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Exercise sheet 1

Submit your solutions in the exercise group on 2011-Apr-11!

Exercise 1: The higher-dimensional Liouville theorem. (*8 points*)
For $n, m \in \mathbb{N}$, prove that every bounded holomorphic function $f\colon \mathbb{C}^n \to \mathbb{C}^m$ is constant.
(You may use the standard theorem of Liouville, i.e. the case $n = m = 1$, without proof.)

Exercise 2: Complex projective spaces. (*12 points*)

a. Prove that $\mathbb{CP}^n$ (with the topology defined in the lecture) is a compact connected second countable Hausdorff space. (*Hint.* There is a continuous surjective map $S^{2n+1} \to \mathbb{CP}^n$.)

b. For $\alpha \in \{0, \ldots, n\}$, prove that the chart $\varphi_\alpha\colon U_\alpha \to \mathbb{C}^n$ defined in the lecture is a homeomorphism with inverse given by $(\zeta_1, \ldots, \zeta_n) \mapsto [\zeta_1, \ldots, \zeta_\alpha, 1, \zeta_{\alpha+1}, \ldots, \zeta_n]$.

c. Prove that $\mathbb{CP}^1$ is diffeomorphic to S^2.

Exercise 3: Complex Grassmannians. (*12 points*)
Let $n \in \mathbb{N}$, let $k \in \{0, \ldots, n\}$.

a. Prove that the subset $\mathrm{Mat}^{\mathrm{reg}}(n, k)$ of rank-k matrices is an open subset of the set $\mathrm{Mat}(n, k)$ of complex $n \times k$ matrices.

b. Prove that $\mathrm{Gr}_k(\mathbb{C}^n)$ is a compact connected second countable Hausdorff space.

c. For $\alpha \in \{0, \ldots, n\}$, let $b(\alpha)$ be the basis $(e_\alpha, e_0, \ldots, e_{\alpha-1}, e_{\alpha+1}, \ldots, e_n)$ of $\mathbb{C}^{n+1}$. Consider $\mathbb{CP}^n = \mathrm{Gr}_1(\mathbb{C}^{n+1})$. Prove that the chart φ_α from Exercise 2 is equal to the chart $\varphi_{b(\alpha)}$ of $\mathrm{Gr}_1(\mathbb{C}^{n+1})$ defined in the lecture. Deduce that the maximal complex atlas of $\mathbb{CP}^n$ which contains the atlas $\{\varphi_\alpha \mid \alpha \in \{0, \ldots, n\}\}$ is equal to the maximal complex atlas of $\mathrm{Gr}_1(\mathbb{C}^{n+1})$ which contains the atlas $\{\varphi_b \mid b \text{ is a basis of } \mathbb{C}^{n+1}\}$.

Exercise 4: The Cauchy-Riemann equations. (*8 points*)
We identify $\mathbb{C}^n$ with $\mathbb{R}^n \oplus \mathbb{R}^n$ by decomposition into real and imaginary parts: $(x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n) \in \mathbb{C}^n$ is identified with $((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \in \mathbb{R}^n \oplus \mathbb{R}^n$.

a. Multiplication by i defines a $\mathbb{C}$-linear map $\mathbb{C}^n \to \mathbb{C}^n$ and is thus described by a 2×2 block matrix each of whose blocks is a real $n \times n$ matrix. Show that this block matrix is $\left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$.

b. For $n, m \in \mathbb{N}$, let A, B, C, D be real $m \times n$ matrices. Then the block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ describes an $\mathbb{R}$-linear map $\mathbb{C}^n \to \mathbb{C}^m$. Show that this map is $\mathbb{C}$-linear if and only if $A = D$ and $B = -C$.

c. Let U be an open subset of $\mathbb{C}^n$, let $f = (f_1, \ldots, f_m)\colon U \to \mathbb{C}^m$ be differentiable, let $z \in U$. The derivative of f at z is an $\mathbb{R}$-linear map $D_z f\colon \mathbb{C}^n \to \mathbb{C}^m$. Check that $D_z f$ is $\mathbb{C}$-linear if and only if the *Cauchy-Riemann equations*

$$\frac{\partial \operatorname{Re} f_k}{\partial x_j} = \frac{\partial \operatorname{Im} f_k}{\partial y_j} \quad , \qquad\qquad \frac{\partial \operatorname{Re} f_k}{\partial y_j} = -\frac{\partial \operatorname{Im} f_k}{\partial x_j}$$

hold for all $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$ at the point z.

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