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Infinite matching theory

Ron Aharoni
Department of Mathematics, Technion—Israel Institute of Technology, Haifa 32000, Israel

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Abstract

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We survey the existing theory of matchings in infinite graphs and hypergraphs, with special attention to the duality between matchings and covers. Some results are presented which have not appeared elsewhere, mainly concerning Menger's theorem for infinite graphs.

1. Introduction

The study of a theory which encompasses, among other concepts, such a basic notion as that of injections between infinite sets, hardly needs further advocacy. But if one is required, an interesting observation is that infinite matching theory can boast a result which is probably as early as any result in finite matching theory, if not earlier.

Theorem 1.1. *If A and B are subsets of the two respective sides of a bipartite graph, and if there exist two matchings covering A and B respectively, then there is a matching covering $A \sqcup B$.*

This is a re-formulation of the famous Cantor-Bernstein Theorem (see, e.g. [20, p. 23]).

There is also no doubt that infinite matching theory, besides its inherent interest, can illuminate some points in finite matching theory. For example, a result saying that the infinite version of a theorem is 'hard' to prove (see [9]) probably reflects on the complexity of the finite problem, although no explicit connection has been established as yet.

Matching theory can be divided very roughly, but usefully, into two parts, circling around two themes: criteria for matchability (exemplified by the theorems of Hall and Tutte) and the duality between matchings and covers (the outstanding results here are the theorems of König and Menger). The two themes are very closely knit together, and an important fact on the link between them should be mentioned here: whereas in the finite case duality results can be proved directly, it seems that in the infinite case they have to be proved via their counterparts on criteria for matchability. The reason is that in the finite case one can start from a maximal matching and produce the dual cover from it, an endeavour which is pretty meaningless in the infinite case (maximal matchings exist with respect to containment, but a stronger notion of maximality is needed here, see Theorem 5.3.)

Here is a somewhat brief historical survey. The first modern treatment of infinite matching theory is in the book of König [21]. It contains an extension due to Erdős of Menger's theorem to the infinite case. In 1935, P. Hall published his theorem [17] which quickly became famous, a fate which befell its predecessors, the theorems of König and Menger (of which it is an easy consequence) only much later.

Theorem 1.2 (P. Hall's Theorem). *In a finite bipartite graph with bipartition (M, W) there exists a matching of M if and only if every subset A of M is connected to at least $|A|$ vertices in W.*

It was well known that the theorem fails in the infinite case. There is a basic example showing this: a bipartite graph whose vertex set is $\{m_i: 0 \leqslant i < \omega\} \sqcup \{w_i: 1 \leqslant i < \omega\}$ and edge set $\{(m_i, w_i): 1 \leqslant i < \omega\} \sqcup \{(m_0, w_i): 1 \leqslant i < \omega\}$ (m_0 is sometimes called the 'playboy'). Every set A of m_i's is connected to at least $|A|$ elements w_i, but there is no matching of the m_i's. It was felt that the example is cannonical in some sense for the countable case. This was confirmed in the early 1970's on apparently two different lines of ideas. One was due to Nash-Williams, who proposed a somewhat complicated function which counts the difference between the size of a set of women (elements of W) and the number of men (elements of M) who must be married (matched) into this set. Damerell and Milner [13] proved a conjecture of Nash-Williams that, in the countable case, there is a marriage of all men if and only if this function is always nonnegative. Later [22] Nash-Williams simplified his own criterion. But there was another line of ideas, which culminated in Podewski and Steffens' criterion on 'critical sets' ([24], see Section 3). In [10] the two criteria were shown to be equivalent. In hindsight it seems that, because of its simplicity, the 'critical sets' criterion is the more useful one, in spite of its apparent circular requirement that the critical set itself should be matchable. For example, Podewski and Steffens were able to prove, using it, the countable case of Erdős' conjecture on the extension of König's theorem to the infinite case.

The general problem was settled in 1983, when a criterion for matchability was proved for general bipartite graphs [10]. Basically, it confirmed a feeling that the most general obstacle for the existence of an injective choice function in the uncountable case is one which forces it to be regressive on a stationary set. This criterion was used to prove König's theorem for general graphs in [3].

As for general graphs, the finite criterion for matchability was found by Tutte [27]. Later he proved that his criterion is valid also for locally finite graphs [28]. (The corresponding extension of P. Hall's theorem, namely that P. Hall's condition is sufficient if the degree of every vertex in the set of 'men' is finite, was proved by M. Hall [18]. But this is nowadays an easy exercise in compactness.) Steffens [26] gave a criterion for matchability in countable graphs, which was later used to prove a Tutte-like result for countable graphs [4]. The general problem was solved in [6].

Although this is a survey paper, we have included a few new results. The survey is not intended in any way to be comprehensive. There is an excellent book on the bipartite case [19] which is much more detailed on this subject.

2. Definitions

A graph will be usually denoted by $G=(V, E)$, where V is the set of vertices and E the set of edges (i.e. $E \subseteq [V]^2$). If G is bipartite with bipartition $V = M \sqcup W$, it will be denoted also by (M, W, E). We then write: $M = M_G$, $W = W_G$, $E = E_G$. If X is a subset of V we write $G[X]$ for the subgraph of G induced by X. The graph $G[V \backslash X]$ is denoted by $G - X$. If $\Gamma = (M, W, E)$ is bipartite and $X \subseteq V$, we assume that $\Gamma[X]$ has the bipartition $(M \cap X, W \cap X)$. A *matching* in a graph is a set of disjoint edges. A *cover* is a set of vertices meeting all edges. For a given subset F of E, a vertex a and a subset A of V, we write

$$F\langle a\rangle = \{v \in V: \{a, v\} \in F\}, \qquad F[A] = \bigcup \{F\langle a\rangle : a \in A\} \quad \text{and}$$

$$F \restriction A = \{f \in F: f \cap A \neq \emptyset\}.$$

If $|F\langle a\rangle| = 1$ we write $F(a)$ for the single element of $F\langle a\rangle$. A matching F is said to *cover* a set A if $A \subseteq F[V]$. If such F exists, then A is said to be *matchable* and if, in such a case, $F[A] \subseteq B$, we say that A is *matchable into* B. For any subset X of V we write $D(X) = \{v \in V: E\langle v\rangle \subseteq X\}$. An *espousal* in a bipartite graph Γ is a matching covering M_Γ. If such a matching exists, then Γ is said to be *espousable*.

We need some terminology concerning paths. The vertex set of a path P is denoted by $V(P)$, and its edge set by $E(P)$. If P has a first vertex, it is denoted by $\mathrm{in}(P)$, and if it has a last vertex, it is denoted by $\mathrm{ter}(P)$. If $\boldsymbol{P}$ is a family of paths, we write $\boldsymbol{P}^f$ for the set of finite paths in $\boldsymbol{P}$. We also write

$$V[\boldsymbol{P}] = \bigcup \{V(P): P \in \boldsymbol{P}\}, \qquad E[\boldsymbol{P}] = \bigcup \{E(P): P \in \boldsymbol{P}\},$$

$$\mathrm{in}[\boldsymbol{P}] = \{\mathrm{in}(P): P \in \boldsymbol{P} \text{ has a vertex}\} \quad \text{and}$$

$$\mathrm{ter}[\boldsymbol{P}] = \{\mathrm{ter}(P): P \in \boldsymbol{P} \text{ has a last vertex}\}.$$

If P and Q are paths such that $V(Q) \cap V(P) = \{\mathrm{in}(Q)\} = \{\mathrm{ter}(P)\}$ then $P * Q$ denotes the concatenation of P and Q. If $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two families of disjoint paths and $V[\boldsymbol{Q}] \cap v[\boldsymbol{P}] = \mathrm{in}[\boldsymbol{Q}] \subseteq \mathrm{ter}[\boldsymbol{P}]$ then $\boldsymbol{P} * \boldsymbol{Q}$ denotes the family $\{P * Q\colon P \in \boldsymbol{P},\ Q \in \boldsymbol{Q} \text{ and } \mathrm{ter}(P) = \mathrm{in}(Q)\}$.

3. Bipartite graphs

Some people regard Hall's Theorem as the cornerstone of finite matching theory, but others, the author included, feel that this title is due to the closely related Theorem of König (which has the advantages of being more symmetrical, of implying Hall's Theorem more easily than the other way round, and also of historical precedence!). Here it is, in a form which is true also in the infinite case:

Theorem 3.1 [3]. *In any bipartite graph there exist a matching F and a cover C such that C consists of the choice of precisely one vertex from each edge in F.*

A pair (F, C) as in the theorem is called *orthogonal.*

Let $\Gamma = (M, W, K)$ be a bipartite graph, and let F and C be as in the theorem. Let $X = C \cap W$ and $D = M \backslash C$. Since C is a cover, there is no edge from D to $W \backslash C$, which means that $K[D] \subseteq X$. Since obviously $X = F[D] \subseteq K[D]$ we have $X = K[D]$. Now assume that Γ is inespousable. Then D is unmatchable, for if I were a matching of D, then $I \sqcup (F \restriction (M \cap C))$ would be an espousal, since $I[D] \subseteq X \subseteq W \backslash F[M \cap C]$. Hence we have the following, which should be considered as the infinite version of the marriage theorem:

Theorem 3.2. *Γ is inespousable if and only if there exists an unmatchable subset D of M such that $K[D]$ is matchable into D.*

Note that in the finite case the conditions on D imply $|K[D]| < |D|$, and thus Hall's Theorem is a particular case of Theorem 3.2.

How does one go about proving Theorem 3.1? Here we are about to present one of the protagonists of the play: the *wave*. A *wave* $\boldsymbol{W}$ is a set of vertex disjoint paths such that $\mathrm{in}[\boldsymbol{W}] \subseteq M$, and $\mathrm{ter}[\boldsymbol{W}]$ is a cover. Note that although Theorem 3.1 is symmetrical in M and W, the notion of a 'wave' is not. In fact, we shall introduce an even greater a-symmetry—we are considering here Γ as a directed graph, in which the edges are directed from M to W. Thus the paths in $\boldsymbol{W}$ consist of either one edge or just one vertex. There exists at least one wave: $\{(m)\colon m \in M\}$. This is called the *trivial* wave. We define an order $\leqslant$ on the waves: $\boldsymbol{W} \leqslant \boldsymbol{U}$ if $\mathrm{in}[\boldsymbol{U}] \subseteq \mathrm{in}[\boldsymbol{W}]$ and every path in $\boldsymbol{U}$ is an extension of some path in $\boldsymbol{W}$. The trivial wave is minimal in this order. A wave $\boldsymbol{W}$ is called *tight* if there does not exist a set $\boldsymbol{V}$ of vertex disjoint paths such that $\mathrm{in}[\boldsymbol{V}] = \mathrm{in}[\boldsymbol{W}]$ and $\mathrm{ter}[\boldsymbol{V}] \subsetneqq \mathrm{ter}[\boldsymbol{W}]$. A wave $\boldsymbol{W}$ is called *full* if $\mathrm{in}[\boldsymbol{W}] = M$. Otherwise it is called a *hindrance*. A tight hindrance is called a *1-obstruction* (a slightly different definition is given in Section 4). If Γ contains a hindrance, it is called *hindered*, and if it contains a 1-obstruction, it is called *1-obstructed*. For a given wave $\boldsymbol{W}$ we denote by ess($\boldsymbol{W}$) the set $\boldsymbol{W}\setminus\{P \in \boldsymbol{W}$: ter$[\boldsymbol{W}\setminus\{P\}]$ is a cover$\}$. It is easy to show that ess($\boldsymbol{W}$) is a wave. If $(\boldsymbol{W}_\alpha$: $\alpha < \theta)$ is a $\leqslant$-ascending sequence of full waves, then $\uparrow_{\alpha<\theta}\boldsymbol{W}_\alpha$ denotes the family of paths

$$\{(m, w)\colon m \in \text{in}[\boldsymbol{W}_\alpha] \text{ for all } \alpha \text{ and } (m, w) \in \boldsymbol{W}_\alpha \text{ for some } \alpha\}$$
$$\sqcup \{(m)\colon (m) \in \boldsymbol{W}_\alpha \text{ for all } \alpha\}.$$

(This is just the result of 'growing' the waves along the sequence. The reason for the restriction of $\leqslant$ to full waves is that for general waves Lemma 3.3 is false. Another way of overcoming this difficulty is the introduction of "billows", as is done in Section 4.)

Lemma 3.3. *$\uparrow_{\alpha<\theta}\boldsymbol{W}_\alpha$ is a wave. Moreover, if each $\boldsymbol{W}_\alpha$ is tight, then so is $\uparrow_{\alpha<\theta}\boldsymbol{W}_\alpha$.*

For the proof of the first part, note that if $\theta = \zeta + 1$, then $\uparrow_{\alpha<\theta}\boldsymbol{W}_\alpha = \boldsymbol{W}_\zeta$, which is, of course, a wave. If θ is a limit ordinal, then $S =$ ter$[\uparrow_{\alpha<\theta}\boldsymbol{W}_\alpha]$ is a cover, since if $(m, w) \in K$ and $m \notin$ ter$[\boldsymbol{W}_\alpha]$ for some α, then $w \in$ ter$[\boldsymbol{W}_\beta]$ for all $\beta > \alpha$ (because $\boldsymbol{W}_\beta$ is a wave and $m \notin$ ter$[\boldsymbol{W}_\beta]$) and hence $w \in S$. The second part is left as an exercise.

By Lemma 3.3 there exists a $\leqslant$-maximal full wave $\boldsymbol{U}$. Let $\boldsymbol{W} =$ ess($\boldsymbol{U}$). The set ter$[\boldsymbol{W}]$ will serve as the set C of Theorem 3.1. (Note how easy it is to find C in comparison to the construction of F!) We also have at hand one part of F: the set of paths in $\boldsymbol{W}$ consisting of an edge. Let

$$A = \text{ter}[\boldsymbol{W}] \cap M, \qquad B = \text{ter}[\boldsymbol{W}] \cap W, \quad \text{and} \quad \Gamma' = \Gamma[A \sqcup W\setminus B].$$

Since $C = A \sqcup B =$ ter$[\boldsymbol{W}]$ is a cover, it suffices to show that Γ' is espousable in order to conclude the proof. For if H is an espousal of Γ', then the matching $(\boldsymbol{W} \upharpoonright B) \sqcup H$ is obviously orthogonal to C. Now, the maximality of $\boldsymbol{U}$ implies that Γ' contains no nontrivial wave. Hence it suffices to show that if a bipartite graph contains no nontrivial wave, then it is espousable. In fact, the following somewhat stronger theorem is true.

Theorem 3.4. *An unhindered graph is espousable.*

Note that this theorem looks like a much weaker version of Theorem 3.2 (to see this assume that a graph Γ is inespousable. By Theorem 3.2 there exists an unmatchable subset D of M such that $K[D]$ has a matching I into D. Then $I \sqcup \{(m)\colon m \in M\setminus D\}$ is a hindrance in Γ.) But in fact, by the preceding argument, it carries the whole weight of Theorem 3.2.

The proof of Theorem 3.4 depends on a criterion for espousability proved by Podewski and Steffens [24] in the countable case, and by Aharoni, Nash-Williams and Shelah [10] in the uncountable case. Podewski and Steffens proved the following stronger result in the countable case.

Theorem 3.5. *A countable bipartite graph is espousable if and only if it is not 1-obstructed.*

(In [9] Theorem 3.4 is given, in the countable case, a more direct proof, although the two proofs are, at base, the same. They both have an alternating paths argument at their core.)

Here is a nice corollary of this theorem, which extends to the infinite case a well-known result of Birkhoff [12].[1]

Corollary 3.5a. *An infinite doubly-stochastic matrix contains a nonzero generalized diagonal.*

In terms of bipartite graphs, what the corollary says is this: if Γ is a bipartite graph and f a nonnegative real valued function on $E = E_\Gamma$ such that $\sum_{x\in e} f(e) = 1$ for every $x \in V$, then Γ has a perfect matching on which f is positive.

Outline of proof. Let $H = \{e \in E: f(e) > 0\}$ and let $\Gamma' = (V, H)$. Every vertex in Γ' has a countable degree, and hence every connected component of Γ' is countable. It suffices to show that every component Δ of Γ' is matchable, and by Theorem 1.1 one has to prove only that M_Δ is matchable. If not then, by the theorem, there exists in Δ a tight wave $\boldsymbol{W}$ and a vertex $a \in M_\Delta \backslash \mathrm{in}[\boldsymbol{W}]$. Let $F = E[\boldsymbol{W}]$ and let $U = F[W_\Delta] \sqcup \{a\}$ (i.e. U consists of a, together with the set of initial vertices of paths in $\boldsymbol{W}$ consisting of an edge.) Define a weight function w on the edges of the complete directed graph D_U on U by: $w(u, v) = f(u, F(v))$ $(u \in U,\ v \in U\backslash\{a\},\ u \neq v)$ and $w(u, a) = 0$ for $u \neq a$. Then, for every vertex $v \in U\backslash\{a\}$ there holds

$$\sum_{u\in U\backslash\{v\}} w(u, v) \leq \sum_{u\in U\backslash\{v\}} w(u, v) \tag{$*$}$$

(the right-hand side is equal to $1 - f(v, F(v))$), while

$$\sum_{u\in U\backslash\{a\}} w(a, u) = 1, \quad \sum_{u\in U\backslash\{a\}} w(a, u) = 0 \tag{$**$}$$

This is easily seen to imply the existence of an infinite path $a, u_1, u_2, \ldots$ in D_U on whose edges w is positive. (A first step towards a proof of this is removing all positive circuits in D_U. This keeps both $(*)$ and $(**)$, and for acyclic graphs the claim is easy.) But then there exists in Δ an infinite F-alternating path starting at a, which contradicts the tightness of $\boldsymbol{W}$ (one sees this by replacing, in $\boldsymbol{W}$, the edges $\{(u_i, F(u_i)): i < \omega\}$ by $\{(u_i, F(u_{i+1})): i < \omega\}$. The resulting matching misses $F(u_1)$.)

[1] Note added in proof: the result was proved by a different method by J.R. Isbell, Birkhoff's problem 111, Proc. Amer. Math. Soc. 6 (1955) 217–218.

In the uncountable case there exist more complicated possible obstacles for espousability. Let Y be the set of all regular uncountable cardinals, together with 1. We define the notion of a *κ-obstruction* for all $\kappa \in Y$ by induction on κ. The notion of a 1-obstruction is already defined. If W is a 1-obstruction and $a \in M\backslash \text{in}[W]$ then, denoting $F = K \cap W$, we call (by an abuse of language) also the graph $\Gamma[\{a\} \sqcup F[V]]$ a 1-obstruction in Γ. For $\kappa > 1$, a *κ-obstructive sequence* is a sequence $((M_\alpha, W_\alpha)\colon \alpha < \kappa)$ of pairs of sets, such that $M_\alpha \subseteq M$, $W_\alpha \subseteq W$, $M_\alpha \cap M_\beta = W_\alpha \cap W_\beta = \phi$ for $\alpha \neq \beta$, $K[M_\alpha] \subseteq \bigcup \{W_\beta\colon \beta \leqslant \alpha\}$; for each $\alpha < \kappa$, either:

(i) $M_\alpha = \phi$ and $|W_\alpha| = 1$, or,

(ii) $\Gamma[M_\alpha, W_\alpha]$ is a μ-obstruction (in itself) for some $\mu < \kappa$,

and the set $\{\alpha\colon \text{(ii) occurs at } \alpha\}$ is stationary.

The graph $O = \Gamma[\bigsqcup_{\alpha<\kappa} M_\alpha \sqcup \bigsqcup_{\alpha<\kappa} W_\alpha]$ is then called a *κ-obstruction in Γ*. (This can also be taken as a special case of a definition in Section 4.) Γ is called *obstructed* if it contains a κ-obstruction for some $\kappa \in Y$.

Theorem 3.6 [10]. *A graph is espousable if and only if it is unobstructed.*

The 'only if' part follows quite easily from Fodor's Lemma. The 'if' part is harder.

It is also not easy to derive Theorem 3.4 from Theorem 3.6. But we can exhibit one main idea in a simple case. Our aim is to show that if Γ is inespousable, then it contains a hindrance. By Theorem 3.6, Γ contains some κ-obstruction T. If $\kappa = 1$, then T is a hindrance, and we are done. So, assume that κ is regular and uncountable. Let Ψ be the set of ordinals α of type (ii) in T. Here we shall greatly simplify matters and assume that, for each $\alpha \in \Psi$, the obstruction $\Gamma[M_\alpha \sqcup W_\alpha]$ is of a particularly simple type: $W_\alpha = \phi$ (i.e. $K[M_\alpha] \subseteq \bigcup \{W_\beta\colon \beta < \alpha\}$) and $|M_\alpha| = 1$. Let $M_\alpha = \{m_\alpha\}$ whenever $\alpha \in \Psi$ and $W_\alpha = \{w_\alpha\}$ for $\alpha \notin \Psi$. For $\alpha \notin \Psi$ define

$$\theta_\alpha = \{\beta \in \Psi\colon \beta > \alpha \quad \text{and} \quad (m_\beta, w_\alpha) \in K\}$$

Let

$$X = \{w_\alpha\colon \theta_\alpha \text{ is stationary}\}, \qquad Y = W_T \backslash X,$$
$$A = K[Y] \cap M_T \quad \text{and} \quad D = M_T \backslash A.$$

(The set X consists of those w_α connected to 'many' m_α's, and D is the set of m_α's connected only to X.)

Assertion 3.7. $\theta_Y := \{\beta \in \Psi\colon m_\beta \in A\}$ *is nonstationary.*

Proof. For each $m_\beta \in A$ choose $\alpha = f(\beta)$ such that $(m_\beta, w_\alpha) \in K$ and $w_\alpha \in Y$. Then $f(\beta) < \beta$ for all $\beta \in \theta_Y$, and hence if θ_Y is stationary, then by Fodor's lemma there exists some $w_\alpha \in Y$ such that $\{\beta\colon f(\beta)\}$ is stationary. But then θ_β is stationary, contradicting the fact that $w_\alpha \in Y$. $\square$

By Assertion 3.7, $\theta_\alpha \backslash \theta_Y$ is stationary for all $w_\alpha \in X$. Hence $|K\langle w_\alpha \rangle \cap D| = \kappa$ whenever $w_\alpha \in X$. Since $|X| \leq \kappa$, this easily implies that there exists a matching of X into D. The definitions imply that $K[D] \subseteq X$, and since $K\langle x \rangle \cap D \neq \phi$ for all $x \in X$, we have $K[D] = X$. Since $\Psi \backslash \theta_Y$ is stationary, $\Gamma[D \sqcup X]$ is a κ-obstruction, and thus D is unmatchable. Hence D satisfies the conditions in Theorem 3.2. As already noted, this proves also Theorem 3.4.

4. Webs

A *web* is a triple $\Gamma = (G, A, B)$ where $G = (V, E)$ is a directed graph, and $A, B \subseteq V(G)$. We write S_Γ for A (the S stands for 'source'). A finite path P is an *A–B path* if $V(P) \cap A = \{\text{in}(P)\}$, $V(P) \cap B = \{\text{ter}(P)\}$. A subset S of V is called *A–B separating* if $V(P) \cap S \neq \phi$ for all A–B paths P. Since we shall be mainly concerned with A–B paths and A–B separating sets, we may assume that there are no edges in Γ which go out of B or go into A, since the deletion of such edges does not alter the above mentioned objects. A web is called *bounded* if it contains no infinite paths.

A *ripple* in Γ is a set of vertex-disjoint paths, each starting at a vertex from A (in [5] this was called a 'warp'). A ripple $\boldsymbol{R}$ is called an *A–B ripple* if all paths in it are A–B paths. If, in addition, $\text{in}[\boldsymbol{R}] = A$, then $\boldsymbol{R}$ is called a *linkage*. If Γ contains a linkage, it is called *linkable*. A ripple $\boldsymbol{W}$ is called a *wave* if $\text{ter}[\boldsymbol{W}]$ is A–B separating. (Note that here we do not require finiteness of the paths—they may be unending. This is different from the terminology of [5]). A wave $\boldsymbol{W}$ is called *tight* if there does not exist a ripple $\boldsymbol{R}$ such that $\text{in}[\boldsymbol{R}] = \text{in}[\boldsymbol{W}]$ and $\text{ter}[\boldsymbol{R}] \subsetneq \text{ter}[\boldsymbol{W}]$.

Given a wave $\boldsymbol{W}$ we write $\Gamma[\boldsymbol{W}]$ for the web $(G, A, \text{ter}[\boldsymbol{W}])$. If $\Gamma[\boldsymbol{W}]$ is not linkable, then $\boldsymbol{W}$ is called an *impediment*, and if Γ contains an impediment it is called *impeded*. Clearly, an impeded web is not linkable. If $\text{in}[\boldsymbol{W}] \neq A$ then $\boldsymbol{W}$ is called a *hindrance*, and if a hindrance exists Γ is called *hindered*.

The notions of 'linkage' and 'A–B separating set' generalize, respectively, the notions of 'espousal' and 'cover' in the bipartite case. Thus, a linkage is an espousal through a "medium" of vertices in $V \backslash (A \sqcup B)$. So, it is natural to ask for the generalizations of the main theorems of Section 3 to webs. The generalization of Theorem 3.1 corresponds to Menger's theorem in the finite case, and is the subject of a famous conjecture of Erdős:

Conjecture 4.1. In any web there exists an A–B ripple $\boldsymbol{R}$ and an A–B separating set S such that S consists of the choice of precisely one vertex from each path in $\boldsymbol{R}$.

As in the bipartite case, the first breakthrough was made by Podewski and Steffens [25]. They proved the conjecture for countable bounded webs. Since then it has become clear that the main obstacle for the proof of Conjecture 4.1, or rather, what makes the conjecture harder than Theorem 3.1, is the possible existence of infinite paths. The difficulty is that trying to 'grow' A–B paths starting from A, one may end up with unending paths, rather than A–B paths. In fact, in [2] it was noted that in the absence of infinite paths, Conjecture 4.1 follows easily from Theorem 3.1 (so, the conjecture is now known for all bounded webs). The way to see this is to assign to Γ a bipartite graph $\hat{\Gamma}$ in the following way. To each vertex $a \in A$ assign a vertex a' of $\hat{\Gamma}$, to each $b \in B$ a vertex b'', and to each $v \in V \backslash (A \sqcup B)$ two vertices, v' and v''. Let

$$V' = \{v' : v \in A \sqcup (V \backslash B)\}, \qquad V'' = \{v'' : v \in B \sqcup (V \backslash A)\},$$

and

$$\hat{E} = \{(x', y''): (x, y) \in E\} \sqcup J, \quad \text{where} \quad J = \{(x', x''): x \in V \backslash (A \sqcup B)\}.$$

Let $\hat{\Gamma} = (V', V'', \hat{E})$. Choose F and C, a matching and a cover in $\hat{\Gamma}$, as in Theorem 3.1. Let

$$S = \{v \in V \backslash (A \sqcup B) : \{v', v''\} \subseteq C\} \sqcup \{a \in A : a' \in C\} \sqcup \{b \in B : b'' \in C\},$$

and let $\boldsymbol{R}$ be the set of maximal paths P in Γ satisfying $(x', y'') \in F$ whenever $(x, y) \in E(P)$, and P contains a vertex from S. It is not hard to show that S is A–B separating; that every $s \in S$ is contained in a path $Q \in \boldsymbol{R}$ such that $\text{ter}(Q) \in B$ if $\text{ter}(Q)$ exists and $\text{in}(Q) \in A$ if $\text{in}(Q)$ exists; and that every path in $\boldsymbol{R}$ contains precisely one vertex from S. If Γ is bounded, then all paths in $\boldsymbol{R}$ must be A–B paths, and thus $\boldsymbol{R}$ is an A–B ripple which satisfies, together with S, the conditions of Conjecture 4.1.

Things are even simpler, in the bounded case, with regard to criteria for linkability (the analogues of Theorems 3.2 and 3.6). For, the web Γ is linkable if and only if $\hat{\Gamma}$ is espousable. To see this, assume that Γ is linkable, and let $\boldsymbol{L}$ be a linkage of Γ. Then the set

$$\{(x', y''): (x, y) \in E[\boldsymbol{L}]\} \sqcup \{(v', v''): v \in V \backslash V[\boldsymbol{L}])\}$$

is an espousal in $\hat{\Gamma}$. Conversely, assume that F is an espousal of $\hat{\Gamma}$. Given any $a \in A$, let v_1 be such that $(a', v_1'') \in F$. If $v_1 \notin B$ then v_1' exists, and $(v_1', v_1'') \notin F$ (since $(a', v_1'') \in F$). Hence $(v_1', v_2'') \in F$ for some v_2. If $v_2 \notin B$ then $(v_2', v_3'') \in F$ for some vertex v_3. Continuing this way, since Γ is bounded, $v_k \in B$ for some k. Connect then a to B by the path $P_a = (a, v_1, \ldots, v_k)$. Since F is a matching, $V(P_{a_1}) \cap V(P_{a_2}) = \phi$ whenever $a_1 \neq a_2$, and thus we have constructed a linkage in Γ. (Note that we have used only the absence of unending paths!)

In order to formulate the criterion for linkability which the above observations yield (together with Theorems 3.2 and 3.6) we have to define "κ-obstructions" for webs, for $\kappa \in Y$. The definition is more involved here than in the special case of bipartite webs, discussed in Section 3. We really have to go into the inner structure of the obstructions.

First, we have to introduce a slight modification of the notion of a wave. A *billow* is a pair $\boldsymbol{B} = (\boldsymbol{W}, \boldsymbol{U} = \in(\boldsymbol{B}))$ of waves where $\text{in}[\boldsymbol{W}] = A$ and $\boldsymbol{U} \subseteq \boldsymbol{W}^f$ (the set of finite paths in $\boldsymbol{W}$). If $\boldsymbol{C}=(\boldsymbol{Z},\boldsymbol{Y})$ is another billow, we write $\boldsymbol{C}\leqslant\boldsymbol{B}$ if $\mathrm{in}[\boldsymbol{Y}]\subseteq\mathrm{in}[\boldsymbol{U}]$ and every path in $\boldsymbol{Z}$ is an extension of some path in $\boldsymbol{W}$. (The idea in these definitions is that although the part $\boldsymbol{W}\setminus\epsilon(\boldsymbol{B})$ is 'redundant' in terms of A–B separation, it is necessary for the definition of the relation $\leqslant$, since it keeps a path P in $\boldsymbol{Z}$ from intersecting the 'redundant' part of $\boldsymbol{W}$, except for the initial part of P which belongs to $\boldsymbol{W}$). We write $\Gamma/\boldsymbol{B}$ for the web $(G-(V[\boldsymbol{W}]\setminus\mathrm{ter}[\boldsymbol{U}]), \mathrm{ter}[\boldsymbol{U}], B)$. If $\boldsymbol{B}_\alpha=(\boldsymbol{W}_\alpha,\boldsymbol{U}_\alpha)$ $(\alpha<\theta)$ is a $\leqslant$-ascending sequence of billows, we write $\uparrow_{\alpha<\theta}\boldsymbol{B}_\alpha$ for the pair $(\boldsymbol{W},\boldsymbol{U})$, where $\boldsymbol{W}$ is the ripple defined by $E[\boldsymbol{W}]=\sqcup\{E[\boldsymbol{W}_\alpha]\colon\alpha<\theta\}$ and $\boldsymbol{U}$ the sub-ripple defined by $\mathrm{in}[\boldsymbol{U}]=\bigcap\{\mathrm{in}[\boldsymbol{U}_\alpha]\colon\alpha<\theta\}\cap\mathrm{in}[\boldsymbol{W}^f]$.

Lemma 4.2. *$\uparrow_{\alpha<\theta}\boldsymbol{B}_\alpha$ is a billow.*

The *trivial billow*, denoted by $\square=\square_\Gamma$, is the pair $(\boldsymbol{W},\boldsymbol{W})$, where $\boldsymbol{W}=\{(a)\colon a\in A\}$.

We next define the *concatenation* of billows. Let $\boldsymbol{B}=(\boldsymbol{W},\boldsymbol{U})$ be a billow in Γ, and let $\boldsymbol{C}=(\boldsymbol{Z},\boldsymbol{Y})$ be a billow in $\Gamma/\boldsymbol{B}$. Then $\boldsymbol{B}*\boldsymbol{C}$ is the pair $(\boldsymbol{R},\boldsymbol{S})$, where $\boldsymbol{R}=\{P*Q\colon P\in\boldsymbol{U},\ Q\in\boldsymbol{Z}$ and $\mathrm{in}(Q)=\mathrm{ter}(P)\}$ and $\boldsymbol{S}=\{R\in\boldsymbol{R}\colon\mathrm{ter}(R)\in\mathrm{ter}[\boldsymbol{Y}]\}$.

Lemma 4.3. *$\boldsymbol{B}*\boldsymbol{C}$ is a billow.*

We can now define inductively the concatentation $*(\boldsymbol{B}_\alpha\colon\alpha<\theta)$ of a sequence $\boldsymbol{B}_\alpha=(\boldsymbol{W}_\alpha,\boldsymbol{U}_\alpha)$, $(\alpha<\theta)$ where each $\boldsymbol{B}_\alpha$ is a billow in $\Gamma/*(\boldsymbol{B}_\beta\colon\beta<\alpha)$. (So, we are already assuming that $*(\boldsymbol{B}_\beta\colon\beta<\alpha)$ is defined for all $\alpha<\theta$.) If $\theta=\gamma+1$ then $*(\boldsymbol{B}_\alpha\colon\alpha<\theta)$ is defined as $(*(\boldsymbol{B}_\alpha\colon\alpha<\theta))*\boldsymbol{B}_\gamma$. If θ is a limit ordinal, then $*(\boldsymbol{B}_\alpha\colon\alpha<\theta)=\uparrow_{\beta<\theta}(*(\boldsymbol{B}_\alpha\colon\alpha<\beta))$. By Lemmas 4.2 and 4.3, $*(\boldsymbol{B}_\alpha\colon\alpha<\theta)$ is a billow. If $*(\boldsymbol{B}_\alpha\colon\alpha<\theta)$ is defined, we say that the sequence $(\boldsymbol{B}_\alpha\colon\alpha<\theta)$ is a *ladder*.

Some definitions needed at this point are the following: For any vertex v we write $e^-(v)$ for the set of edges going into v. If F is a set of vertices, we write $e^-[F]=\bigcup\{e^-(v)\colon v\in F\}$.

If $F\subseteq V\setminus A$ then $\Gamma+(s)F$ denotes the web $(G, A\sqcup F, B)$, $\Gamma+(d)F$ denotes the web $(G, A, B\sqcup F)$, and $\Gamma-(d)F$ denotes the web (G', A, B) where $G'=(V, E\setminus e^-[F])$ (here (s) stands for 'source' and (d) for 'destination'. In $\Gamma-(d)F$ the vertices of F cannot be reached.) We also write $\Gamma\bullet F$ for $\Gamma-(d)F+(s)F$.

In our new terminology, a 1-*obstruction* is a pair $(\boldsymbol{B},\phi)$, where $\boldsymbol{B}=(\boldsymbol{W},\boldsymbol{U})$ is a billow, $\boldsymbol{W}$ is tight and $\boldsymbol{U}\subsetneq\boldsymbol{W}$. Assume now that $\kappa\in Y$ and that the notion of μ-obstruction is defined for all $\mu\in Y\cap\kappa$. A pair $(\boldsymbol{B},F)$ is said to be a *κ-obstruction* if there exists a sequence $((\boldsymbol{B}_\alpha,F_\alpha)\colon a<\kappa)$ of pairs, such that $\boldsymbol{B}_\alpha$ is a billow in $\Gamma_\alpha=(\Gamma\bullet\hat{F}_\alpha)/\hat{\boldsymbol{B}}_\alpha$ (where $\hat{F}_\alpha=\bigsqcup\{F_\beta\colon\beta<\alpha\}$ and $\hat{\boldsymbol{B}}_\alpha=*(\boldsymbol{B}_\beta\colon\beta<\alpha)$); $\boldsymbol{B}=*(\boldsymbol{B}_\alpha\colon\alpha<\kappa)$, $F=\bigsqcup\{F_\alpha\colon\alpha<\kappa\}$, and for each $\alpha<\kappa$ either:

(i) $(\boldsymbol{B}_\alpha,F_\alpha)$ is a μ-obstruction in Γ_α for some $\mu<\kappa$, or,

(ii) $\boldsymbol{B}_\alpha=\square_{\Gamma_\alpha}$, $|F_\alpha|=1$, $F_\alpha=\{x\}$ for some $x\in V(\Gamma_\alpha)-S_{\Gamma_\alpha}$,

and the set $\{\alpha<\kappa\colon$ (i) occurs at $\alpha\}$ is stationary. We say that Γ is *obstructed* (or specifically *κ-obstructed*), if it contains a κ-obstruction for some $\kappa\in Y$.

Theorem 4.4. *A web containing no unending path is linkable if and only if it is unobstructed.*

Conjecture 4.5. The same is true for all webs.

For countable graphs the following weaker version of the conjecture is known.

Theorem 4.6. *A countable unhindered web is linkable.*

The main step in the proof of Theorem 4.6 is the following.

Theorem 4.7. *If Γ is countable and unhindered, then for every $a \in A$ there exists an a–B path P such that $\Gamma - V(P)$ is unhindered.*

This theorem should be true also for uncountable webs, but countability is so far used very essentially in the proof. Its proof in the uncountable case would probably clear the way to the proof of Conjecture 4.1. On the other hand, if the term 'unhindered' could be replaced in the theorem by 'not 1-obstructed' then Conjecture 4.5 would follow for countable webs.

Proof of Conjecture 4.1 for countable webs. Let $\boldsymbol{B} = (\boldsymbol{W}, \boldsymbol{U})$ be a $\leqslant$-maximal billow in Γ. By Lemma 4.3, $\Gamma/\boldsymbol{B}$ contains no nontrivial billow, and hence, by Theorem 4.6, it is linkable. Let $\boldsymbol{L}$ be a linkage in $\Gamma/\boldsymbol{B}$. Then the ripple $\boldsymbol{U} * \boldsymbol{L}$ and the A–B separating set ter$[\boldsymbol{U}]$ satisfy the conditions of Conjecture 4.1. $\square$

5. General graphs

A matching F in a graph $G = (V, E)$ is called *perfect* if $F[V] = V$. If G contains a perfect matching it is called *matchable*. In 1947 Tutte proved his celebrated theorem on a necessary and sufficient condition for matchability [27]. We shall bring it here in a stronger form, proved by Gallai [15].

A graph P is called *factor-critical* if it is unmatchable, but $P - x$ is matchable for every $x \in V(P)$. Clearly, a finite factor critical graph is of odd cardinality (for which reason it is called in [6] "peculiar"). Given a graph $G = (V, E)$ and a subset S of V, we write $\boldsymbol{P}(S) = \boldsymbol{P}(G, S)$ for the set of factor-critical connected components in $G - S$. We associate with S a bipartite graph $\Pi = \Pi(G, S) = (\boldsymbol{P}(G, S), S, H)$, where $(P, s) \in H$ (here $P \in \boldsymbol{P}(S)$ and $s \in S$) if $(v, s) \in E$ for some $v \in V(P)$. We can now state Gallai's theorem in a form which is true also for infinite graphs.

Theorem 5.1 [6]. *A graph G is matchable if and only if $\Pi(G, S)$ is espousable for every set of vertices S.*

If G is finite and unmatchable, then, by Theorem 5.1 and Hall's theorem, there exists a subset $\boldsymbol{P}'$ of $\boldsymbol{P}(S)$ connected in $\Pi(G, S)$ to a subset S' of S containing fewer than $|\boldsymbol{P}'|$ vertices. Then $\boldsymbol{P}' \subseteq \boldsymbol{P}(S')$, and hence $|\boldsymbol{P}(S')| > |S'|$. This, in particular, implies Tutte's theorem, which states that if G is unmatchable, then it contains a set S' of vertices such that the number of odd components in $G - S'$ is larger than $|S'|$. It is also worth while noting that a locally finite factor-critical graph P is finite (this is proved by the use of König's Lemma and the fact that if I is a matching of $P - x$ for some vertex x, then there is no infinite I-alternating path.) This proves the result first proved by Tutte himself [28], that his theorem holds also for locally finite graphs.

We shall not attempt any description of the proof of this theorem. Instead, we give a structure theorem for graphs, which follows easily from it. (In the finite case this is the Gallai-Edmonds decomposition theorem.)

Theorem 5.2 [11]. *The vertex set V of any graph $G = (V, E)$ can be decomposed as a disjoint union $V = A \sqcup B \sqcup T$, where:*

(i) *$G[B]$ is matchable;*
(ii) *$A = \bigsqcup\{V[P]: P \in \boldsymbol{P}\}$ where $\boldsymbol{P} = \boldsymbol{P}(G, T)$; and*
(iii) *T is matchable into $\boldsymbol{P}$ in $\Pi(G, T)$.*

Proof. Let $\boldsymbol{Z}$ be the set of pairs (Y, I), where $Y \subseteq V$ and I is a matching of S into $\boldsymbol{P}(G, Y)$ in $\Pi(G, Y)$. Define an order $<$ on $\boldsymbol{Z}$ by $(Y, I) < (Z, H)$ if (1) $Y \subseteq Z \subseteq V \backslash V[\boldsymbol{P}(G, Y)]$ and (2) $I \subseteq H$ (note that (1) implies that $\boldsymbol{P}(G, Y) \subseteq \boldsymbol{P}(G, Z)$). Applying Zorn's Lemma, it is not hard to see that $\boldsymbol{Z}$ contains a maximal element (T, I). Let $A = V[\boldsymbol{P}(G, T)]$ and let $B = V \backslash (A \sqcup T)$. If $G' = G[B]$ were not matchable, then, by Theorem 5.1 there exists a subset S of B such that $\Pi = \Pi(G', S)$ is inespousable. By Theorem 3.2 there exists $X \subseteq S$ which has a matching J into $D_\Pi(X)$, and $D_\Pi(X)$ is unmatchable. Since $D_\Pi(X)$ is unmatchable and $D_\Pi(\phi) = \phi$, it follows that $X \neq \phi$. Then $T \sqcup X$ is matchable into $\boldsymbol{P}(G, T \sqcup X) = \boldsymbol{P}(G, T) \sqcup D_\Pi(X)$ by the matching $I \sqcup J$, contradicting the maximality of the pair (T, I). We have thus shown that $G[B]$ is matchable, as required in the theorem. □

In [11] Theorem 5.2 was used to prove a duality result on fractional covers and matchings in graphs (see Section 6). Here we shall use it to prove another result. A matching F is called strongly *maximal* if there does not exist a matching H such that $|H \backslash F| > |F \backslash H|$. Since for any matching H the set of edges in $F \Delta H$ forms a set of disjoint alternating paths and cycles, it is clear that F is strongly maximal if and only if there does not exist a finite improving F-alternating path, i.e. one starting and ending at $V \backslash F[V]$.

Theorem 5.3. *Every graph contains a strongly maximal matching.*

Proof. Let A, B, T and $\boldsymbol{P}$ be as in Theorem 5.2, let J be a perfect matching in $G[B]$, and let I be a matching of T into $\boldsymbol{P}$ in $\Pi(G, T)$. For each $t \in T$ choose a vertex $f(t) \in E\langle t \rangle \cap V(I(t))$, and a matching K_t of $I(t) - f(t)$. For each $P \in \boldsymbol{P}' =$

$\boldsymbol{P}\backslash I[T]$ choose a vertex $x = x(P) \in V(P)$, and a matching L_P of $P - x$. Let

$$F = J \sqcup \{(t, f(t)) \colon t \in T\} \sqcup \bigcup \{K_t \colon t \in T\} \sqcup \bigsqcup \{L_P \colon P \in \boldsymbol{P}'\}.$$

Then the set $V \backslash F[V]$ of vertices not covered by F is $\{x(P) \colon P \in \boldsymbol{P}'\}$. Suppose that $(y_1, y_2, \ldots, y_n)$ is an improving F-alternating path. Then $y_1 = x(P_1)$ for some $P_1 \in \boldsymbol{P}$. Since $E[V(P_1)] \subseteq V(P_1) \sqcup T$, there exists k_1 such that $y_i \in V(P_1)$ for $i < k_1$ and $y_{k_1} \in T$. Then $y_{k+1} = f(y_k) \in V(P_2)$, where $P_2 \in \boldsymbol{P}\backslash\boldsymbol{P}'$. Then there exists $k_2 > k_1 + 1$ such that $y_i \in V(P_2)$ for $k_1 < i < k_2$ and $y_{k_2} \in T$. Continuing this way we see that for no $i > 1$ does there hold $y_i \in V[\boldsymbol{P}']$. Hence y_n cannot belong to $V[\boldsymbol{P}']$, a contradiction. $\square$

Problem 5.4. For which finite graphs H is it true that every infinite graph G contains a strongly maximal set of edges not spanning H? (In Theorem 5.3 H is a path of length 2.)

Problem 5.5. Does every hypergraph with finite edges contain a strongly maximal matching?

6. Hypergraphs and linear programming duality

So far there have been two (very closely related) main themes in our survey: conditions for matchability, and the duality between matchings and covers; all, of course, in graphs. When we turn to hypergraphs, we see that very little has been done on these two themes, even in the finite case. As is well known, one cannot expect a 'good characterization' of matchability in hypergraphs (which means that it is probably not a CO-NP problem). Yet it is quite possible that the theorems of König, Hall and Tutte can be subsumed in a more comprehensive theory on matchings in hypergraphs. Better understood in the finite case are fractional matchings and covers, and perhaps this is the first direction which should be investigated in the infinite case.

First let us define the above concepts. A *hypergraph* will mean here a pair $H = (V, E)$, where E is a set of subsets of V, called 'edges'. A *matching* in H is a set of disjoint edges, and a *cover* is a set of vertices (elements of V) which meets all edges. The subjects of investigation in matching theory are maximal matchings and minimal covers (it is easy to find large covers and small matchings—for example, the empty set of edges is a matching (see Fig. 1)).

Fig. 1. The empty matching.

In the finite case 'maximal' is with respect to cardinality. In the infinite case stronger notions are needed, which are given by the duality between matchings and covers. In this section we give another direction from which this duality can be viewed.

A *fractional matching* is a function $f : E \to \mathbb{R}^+$ such that $\sum \{f(e): v \in e\} \leq 1$ for every $v \in V$. A *fractional cover* is a function $g : V \to \mathbb{R}^+$, such that $\sum \{g(v): v \in e\} \geq 1$ for every $e \in E$. For any function $h : X \to \mathbb{R}^+$ write $|h| = \sum \{h(x): x \in X\}$ ($|h|$ may be an infinite cardinality). If f and g are a fractional matching and a fractional cover, respectively, then, by the definition of these objects,

$$|g| \geq \sum_{v \in V} g(v) \sum_{e \ni v} f(e) = \sum_{e \in E} f(e) \sum_{v \in e} g(v) \geq |f|. \tag{1}$$

Thus, if we write

$$\nu^* = \nu^*(H) = \sup\{|f|: f \text{ is a fractional matching in } H\}$$

and

$$\tau^* = \tau^*(H) = \inf\{|g|: g \text{ is a fractional cover}\},$$

there holds $\nu^* \leq \tau^*$. The duality theorem of linear programming implies the following.

Theorem 6.1. *In a finite hypergraph $\nu^* = \tau^*$.*

Of course, in a finite hypergraph ν^* and τ^* are attained, that is, there exist a fractional matching f and a fractional cover g such that $|f| = |g| = \nu^* = \tau^*$. By (1) it follows that:

(a) $\sum \{f(e): v \in e\} = 1$ whenever $g(v) > 0$; and

(b) $\sum \{g(v): v \in e\} = 1$ whenever $f(e) > 0$.

These are the so-called 'complementary slackness conditions' of linear programming. In the infinite case (a) and (b) are strictly stronger than the condition $\nu^* = \tau^*$. We say that a hypergraph H has the *strong duality* property if it has a fractional matching f and a fractional cover g satisfying (a) and (b), and that H has the *weak duality* property if $\nu^*(H) = \tau^*(H)$. We say that H has König's Property (or *integral strong duality*) if it has a matching F and a cover G such that G consists of the choice of precisely one vertex from each edge in F. Clearly, this means that H satisfies the strong duality property, with f and g being $(0, 1)$ functions.

The question now arises—which hypergraphs satisfy the weak, strong and integral strong duality properties? As usual, the situation is much more pleasant in the case of graphs.

Theorem 6.2 [11]. *Every graph has the strong duality property, with f and g taking 0, $\frac{1}{2}$, 1 values.*

The proof uses Theorem 5.2.

A hypergraph $H=(V, E)$ is said to be of *finite character* if each of its edges is finite. It is said to be *locally finite* if every vertex belongs to finitely many edges. Easy compactness arguments yield the following.

Theorem 6.3. (i) *If H is of finite character, then there exists a fractional cover g with $|g|=\tau^*$, and $\nu^*=\tau^*$.*

(ii) *If H is of finite character and locally finite, then it has the strong duality property.*

Here is an example (taken from [7]) of a hypergraph of finite character, which does not have the strong duality property.

Let $V=\{a_i\colon 0\leqslant i<\omega\}\sqcup\{b_i\colon 0\leqslant i<\omega\}$, and for each $1\leqslant k<\omega$ let $e_k=\{a_0, a_1, \ldots, a_k, b_k\}$ and $d_k=\{b_0, b_1, \ldots, b_k, a_k\}$. Let $E=\{e_k\colon 1\leqslant k<\omega\}\sqcup\{d_k\colon 1\leqslant k<\omega\}$, and let $H=(V, E)$.

Assertion 6.4. *Every fractional matching f in H satisfies $|f|<2$.*

Proof. Let $m=\min\{k\colon f(e_k)>0 \text{ or } f(d_k)>0\}$, say $\alpha=f(e_m)>0$. Then each d_i for which $f(d_i)>0$ contains b_m, and hence $\sum f(d_i)\leqslant 1-\alpha$. Since each e_i contains a_1, we have $\sum f(e_i)\leqslant 1$. Thus $|f|\leqslant 2-\alpha$. □

Assertion 6.5. $\nu^*=2$.

Proof. For each $1\leqslant n<\omega$ let $f=f_n$ be defined by: $f(e_k)=f(d_k)=(\frac{1}{2})^{n-k}$ for $k<n$, $f(e_k)=f(d_k)=0$ for $k\geqslant n$. Then f_n is a fractional matching with $|f_n|=2-(\frac{1}{2})^{n-1}$. The assertion now follows from Assertion 6.4. □

By Theorem 6.3(i) $\tau^*(H)=\nu^*(H)=2$. Since, by Assertion 6.4, ν^* is not attained, we see by (1) that H does not have the strong duality property.

This example can be adapted to yield also a locally finite hypergraph without the strong duality property (and in which ν^* is not attained). However, the following problem is as yet unsettled.

Problem 6.6. Does the strong duality property hold for hypergraphs whose edges are of *bounded* size (i.e., $|e|\leqslant k$ for all $e\in E$ for some $k\in N$)? Or does it hold for hypergraphs with bounded vertex valencies?

In [7] this was settled in the affirmative for the case that ν^* is finite.

Let us conclude this section with a basic open problem concerning König's Property.

Problem 6.7. Suppose that every finite subhypergraph of H (i.e. $H'=(V, E')$, where E' is a finite subset of E) satisfies König's Property. Does it necessarily follow that H satisfies König's Property?

7. Posets

It was probably Fulkerson who first used the trick of replacing a partially ordered set (or, for that matter, any directed graph) by a bipartite graph by splitting its vertices. The splitting is similar to that in Section 4, the difference being that you do not add the edges (x', x''). Thus the bipartite graph attained from a poset P is $\Gamma = (V', V'', E)$ where $V = V(P)$ and $(x', y'') \in E$ if and only if $x > y$ in P. A matching F in Γ corresponds in a natural way to a decomposition $\boldsymbol{C}_F$ of P into paths: an edge (x, y) will appear in some path in $\boldsymbol{C}_F$ if and only if $(x', y'') \in F$. (Thus, if $\{x', x''\} \cap \bigcup F = \emptyset$ then (x) is a single vertex path in $\boldsymbol{C}_F$). Fulkerson used this to derive Dilworth's theorem from König's theorem [14]. Oellrich and Steffens used the same method to prove a 'strong duality' version of Dilworth's theorem for posets containing no infinite chains.

Theorem 7.1 [23]. *If a poset $\boldsymbol{P}$ does not contain infinte chains then there exist a decomposition $\boldsymbol{C}$ of $\boldsymbol{P}$ into disjoint chains ($=$paths) and an independent set A such that $A \cap V(Q) \neq \emptyset$ for every $Q \in \boldsymbol{C}$.*

In [23] this was proved for countable posets, using the countable version of Theorem 3.1, but their proof applies also to the general case. The elegant proof is worth repeating.

Proof. Let F and C be a matching and a cover in Γ, as in Theorem 3.1. Let $\boldsymbol{C}_F = (Q_i: i \in I)$ and let $A = \{x \in V(P): \{x', x''\} \cap C = \emptyset\}$. Then A is independent since if, say, $x > y$ for $x, y \in A$ then $(x', y'') \in E(\Gamma)$, contradicting the fact that x', y'' do not belong to the cover C.

Let $Q \in \boldsymbol{C}$, and suppose that Q and k edges, hence $k+1$ vertices. By the properties of C and F, the set $\{x', x'': x \in V(Q)\}$ contains k elements from C (one for each edge in $E(Q)$.) Hence there is a vertex $x \in V(Q)$ such that $\{x', x''\} \cap C = \emptyset$, proving that $A \cap V(Q) \neq \emptyset$. □

In [8] the splitting trick was used to prove an infinite analogue of Greene–Kleitman's theorem, which, in the finite case, is a generalization of Dilworth's theorem.

Theorem 7.2 [8]. *If a poset $\boldsymbol{P}$ does not contain infinite chains and k is a positive integer, then there exist a decomposition $\boldsymbol{C}$ of $\boldsymbol{P}$ into chains and disjoint antichains $A_1, \ldots, A_k$ such that each path $Q \in \boldsymbol{C}$ meets* $\min(|V(Q)|, k)$ *antichains A_i.*

In the finite case Theorem 7.2 remains true if we replace everywhere the word 'chain' by 'antichain' and vice versa (this is a theorem of Greene [16]). We conjecture that this is true for Theorem 7.2 also in the infinite case, but this seems to be much harder to prove than Theorem 7.2. Even the case $k = 1$ which is trivial for finite posets (and, in fact, for all well-founded posets) is unknown, and we state it here as follows.

Conjecture 7.3. If a poset P contains no infinite antichain then there exist a chain C and a decomposition of P into disjoint antichains $(A_i : i \in I)$ such that $A_i \cap C \neq \emptyset$ for all i.

In [8] this was proved for posets containing no antichain of size larger than 2.

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