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AN INTRODUCTION TO SUPERSYMMETRY

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ABSTRACT. This is a short introduction to supersymmetry based on the first of two lectures given at the II Workshop in Differential Geometry, La Falda, Córdoba, 2005.

OUTLINE

The aim of this note is to explain — in mathematical terms and based on simple examples — some of the basic ideas involved in classical supersymmetric field theories. It should provide some helpful background for the, more advanced, discussion of geometrical aspects of supersymmetric field theories on Euclidian space, which is the theme of a second paper in this volume. Supersymmetric field theories on Minkowski space are discussed in great detail in the paper [DF], written for mathematicians. We shall not attempt here to give a reasonably complete list of papers written for physicists.

Our exposition starts with the simplest supersymmetric field theory on a pseudo-Euclidian space: the free supersymmetric scalar field. A straightforward generalisation is the linear supersymmetric sigma-model, the target manifold of which is flat. The generalisation to curved targets is non-trivial and leads to geometrical constraints imposed by supersymmetry.

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1. THE FREE SUPERSYMMETRIC SCALAR FIELD

The bosonic scalar field. Let $\mathbb{M} = V = (\mathbb{R}^d, \eta =< \cdot, \cdot >)$ be a pseudo-Euclidian vector space, e.g. $\mathbb{M}$ = Minkowski space, the space-time of special relativity. A *scalar field* on $\mathbb{M}$ is a function $\phi : \mathbb{M} \to \mathbb{R}$. The simplest Lagrangian for a scalar field is

$$\mathcal{L}_{bos}(\phi) = \langle \mathrm{grad}\phi, \mathrm{grad}\phi \rangle = \eta^{-1}(d\phi, d\phi) =: |d\phi|^2.$$

It is invariant under any isometry $\varphi \in \mathrm{Isom}(\mathbb{M})$, since $d(\varphi^*\phi) = \varphi^* d\phi$. The corresponding Euler-Lagrange equations are linear:

$$0 = \mathrm{div}\ \mathrm{grad}\phi =: \Delta\phi.$$

Δ is the pseudo-Euclidian version of the Laplacian.

The supersymmetry algebra. Suppose now that we have a non-degenerate bilinear form β on the spinor module S of V such that there exist $\sigma, \tau \in \{\pm 1\}$ such that:

(i) $\beta(s, s') = \sigma\beta(s', s)$ and
(ii) $\beta(\gamma_v s, s') = \tau\beta(s, \gamma_v s')$,

for all $s, s' \in S, v \in V$, where $\gamma_v : S \to S$ is the Clifford multiplication by $v \in V$. All such forms have been determined in [AC].

If $\sigma\tau = +1$, which will be assumed from now on, we can define a *symmetric* vector-valued bilinear form

$$\Gamma = \Gamma_\beta : S \times S \to V$$

by the equation

$$\langle \Gamma(s, s'), v \rangle = \beta(\gamma_v s, s') \quad \forall s, s' \in S,\, v \in V.$$

Γ is equivariant with respect to the connected spin group and defines an extension of the Poincaré algebra

$$\mathfrak{g}_0 = \text{Lie Isom } (\mathbb{M}) = so(V) + V$$

to a Lie superalgebra

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 \text{ with } \mathfrak{g}_1 = S.$$

Such Lie superalgebras are called *super-Poincaré algebras*. (More generally, $\mathfrak{g}_1$ could be a sum of spinor and semi-spinor modules.)

The supersymmetric scalar field. It turns out that the Lagrangian $\mathcal{L}_{bos}(\phi)$ for a scalar field ϕ can be extended to a Lagrangian $\mathcal{L}(\phi, \psi)$ depending on the additional spinor field $\psi : \mathbb{M} \to S$ in such a way that the action of $\text{Isom}_0(\mathbb{M}) = SO_0(V) \ltimes V$ on scalar fields ϕ is extended to an action of its double covering $Spin_0(V) \ltimes V$ on fields (ϕ, ψ) preserving the Lagrangian $\mathcal{L}(\phi, \psi)$. Moreover, the infinitesimal action of $\mathfrak{g}_0$ extends, roughly speaking, to an infinitesimal action of $\mathfrak{g}$ preserving $\mathcal{L}(\phi, \psi)$ up to a divergence.

The formula for the Lagrangian is the following:

$$\mathcal{L}(\phi, \psi) = \eta^{-1}(d\phi, d\phi) + \beta(\psi, D\psi),$$

where D is the *Dirac operator*

$$D\psi = \sum \gamma^\mu \partial_\mu \psi, \quad \gamma^\mu = \sum \eta^{\mu\nu}\gamma_\nu \text{ with } \gamma_\nu = \gamma_{\partial_\nu}.$$

In this formula ψ has to be understood as an *odd* element of

$$\Gamma_A(\Sigma) := \Gamma(\Sigma) \otimes A,$$

where $\Sigma = \mathbb{M} \times S \to \mathbb{M}$ is the trivial spinor bundle and $A = \Lambda E$ is the exterior algebra of some auxiliary finite dimensional vector space E.

The bilinear form $\beta : S \times S \to \mathbb{R}$ extends as follows to an even $C^\infty_A(\mathbb{M})$ -bilinear form

$$\beta : \Gamma_A(\Sigma) \times \Gamma_A(\Sigma) \to C^\infty_A(\mathbb{M}) = C^\infty(\mathbb{M}) \otimes A.$$

Let (ϵ_a) be a basis of S, $\beta_{ab} := \beta(\epsilon_a, \epsilon_b)$ and

$$\psi = \sum \epsilon_a \psi_a\,,\ \psi' = \sum \epsilon_a \psi'_a \in \Gamma_A(\Sigma) = \Gamma(\Sigma) \otimes A = S \otimes C^\infty_A(\mathbb{M}).$$

Then on defines

$$\beta(\psi,\psi') := \sum \beta_{ab}\psi_a\psi'_b.$$

For homogeneous elements ψ,ψ' of degree $\tilde{\psi},\tilde{\psi}' \in \{0,1\}$ we obtain

$$\beta(\psi,\psi') = (-1)^{\tilde{\psi}\tilde{\psi}'}\sigma\beta(\psi',\psi). \tag{1}$$

This implies

$$\begin{aligned}\beta(\psi,D\psi') &= \sum \beta(\psi,\gamma^\mu\partial_\mu\psi') = \tau\sum\beta(\gamma^\mu\psi,\partial_\mu\psi') \equiv -\tau\beta(D\psi,\psi') \pmod{div}\\ &= -\underbrace{\tau\sigma}_{=+1}(-1)^{\overset{\widetilde{D\psi}\tilde{\psi}'}{(=\tilde{\psi}\tilde{\psi}')}}\beta(\psi',D\psi) = -(-1)^{\tilde{\psi}\tilde{\psi}'}\beta(\psi',D\psi).\end{aligned}$$

In particular,

$$\beta(\psi,D\psi) \equiv -(-1)^{\tilde{\psi}}\beta(\psi,D\psi) \pmod{div}.$$

Hence $\beta(\psi,D\psi)$ is a divergence if ψ is even. The Euler-Lagrange equations are again linear:

$$\begin{cases}\Delta\phi = 0,\\ D\psi = 0.\end{cases}$$

This is why the classical field theory defined by the Lagrangian $\mathcal{L}(\phi,\psi)$ for the scalar field ϕ and its fermionic superpartner ψ is called *free*. It is easy to check the $Spin_0(V) \ltimes V$-invariance of $\mathcal{L}(\phi,\psi)$.

Verification of supersymmetry. We shall now define the supersymmetry transformations and check the invariance of the Lagrangian $\mathcal{L}(\phi,\psi)$ up to a divergence.

For any odd constant spinor

$$\lambda = \sum \epsilon_a\lambda^a \in S\otimes\Lambda^{odd}E\ (\cong \mathfrak{g}_1\otimes\Lambda^{odd}E \subset (\mathfrak{g}\otimes\Lambda E)_0)$$

we define a vector field X on the the infinite-dimensional vector space of fields. The value $X_{(\phi,\psi)} = (\delta\phi,\delta\psi)$ of X at (ϕ,ψ) is

$$\begin{cases}\delta\phi := -\beta(\psi,\lambda) \in C^\infty_A(\mathbb{M})_0\\ \delta\psi := \gamma_{\mathrm{grad}\phi}\lambda \in \Gamma_A(\Sigma)_1\end{cases}$$

Let us check that this infinitesimal transformation preserves the Lagrangian up to a divergence:

$$\delta\mathcal{L}(\phi,\psi) \equiv 2\eta^{-1}(d\delta\phi,d\phi) + 2\beta(\delta\psi,D\psi) \pmod{div}. \tag{2}$$

Here we have used that, by (1):

$$\beta(\psi,D\delta\psi) \equiv \underbrace{-(-1)^{\tilde{\psi}\widetilde{\delta\psi}}}_{=+1}\beta(\delta\psi,D\psi) \pmod{div} = \beta(\delta\psi,D\psi).$$

The calculation of the two terms in (2) yields:

$$\begin{aligned}
\eta^{-1}(d\delta\phi, d\phi) &= -\eta^{-1}(\beta(d\psi,\lambda), d\phi) = -\sum \eta^{\mu\nu}\beta(\partial_\mu\psi,\lambda)\partial_\nu\phi \\
\beta(\delta\psi, D\psi) &= \sum \beta(\gamma_{\mathrm{grad}_\phi}\lambda, \gamma^\mu\partial_\mu\psi) = \tau\sum \beta(\gamma^\mu\gamma_{\mathrm{grad}_\phi}\lambda, \partial_\mu\psi) \\
&= -\tau\sum \eta^{\mu\nu}(\partial_\nu\phi)\beta(\lambda,\partial_\mu\psi) + \frac{\tau}{2}\sum \beta((\gamma^\mu\gamma_{\mathrm{grad}_\phi} - \gamma_{\mathrm{grad}_\phi}\gamma^\mu)\lambda, \partial_\mu\psi) \\
&\quad \text{(by the Clifford relation)} \\
&= +\tau\sigma\sum \eta^{\mu\nu}(\partial_\nu\phi)\beta(\partial_\mu\psi,\lambda) + \frac{\tau}{2}\sum(\partial_\nu\phi)\beta([\gamma^\mu,\gamma^\nu]\lambda,\partial_\mu\psi) \\
&\equiv \sum \eta^{\mu\nu}(\partial_\nu\phi)\beta(\partial_\mu\psi,\lambda) - \frac{\tau}{2}\underbrace{\sum(\partial_\mu\partial_\nu\phi)}_{symm.}\beta(\underbrace{[\gamma^\mu,\gamma^\nu]}_{skew-symm.}\lambda,\psi) \quad (\mathrm{mod}\ div) \\
&= \sum \eta^{\mu\nu}(\partial_\nu\phi)\beta(\partial_\mu\psi,\lambda) = -\eta^{-1}(d\delta\phi, d\phi).
\end{aligned}$$

This shows that $\delta\mathcal{L}(\phi,\psi) \equiv 0 \pmod{div}$.

2. Sigma-models

The linear supersymmetric sigma-model. Instead of considering one scalar field ϕ and its superpartner ψ we may consider n scalar fields ϕ^i and n spinor fields ψ^i on $\mathbb{M}$ $(i = 1, \dots, n)$. The following Lagrangian is supersymmetric:

$$\mathcal{L}(\phi^1,\dots,\phi^n,\psi^1,\dots,\psi^n) = \sum_{i,j=1}^{n} g_{ij}(\eta^{-1}(d\phi^i, d\phi^j) + \beta(\psi^i, D\psi^j)),$$

where g_{ij} is a constant symmetric matrix, which we assume to be non-degenerate. The above Lagrangian is called the *linear supersymmetric sigma-model*. The Euler Lagrange equations for the scalar fields imply that the map

$$\phi = (\phi^1,\dots,\phi^n) : \mathbb{M} \to \mathbb{R}^n$$

is harmonic, where the target carries the flat metric $g = (g_{ij})$.

Non-linear supersymmetric sigma-models. Next we consider maps

$$\phi : \mathbb{M} \to (M, g)$$

into a *curved* pseudo-Riemannian manifold (M, g). The Lagrangian

$$\mathcal{L}_{bos}(\phi) = |d\phi|^2 := (g_\phi \otimes \eta^{-1})(d\phi, d\phi)$$

is called the *non-linear bosonic sigma-model*. The Euler-Lagrange equation of $\mathcal{L}_{bos}$ is the harmonic map equation for ϕ.

It is natural to ask:
Does there exist a *supersymmetric non-linear sigma-model*, i.e. a supersymmetric extension $\mathcal{L}(\phi,\psi)$ of the bosonic sigma-model $\mathcal{L}_{bos}(\phi)$?

It turns out that one cannot expect a positive answer for arbitrary target (M, g).

Restrictions on the target geometry. Supersymmetry imposes restrictions on the target geometry, which depend on the dimension d of space-time and on the signature of the space-time metric η. In the case of 4-dimensional *Minkowski* space the restriction is that (M, g) is a (possibly indefinite) *Kähler* manifold [Z]. The corresponding supersymmetric sigma-model is of the form

$$\mathcal{L}(\phi, \psi) = (g_\phi \otimes \eta^{-1})(d\phi, d\phi) + (g_\phi \otimes \beta)(\psi, D^\phi \psi) + Q(\phi, \psi),$$

where $\psi \in \Gamma_A(\phi^* TM \otimes_{\mathbb{R}} \Sigma)$, $\psi = \psi^{1,0} \oplus \overline{\psi^{1,0}}$, with $\psi^{1,0} \in \Gamma_A(\phi^* TM^{1,0} \otimes_{\mathbb{C}} \Sigma)$ and $D^\phi = \sum \gamma^\mu \nabla^\phi_{\partial_\mu}$, where ∇^ϕ is the natural connection in $\phi^* TM \otimes \Sigma$ and Q is a term *quartic in the fermions* constructed out of the curvature-tensor R^g of g, using that (M, g) is Kähler and $S = \mathbb{C}^2$.

Extended supersymmetry, special geometry. The super-Poincaré algebra $\mathfrak{g} = \mathfrak{g}_{N=1} = \mathfrak{g}_0 + \mathfrak{g}_1$ underlying the above non-linear supersymmetric sigma-model on four-dimensional Minkowski space is *minimal*, in the sense that $\mathfrak{g}_1 = S$ is an irreducible $Spin(1, 3)$-module. The real dimension of S is four.

There exists another super-Poincaré algebra $\mathfrak{g} = \mathfrak{g}_{N=2} = \mathfrak{g}_0 + \mathfrak{g}_1$ for which $\mathfrak{g}_1 = S \otimes \mathbb{R}^2$ is a sum of two irreducible submodules. Note that $\mathfrak{g}_{N=1}$ is not a subalgebra of $\mathfrak{g}_{N=2}$. In fact, the $Spin(V)$-submodules $S \otimes v \subset S \otimes \mathbb{R}^2$, $v \in \mathbb{R}^2$, are commutative subalgebras, i.e. $[S \otimes v, S \otimes v] = 0$.

Field theories admitting the extended super-Poincaré algebra $\mathfrak{g}_{N=2}$ as supersymmetry algebra are called $N = 2$ supersymmetric theories. The target geometry of such theories is called *special geometry*. The geometry depends on the field content of the theory. There are two fundamental cases:

(i) Theories with *vector multiplets*: the target geometry is (affine) *special Kähler* [DV], see [C] for a survey on special Kähler manifolds.
(ii) Theories with *hypermultiplets*: the target geometry is *hyper-Kähler*, as follows from results about two-dimensional sigma-models [AF].

There exists also a Euclidian version $\mathfrak{g}'_{N=2} = \mathfrak{g}'_0 + \mathfrak{g}'_1$ of the Minkowskian $N = 2$ super-Poincaré algebra $\mathfrak{g}_{N=2}$, for which $\mathfrak{g}'_0 = so(4) + \mathbb{R}^4$ is the Lie algebra of Killing vector fields of the four-dimensional Euclidian space and $\mathfrak{g}'_{N=2} \otimes \mathbb{C} \cong \mathfrak{g}_{N=2} \otimes \mathbb{C}$. The special geometry associated with field theories admitting the Lie superalgebra $\mathfrak{g}'_{N=2}$ as supersymmetry algebra is discussed in a second contribution to this volume.

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