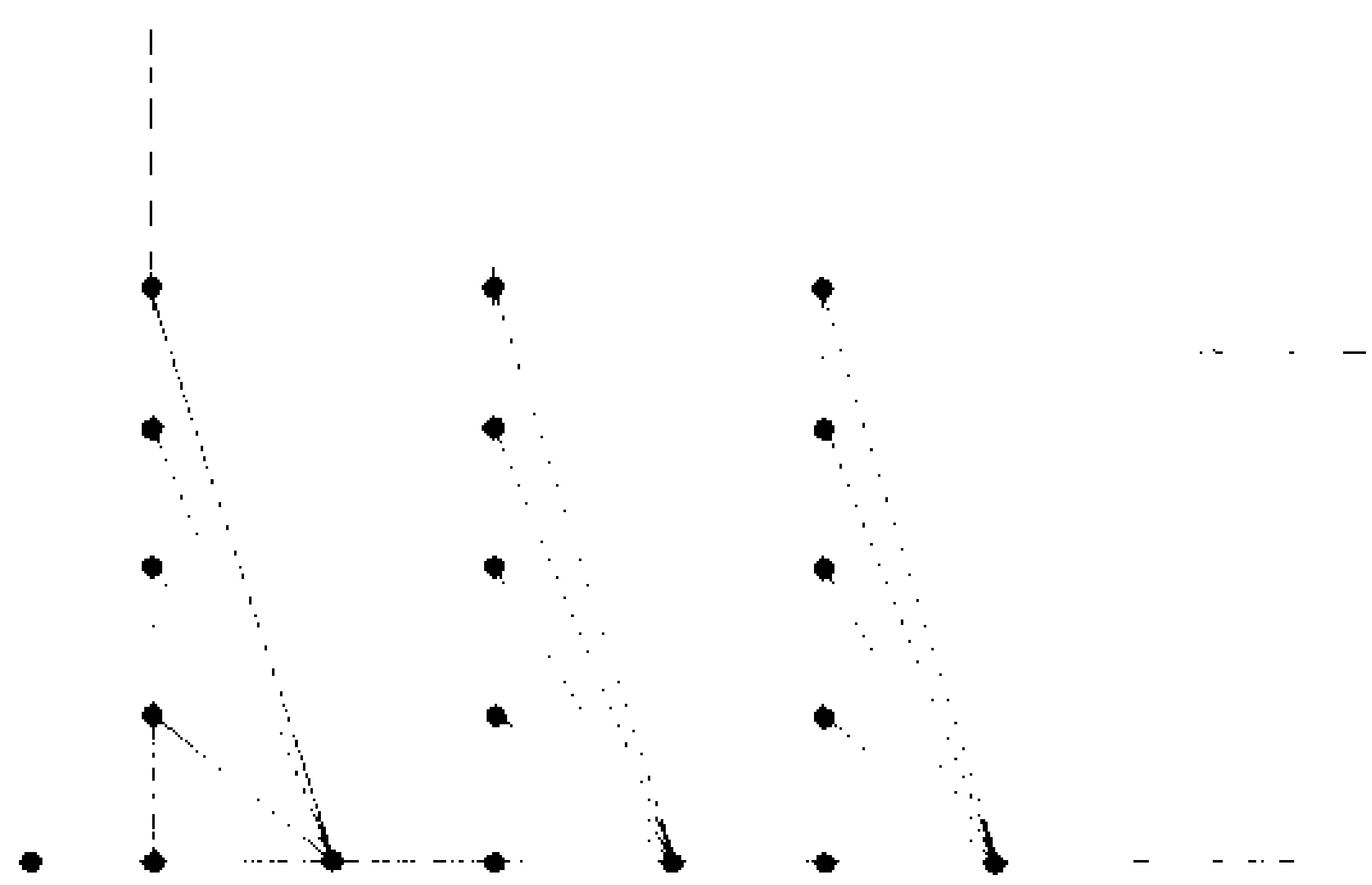

Fig. 2.

of G, by choice of the sequence (b_i). Hence G cannot be universal in the class of countable locally finite graphs.

In the class of all countable graphs the bounded ones form a proper sub-class, itself containing the class of locally finite countable graphs properly. The main problem in this context is, naturally, how the bounded graphs can be characterized, especially whether there is a criterion in terms of forbidden configurations.

As we have seen, the bundle graph, the fan graph, the ω-regular tree and the disjoint union of uncountably many rays are forbidden in a bounded graph. The following is an old conjecture of the author, first proposed (for countable graphs) in 1964, in connection with [6].

Bounded Graph Conjecture (BGC). Assume the Continuum Hypothesis. A graph G is bounded if and only if it contains neither an uncountable system of disjoint rays nor a subdivision of the bundle graph, the fan graph or the ω-regular tree T_ω.

BGC would imply that an uncountable graph which does not contain uncountably many disjoint rays is bounded if and only if every countable subgraph is bounded. Also it would follow from BGC that a countable graph G is already bounded if there exists at least one injective function $f: V(G) \to \mathbb{N}$ which is ray-majorized by some sequence of natural numbers. (This statement is easily derived from the fact that no injective $\mathbb{N}$-valued function on one of the forbidden configurations can be ray-majorized by a sequence of naturals.)

We shall prove BGC for two special classes of infinite graphs. First we state a further lemma.

Lemma 5. *Let R, S be disjoint finite sets of vertices in a graph G, and let H denote the union of all $R-S$ paths in G (i.e. of those paths which connect an $r \in R$ with an $s \in S$ and have only their endvertices in common with $R \cup S$). If H is infinite, then either it contains a ray, or there exists an $R-S$ path $P = r \cdots s$ with vertices $x, y \in P$ whose Menger (or local connectivity) number is infinite. That is to say, we can find infinitely many internally disjoint $x-y$ paths in H which have only x and y in common with P.* (Such a configuration will be called an $R-S$ *bundle* with endpoints r, s; see Fig. 3.)

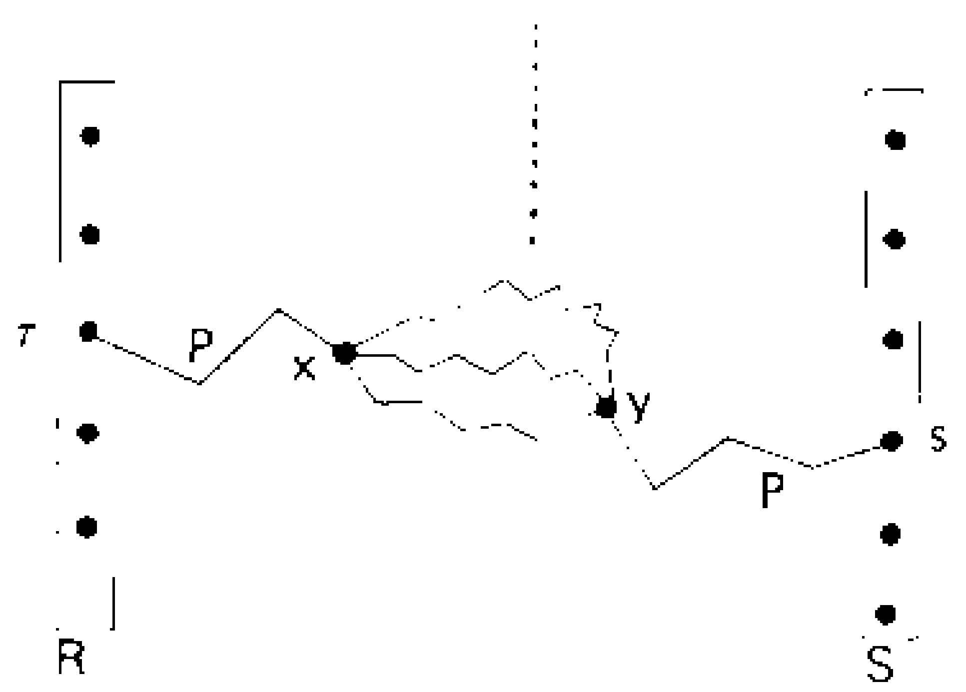

Fig. 3. An $R-S$ bundle.

Proof. Let G_{rs} be obtained from G by adding two new vertices r and s, joining r to the vertices of R and s to those of S. Similarly, let H_{rs} consist of H, the new vertices r and s, and the $r-H$ and $s-H$ edges of G_{rs}.

Of course, H_{rs} is connected. If it does not contain a ray (which would have a subray in H), there must exist a vertex $x \notin \{r, s\}$ of infinite degree (with respect to H_{rs}). Let N be the set of neighbours of x. For each $w \in N$ the edge $[x, w]$ lies on an $r-s$ path P_w which we consider to be oriented from r to s. There is an infinite subset N' of N such that, for all $w \in N'$, the vertices x, w have the same position with respect to the orientation of P_w; by symmetry we may assume that x preceeds w on P_w for all $w \in N'$. Let H' be the (infinite) union of the $x-s$ paths $P \subset P_w$ for $w \in N'$, and put $H'' := H' - x$. Note that H'' is still connected. If there is a finite $F \subset V(H'')$ such that $H'' - F$ has infinitely many components $C_\nu (\nu \in \mathbb{N})$ each containing a $w_\nu \in N'$, then choose paths Q_ν from w_ν to F. There must be a vertex $y \neq s$ of F which is the endvertex of infinitely many such Q_ν; let $w \in N'$ be such that $y \in P_w$, and let $\mu \in \mathbb{N}$ be such that $P_w \cap C_\nu = \emptyset$ for every $\nu > \mu$. The paths $[x, w_\nu] \cap Q_\nu$ for $\nu > \mu$, together with P_w, then give the desired configuration. On the other hand, if there is no such finite F, we choose an arbitrary vertex v_0 of H''. $H'' - v_0$ must have a component C_0 which contains infinitely many vertices from N'. Let v_1 be a neighbour of v_0 in C_0. Then again $C_0 - v_1$ must have a component C_1 containing infinitely many vertices from N', and we choose v_2 as a neighbour of v_1 in C_1. Continuing in this way we obtain a ray in H_{rs}, which has a subray in H. Our lemma follows. □

Theorem 2. *BGC holds for every graph G all blocks of which are rayless.*

Proof. We suppose that G is unbounded, and show that G contains one of the configurations listed in BGC. This is clear if G has uncountably many components each containing a ray; we may therefore assume that G has only countably many such components. By Lemma 4, then, one of these components must be unbounded. Since it suffices to find one of the desired configurations in this component, we may thus assume that G is connected. Let as arrange the blocks of G in such a way that they form a simplicial decomposition of G (see [1], or [2, p. 224]). This means that the blocks are indexed as subgraphs G_λ of G where λ runs through all ordinals less than some fixed ordinal σ, and for each τ (with $0 < \tau < \sigma$) G_τ has exactly one vertex a_τ (an articulation) in common with the union of the G_λ with $\lambda < \tau$. Let τ_- denote the smallest λ with $a_\tau \in V(G_\lambda)$. Note that unless $\tau_- = 0$, a_τ and a_{τ_-} are distinct vertices of G_τ, and hence joined by a nontrivial path in G_τ. Take the ordinals $< \sigma$ to be the vertices of graph $\mathcal{T}$ whose edges are the (unordered) pairs $[\tau, \tau_-]$. Each sequence $\tau, \tau_-, (\tau_-)_-, \ldots$ ends at 0 after finitely many steps; so $\mathcal{T}$ is connected. Further, if $\tau_1 < \tau_2 < \cdots < \tau_k$ are finitely many vertices of $\mathcal{T}$, then τ_k can have at most one neighbour with respect to $\mathcal{T}$ among $\tau_1, \ldots, \tau_{k-1}$; hence $\mathcal{T}$ is without circuits. We see that $\mathcal{T}$ is a tree. Consider the ordinal 0 as its root; we denote the natural order of the tree $\mathcal{T}$ (with respect to the root 0) by $\ll$ and carry it over to the blocks of G (by the correspondence $\lambda \to G_\lambda$). We fix a vertex r in G_0. By Lemma 2, in order to test the boundedness of G, we need only consider rays starting in r. Each such ray R traces out a unique ray R^* of $\mathcal{T}$ starting in 0, and vice versa, if a ray T of $\mathcal{T}$ starting in 0 is given, we find at least one ray $R \subseteq G$ starting in r with $R^* = T$; all rays R of this kind traverse the same blocks, namely those corresponding to the vertices of T.

If, for some λ with $0 < \lambda < \sigma$, there is no ray $\subseteq \mathcal{T}$ starting in 0 and containing λ, then in G there is no ray starting in r and meeting G_λ in a vertex different from a_λ. So we may omit this block G_λ because it is not relevant for our problem of boundedness; we see that we may assume, without loss of generality, that $\mathcal{T}$ has no endvertex (with the possible exception of 0). Further, if $\mathcal{T}$ has a vertex of uncountable degree, we find uncountably many disjoint rays in $\mathcal{T}$ and therefore in G. So we can also assume that $\mathcal{T}$ is countable.

If there is a ray T in $\mathcal{T}$ such that infinitely many of the G_λ with $\lambda \in V(T)$ are infinite, then by applying Lemma 5 to each such G_λ we find a subdivision of the bundle graph in G. (Note that, since G_λ is 2-connected, every vertex or edge of G_λ lies on a path in G connecting the first and the last vertex in G_λ of any ray $R \subset G$ with $R^* = T$.) So we may assume that for every ray T of $\mathcal{T}$ all the G_λ with $\lambda \in V(T)$, apart from finitely many, are finite.

Let us show that $\mathcal{T}$ is unbounded. Suppose $\mathcal{T}$ is bounded; we shall prove that then G is also bounded, contrary to our assumption. Let $f: V(G) \to \mathcal{N}$ be given. For each finite G_λ put $f^*(\lambda) = \max(f(x): x \in V(G_\lambda))$; for infinite G_λ let $f^*(\lambda) = 1$. By assumption there is a monotonically increasing sequence (c_ν) ray-majorizing f^* on $\mathcal{T}$. Now let $U = (u_0, u_1, \ldots)$ with $u_0 = r$ be a ray in G. U determines a ray $U^* = (\lambda_0, \lambda_1, \ldots)$ in $\mathcal{T}$; each u_j is in a G_{λ_i} with $i \leq j$. There exists a k such that G_{λ_i} is finite and $f^*(\lambda_i) < c_i$ for each $i \geq k$. Since, for sufficiently large n, $u_n \in G_{\lambda_i}$ implies $i \geq k$, we therefore have $f(u_n) \leq f^*(\lambda_i) < c_i \leq c_n$ for all such n, showing that (c_ν) ray-majorizes G. As this implies that G is bounded (contrary to our assumption), we deduce that $\mathcal{T}$ is unbounded, as claimed.

Call a vertex α of $\mathcal{T}$ *essential* if the branch above α, i.e. the subtree induced by α and all $\lambda \gg \alpha$, is unbounded. The essential vertices induced a subtree $\mathcal{T}_0$ of $\mathcal{T}$, which includes the root 0. Since $\mathcal{T}$ is countable, $\mathcal{T} - \mathcal{T}_0$ has at most countably many components $\mathcal{C}_i$, which are all bounded. If $\mathcal{T}_0$ is also bounded, then let an arbitrary mapping $f: V(\mathcal{T}) \to \mathbb{N}$ be given. Since the $\mathcal{C}_i$ and $\mathcal{T}_0$ are all bounded, for each of these trees there is a sequence ray-majorizing the restriction of f to that tree, and by Lemma 1 we get a sequence which majorizes all these sequences simultaneously. Since each ray of G ends in $\mathcal{T}_0$ or in some $\mathcal{C}_i$, we find that the latter sequence ray-majorizes f on $\mathcal{T}$. As this contradicts the fact that $\mathcal{T}$ is unbounded, we find that $\mathcal{T}_0$ too is unbounded.

Hence, by Theorem 1, $\mathcal{T}_0$ cannot be locally finite; let τ_0 be a vertex of infinite degree with respect to $\mathcal{T}_0$. The neighbours $\gg \tau_0$ of τ_0 in $\mathcal{T}_0$ determine infinitely many branches of $\mathcal{T}$ which are themselves unbounded. In each of these branches we analogously find a vertex of infinite degree, and repeating this procedure again and again we get a subdivision $\mathcal{U}$ of T_ω in $\mathcal{T}$; let ξ be the minimal vertex of $\mathcal{U}$ with respect to $\ll$.

If τ is a vertex of infinite degree in $\mathcal{U}$, then there are infinitely many λ in $\mathcal{U}$ such that a_λ is in G_τ. If $\tau \gg \xi$ then also a_{τ_-} is in G_τ. As G_τ is connected and rayless, we can find a vertex z_τ in G_τ and a system of paths (having pairwise only z_τ in common) from z_τ to infinitely many of the a_λ. (z_τ and some of the a_λ may coincide; then the paths in question are of length 0.) Further, by adding a $z_\tau - a_\tau$ path and 'sacrificing' finitely many of the other paths if necessary, we may assume that a_{τ_-} is among these a_λ. We now apply this observation repeatedly, starting with ξ and performing the same procedure step by step to subsequent vertices of infinite degree in U. We finally get a subdivision of T_ω in G, and our proof is complete. □

Theorem 2 is a sharpening of Theorem 6.6 in [5], in which BGC was proved for countable trees. In the light of the graphs in Figs. 1b and 2 it seems rather hopeless to apply the method of our last proof for a more general verification of BGC, e.g. for all graphs which have a simplicial decomposition with finite members.

In [4] it was proved that in every graph G there is a system of disjoint rays with maximal cardinality; the latter invariant is denoted by $m_1(G)$.

Theorem 3. *BGC holds for all graphs G with $m_1(G) < \infty$.*

Proof. Assume that $m_1(G) = n \geq 1$ (the case $n = 0$ is trivial). Suppose G is unbounded; our aim is to establish the existence of a subdivision of the bundle graph in G. Since there are only finitely many ends in G, there exists a finite $T \subset G$ such that each component of $G - T$ has at most one end (see [3] for the definition of an end). If each of the finitely many components of $G - T$ containing rays were bounded, their union would form a bounded subgraph of G in which each ray ends, and G itself would be bounded by Lemma 3. Therefore at least one of these components must be unbounded, so we may assume without loss of generality that G itself has only one end. By *Satz 2* in [4] there exists a finite $F \subset G$ such that $G - F$ is the union of an infinite sequence $G_0, G_1, G_2, \ldots$ of (induced) rayless subgraphs such that each G_i ($i \geq 1$) has a subgraph T_i of n vertices in common with G_{i-1} and empty intersection with each G_ν, $\nu \leq i - 2$, and further there is a path matching of T_i and T_{i+1} in G_i (see Fig. 4).

Note that distinct T_i must be disjoint. Moreover, each T_i separates (in $G - F$) the vertices of $(G_0 \cup \cdots \cup G_{i-1}) - T_i$ from those of $(G_i \cup G_{i+1} \cup \cdots) - T_i$, and the union of the aforementioned path matchings forms a system of n disjoint rays $R_1, \ldots, R_n$. We shall assume that there is no rayless component of $G - F$; otherwise it could be omitted by Lemma 3.

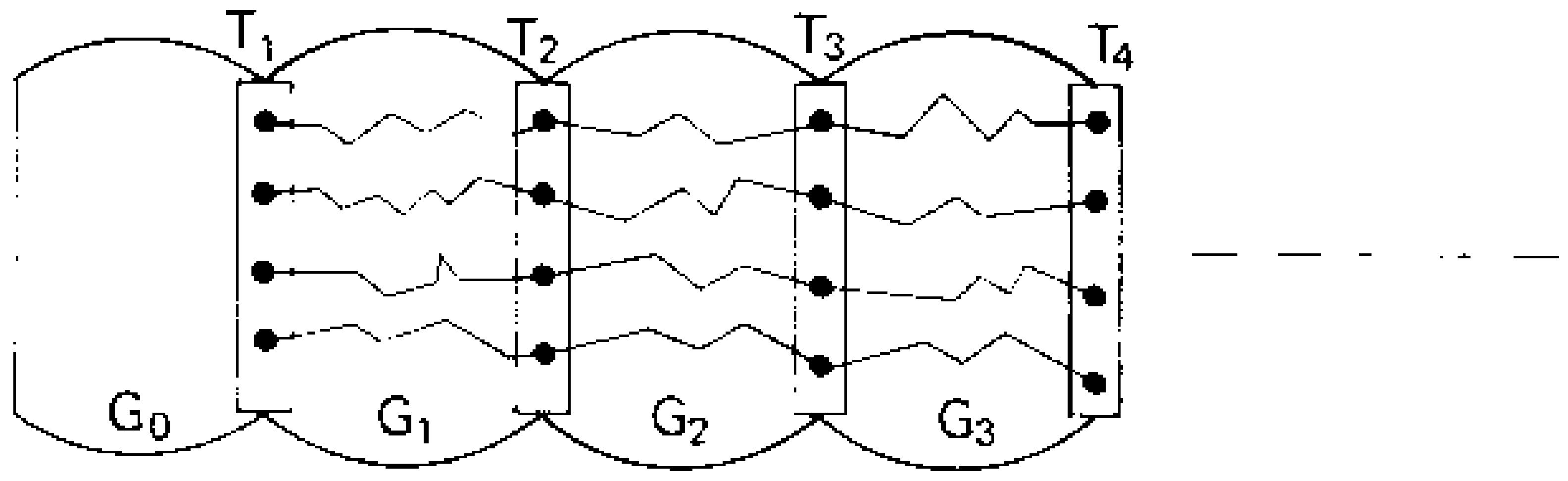

Fig. 4.

As $R_1, \ldots, R_n$ all belong to the same end, for any pair R_i, R_j there are infinitely many disjoint paths each connecting a vertex of R_i with a vertex of R_j, and we find an infinite sequence of integers $0 = i_0 < i_1 < i_2 < \cdots$ such that

$$G'_\nu := G_{i_\nu} \cup G_{i_\nu+1} \cup \cdots \cup G_{i_{\nu+1}-1}$$

is connected, for $\nu = 0, 1, 2, \ldots$.

We see that the G'_ν form an analogous representation of $G - F$; we may thus assume that already every G_i is connected. Choose a fixed vertex r of $G_0 - T_1$ (as a 'root'), and let V' be the set of vertices of $G - F$ which lie on a ray starting in r; put $V'_i = V(G_i) \cap V'$. Obviously $T_i \subseteq V'$ for all $i \in \mathbb{N}$, and every ray of G has a subray containing only vertices of V'.

If there is a $k \in \mathbb{N}$ such that all V'_j with $j \geq k$ are finite, then let

$$H := G_k \cup G_{k+1} \cup G_{k+2} \cup \cdots,$$

and let an arbitrary function $f : V(H) \to \mathbb{N}$ be given. Let r' be a fixed vertex ('root') of T_k, and put

$$c_i := 1 + \max(f(v) : v \in V'_k \cup \cdots \cup V'_{k+i})$$

for all $i \in \mathbb{N}$. If $U = (u_0, u_1, u_2, \ldots)$ is a ray $\subseteq H$ starting in $r' = u_0$, then obviously $u_i \in V'_k \cup \cdots \cup V'_{k+i}$ for each i, implying $f(u_i) < c_i$. Therefore H is bounded by Lemma 2. Since every ray of G ends in H, Lemma 3 then implies that G is bounded, a contradiction. Hence, there are infinitely many $i \in \mathbb{N}$ such that V'_i is infinite. Let V''_i denote the set of those vertices in G_i which lie on a $T_i - T_{i+1}$ path ($i \geq 1$). Then $V''_i \subseteq V'_i$, but there may be vertices in V'_i which do not belong to V''_i. Each such vertex x must lie on a path $P \subseteq G_i$ connecting a pair of vertices of T_i or a pair of vertices of T_{i+1}, and is separated from V''_i by T_i (or T_{i+1}, respectively). We then say that x is *attached* to T_i (or T_{i+1}, respectively).

For each i such that V''_i is infinite we may use Lemma 5 to obtain a $T_i - T_{i+1}$ bundle B_i with certain endpoints s_i in T_i and t_i in T_{i+1}. Now suppose that the set I of these i is infinite. We choose an infinite sequence i_ν ($\nu \in \mathbb{N}$) of elements of I such that $i_\nu + 2 \leq i_{\nu+1}$ for all $\nu \in \mathbb{N}$. Further we choose a path P_1 from r to s_{i_1} through $G_0 \cup \cdots \cup G_{i_1-1}$; then we add B_{i_1} and select a path P_2 from t_{i_1} to s_{i_2} through $G_{i_1+1} \cup \cdots \cup G_{i_2-1}$. Then we add B_{i_2} and select a path from t_{i_2} to s_{i_3} through $G_{i_2+1} \cup \cdots \cup G_{i_3-1}$. Continuing in this way we get a subdivision of the bundle graph in $G - F$.

If there are only finitely many i for which V_i'' is infinite, then there must be infinitely many i such that infinitely many elements of V_i' are attached to T_i or T_{i+1}. But every vertex of V_i' attached to T_i or T_{i+1} lies on a $T_{i-1} - T_{i+2}$ path. Putting $R := T_{i-1}$ and $S := T_{i+2}$ in Lemma 5 for infinitely many such i (sufficiently far apart), we again obtain a bundle graph. □

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