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DIVERGENCE-MEASURE FIELDS: GENERALIZATIONS OF GAUSS-GREEN FORMULA WITH APPLICATIONS

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Alla mia famiglia

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Introduction

In this thesis, we will study recent generalizations of the classical divergence theorem which relax considerably the regularity assumptions made on both the vector fields and the domains of integration. The vector fields, whose divergence will be interpreted as a Radon measure, may have discontinuities and ultimately we will obtain a representation of the jump component of their divergence which is suitable for the description of shocks, in terms of a generalized notion of normal traces. As for boundary geometry, we will work in the context of sets with finite perimeter, which include domains with Lipschitz boundaries. We will present a self-contained synthesis of many related approaches which will yield variants of known results and indicate some first applications of these variants, through the related notion of normal trace, to nonlinear hyperbolic conservation laws.

The classical statement of the divergence theorem, the so-called Gauss-Green formula, has rather old origins in the history of mathematics. The first formulations date back to Lagrange (1762), Gauss (1813), Green (1825) and Ostrogradskij (1831), who presented a first proof of it.
In its classical form, the statement is the following theorem.[1]

Theorem 0.0.1. (Classical Gauss-Green formula)
Let $E \subset \mathbb{R}^N$ be an open regular set; that is, E is bounded, $(\bar{E})^\circ = E$ and ∂E is an $(N-1)$-manifold of class C^1. Then $\forall \phi \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$

$$\int_E \mathrm{div}\phi \, dx = -\int_{\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1},$$

where ν_E is the interior unit normal to ∂E.

The class of open regular sets is actually too restrictive, since we see that it does not include bounded open sets with Lipschitz boundary. Indeed, it is not difficult to show that this theorem may be extended to certain open sets which do not satisfy the boundary regularity requirements. For example, using convergence theorems for integrals (such as Lebesgue's dominated convergence theorem) one can prove the formula for cones and cubes.

[1] Here and in what follows, $\mathcal{H}^s$ is the s-dimensional Hausdorff measure.

The need for a characterization of a wider class of sets for which this theorem was valid was satisfied by the theory of functions of bounded variation (BV), and, in particular, by the concept of set of (locally) finite perimeter, due to Caccioppoli (1928) and De Giorgi (1952).
As we will briefly recall in Chapter 1, a function u is in $BV(\Omega)$, for $\Omega \subset \mathbb{R}^N$, if $u \in L^1(\Omega)$ and its distributional derivative Du is a Radon measure; that is, a vector valued Borel measure with finite total variation on compact sets. A set of (locally) finite perimeter E is a set whose characteristic function χ_E is a (locally) BV function.
For a set of finite perimeter, it is useful to consider two particular subsets of the topological boundary: the reduced boundary, $\partial^* E$, on which it is well defined a unit vector ν_E, called (up to a sign) measure theoretic interior unit normal; and the measure theoretic boundary, $\partial^m E$, which coincides up to a set of $\mathcal{H}^{N-1}$-measure zero with $\partial^* E$.
This theory, as presented in [EG], for example, yields the following version of Gauss-Green formula.

Theorem 0.0.2. (Gauss-Green formula on sets of finite perimeter)
Let $E \subset \mathbb{R}^N$ be a set of locally finite perimeter. Then $\forall \phi \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$

$$\int_E \mathrm{div}\phi \, dx = -\int_{\partial^m E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}.$$

This result, although important for the large family of domains of integration which are allowed, is however restricted to a class of integrands whose heavy regularity demands can prove to be inconvenient for applications. If we require less regularity, we have to find a way to recover the meaning of $\mathrm{div}\phi$ and of the normal trace $\phi \cdot \nu_E$. The solution to the first problem is found by considering special classes of distributional derivatives (i.e. distributional derivative which can be represented by L^p functions or by Radon measures), the solution to the second is rather more delicate (in the case of the space BV it is of great importance the fact that any BV function admits a representative which is well defined almost everywhere (a.e.) with respect to the measure $\mathcal{H}^{N-1}$) and it will be handled through approximation arguments.
Important progress in this direction have been made by De Giorgi and Federer, who proved the same theorem for Lipschitz vector fields F, and later by Vol'pert ([VH]), who stated the following theorem.

Theorem 0.0.3. (Gauss-Green formula for BV vector fields)
Let $\Omega \subset \mathbb{R}^N$ be an open set, $u \in BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ and $E \subset\subset \Omega$ be a set of finite perimeter,

$$\int_{E^1} d\,\mathrm{div}(u) = \mathrm{div}u(E^1) = -\int_{\partial^m E} u_{\nu_E} \cdot \nu_E \, d\mathcal{H}^{N-1}$$

where E^1 is the measure theoretic interior of the set E and u_{ν_E} is the interior trace, that is, the approximate limit at $x \in \partial^m E$ restricted to $\Pi_{\nu_E}(x) := \{y \in \mathbb{R}^N : (y - x) \cdot \nu_E \geq 0\}$.

We also briefly mention a related result due to Fuglede ([Fu2]), for vector fields $F \in L^p(\mathbb{R}^N; \mathbb{R}^N)$ such that their distributional divergence is in $L^p(\mathbb{R}^N)$.
Using the concept of module of order p of a family of Radon measures (connected to the theory of extremal length), he defined a collection of sets of finite perimeter $\mathcal{E}$ to be p-exceptional (p-exc) if there is a nonnegative function $f \in L^p(\mathbb{R}^N)$ such that $\int_{\partial^* E} f(x)\, d\mathcal{H}^{N-1}(x) = +\infty$, $\forall E \in \mathcal{E}$.
He then stated the following result.

Theorem 0.0.4. (Fuglede)
Let $F \in L^p(\mathbb{R}^N; \mathbb{R}^N)$, $1 \leq p < \infty$, with $\mathrm{div} F \in L^p(\mathbb{R}^N)$. Then

$$\int_E \mathrm{div} F dx = - \int_{\partial^* E} F(x) \cdot \nu_E(x) d\mathcal{H}^{N-1}(x)$$

for each set E of finite perimeter except those in a p-exc collection $\mathcal{E}$.

The purpose of this work is to examine recent generalizations that concern vector fields $F \in L^p(\Omega; \mathbb{R}^N)$ such that $\mathrm{div} F$ is a Radon measure μ. These fields are called divergence-measure fields, and their space is denoted by $\mathcal{DM}^p(\Omega; \mathbb{R}^N)$.
They were studied in the last years by, among the others, Anzellotti ([A]), who investigated the properties of the normal trace as a functional defined on suitable function spaces, and Chen and Frid ([CF1], [CF2], [CF3]), because of the interest in possible applications in the context of nonlinear hyperbolic conservation laws. They established a Gauss-Green formula and a way to define the normal trace over the boundary $\partial\Omega$ of a bounded open set with Lipschitz deformable boundary. This kind of set Ω is such that its boundary is locally the graph of a Lipschitz function and there exists a bi-Lipschitz homeomorphism over its image $\Psi : \partial\Omega \times [0, 1] \to \overline{\Omega}$ which satisfies $\Psi(x, 0) = x \ \forall x \in \partial\Omega$.
We notice that this definition of admissible domains allows for open sets which need not to be regular, but is not as wide as the class of sets with finite perimeter. Indeed, we may consider a set with C^1 boundary except for a point where there is a cusp, for example in $\mathbb{R}^2$ the set $E = (\overline{B(1, 1)} \cup \overline{B(-1, 1)} \cup (B(0, 2) \cap \{(x, y) : y \leq 0\}))^\circ$ has a cusp in $(0, 0)$, and so it cannot have a Lipschitz boundary, whereas $\mathcal{H}^1(\partial^* E) = 4\pi$ and so it is a set of finite perimeter.

In Chapter 2 we shall give some of the basic properties of divergence-measure fields, which will be shown to be closely related to those of BV functions. Indeed, it is easy to see that if $F = (F_1, ..., F_N)$ with $F_i \in BV(\Omega) \cap L^p(\Omega)\ \forall i$, then $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^N)$.
However, in general, the condition $\mathrm{div} F = \mu$ allows for cancellations, which thus

make the space $\mathcal{DM}^p$ larger and therefore more interesting. Indeed, an easy example of this fact is $F(x,y) = (\sin\frac{1}{x-y}, \sin\frac{1}{x-y})$: then $F \in \mathcal{DM}^\infty(\mathbb{R}^2;\mathbb{R}^2) \setminus BV_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^2)$ and $\text{div}F = 0$ in $\mathcal{M}(\mathbb{R}^2)$.

We also prove that, if $F \in \mathcal{DM}^p_{\text{loc}}(\Omega;\mathbb{R}^N)$ and $\frac{N}{N-1} \le p \le \infty$, then $||\text{div}F||$ is absolutely continuous with respect to the $\mathcal{H}^{N-q}$-measure, with $q = \frac{p}{p-1}$. From this result, we can see the particular importance of the case $p = \infty$, since we have $||\text{div}F|| \ll \mathcal{H}^{N-1}$ for essentially bounded divergence-measure fields, which is a result analogous to the one we know about the gradient of a BV function. Indeed, we also show that for $p \in [1,\infty)$ we cannot in general expect to recover a Gauss-Green formula: thus, our study will concentrate on the space $\mathcal{DM}^\infty$.

In Chapter 3, we show two versions of the divergence theorem for essentially bounded divergence-measure fields.
First, following [CTZ1], we give a self contained and geometrical proof of the theorem for bounded open sets I with C^1 (orientable) compact boundary. Through an approximation by the interior of I, we show the existence of a normal trace, which is an essentially bounded function on ∂I.
Then, after having established Leibniz rules for essentially bounded divergence-measure fields and BV functions, following in the footsteps of Vol'pert's work, we prove the Gauss-Green formula over bounded sets of finite perimeter.

Theorem 0.0.5. (Gauss-Green formula for $\mathcal{DM}^\infty$ fields on bounded sets of finite perimeter) *Let $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^N)$. If $E \subset\subset \Omega$ is a bounded set of finite perimeter, then there exist interior and exterior normal traces of F on $\partial^* E$; that is, $(\mathcal{F}_i \cdot \nu), (\mathcal{F}_e \cdot \nu) \in L^\infty(\partial^* E;\mathcal{H}^{N-1})$ such that*

$$\text{div}F(E^1) = -2\overline{\chi_E F \cdot D\chi_E}(\partial^* E) = -\int_{\partial^* E} \mathcal{F}_i \cdot \nu \, d\mathcal{H}^{N-1}$$

and

$$\text{div}F(E) = -2\overline{\chi_{E^0} F \cdot D\chi_E}(\partial^* E) = -\int_{\partial^* E} \mathcal{F}_e \cdot \nu \, d\mathcal{H}^{N-1},$$

where $E = E^1 \cup \partial^m E$, $\overline{\chi_E F \cdot D\chi_E}$ and $\overline{\chi_{E^0} F \cdot D\chi_E}$ are the weak star limits, respectively, of the sequences $\chi_E F \cdot \nabla(\chi_E * \rho_\delta)$ and $\chi_{E^0} F \cdot \nabla(\chi_E * \rho_\delta)$ as $\delta \to 0$. Moreover,

$$||\mathcal{F}_i \cdot \nu||_{L^\infty(\partial^* E;\mathcal{H}^{N-1})} \le ||F||_{L^\infty(E^1;\mathbb{R}^N)}$$

and

$$||\mathcal{F}_e \cdot \nu||_{L^\infty(\partial^* E;\mathcal{H}^{N-1})} \le ||F||_{L^\infty(\Omega\setminus E;\mathbb{R}^N)}.$$

Moreover, as a corollary, we gain a representation formula of jump component in the divergence of F; that is, for any bounded set of finite perimeter E we have

$$\chi_{\partial^* E}\text{div}F = 2\overline{\chi_E F \cdot D\chi_E} - 2\overline{\chi_{E^0} F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu - \mathcal{F}_e \cdot \nu)\, \mathcal{H}^{N-1} \llcorner \partial^* E$$

in the sense of Radon measures on Ω. Therefore we have

$$||\text{div}F||(\partial^* E) = \int_{\partial^* E} |\mathcal{F}_i \cdot \nu - \mathcal{F}_e \cdot \nu| \, d\mathcal{H}^{N-1}$$

and, for any Borel set $B \subset \partial^* E$,

$$\text{div}F(B) = \int_B (\mathcal{F}_i \cdot \nu - \mathcal{F}_e \cdot \nu) \, d\mathcal{H}^{N-1}.$$

We then show that if F is also continuous, the interior and exterior normal traces on $\partial^* E$ coincide, as essentially bounded functions, and admit a representative which is in fact the classical dot product $F \cdot \nu_E$, where ν_E is the measure theoretic interior normal. It follows also that continuous fields have no jump component in the divergence.
Then we examine the special case of an essentially bounded divergence-measure vector field with constant direction $F = fv$, with $v \in \mathbb{S}^{N-1}$.

In Chapter 4, we show some consequences and applications of the Gauss-Green formula.
We obtain gluing and extension theorems for essentially bounded divergence-measure fields: if $F_1 \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$, $F_2 \in \mathcal{DM}^\infty(\mathbb{R}^N \setminus \overline{W}; \mathbb{R}^N)$ and $W \subset\subset E \subset\subset \Omega$ for a set of finite perimeter E, then we can glue F_1 and F_2 over the boundary of E; on the other hand, if $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$ and $\mathcal{H}^{N-1}(\partial\Omega) < \infty$, then we can extend F to 0 outside Ω.
Using techniques of harmonic analysis, we also prove an existence result for the equation $\text{div}F = \mu$ in $\mathbb{R}^N$, with μ positive Radon measure in the subcritical case; that is, for $F \in \mathcal{DM}^p(\mathbb{R}^N; \mathbb{R}^N)$ with $1 \leq p \leq \frac{N}{N-1}$. This equation is indeed of great interest in the context of continuum mechanics and conservation laws, and it has been studied by Phuc and Torres ([PT]) and Šilhavý ([S]).
Finally, we illustrate an application of the theory developed to nonlinear hyperbolic systems of conservation laws

$$\partial_t u + \text{div}_x f(u) = 0 \quad \text{in} \quad \mathbb{R}^{d+1}_+ := (0, +\infty) \times \mathbb{R}^d.$$

It is well known that in order to have a unique weak solution of such a system, it is natural to select only those solutions which satisfy the Lax entropy inequality

$$\partial_t \eta(u) + \text{div}_x q(u) \leq 0$$

in the sense of distributions for any convex entropy pair (η, q).
We show that for any weak solution $u(t, x) \in L^\infty_{\text{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^m)$ the field $(\eta(u), q(u))$ is indeed in $\mathcal{DM}^\infty_{\text{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^{d+1})$ and, in particular, there exists a positive Radon measure μ_η such that $\text{div}_{(t,x)}(\eta(u), q(u)) = -\mu_\eta$. This was first shown by Chen and Frid and motivated the beginning of their investigations on these function spaces.

Using the Gauss-Green formula, we prove that, for any $\tau > 0$, if a weak entropy solution u satisfies a vanishing mean oscillation property on the half balls $B^+((\tau,y),r) := B((\tau,y),r) \cap \{(t,x) \in \mathbb{R}^{d+1} : t > \tau\}$, then $\eta(u)$ has an essentially bounded trace $\eta(u)_{tr}$ $\mathcal{H}^d$-a.e. on the hyperplane $\{(t,x) \in \mathbb{R}^{d+1} : t = \tau\}$; that is,

$$\lim_{r\to 0} \frac{1}{\omega_d r^{d+1}} \int_{C^+((\tau,y),r)} \eta(u(t,x))\, dt\, dx = \eta(u)_{tr}(\tau,y),$$

where $C^+((\tau,y),r)$ is the cylinder $\{(t,x) \in \mathbb{R}^{d+1} : 0 < t-\tau < r, |x-y| < r\}$. In particular, if we choose $\eta(u) = u_j$, $j = 1,...,m$, we obtain the trace for each component of u.

Chapter 1

Preliminaries

In this chapter, we shall introduce some basic notions and tools from measure theory, Sobolev spaces and BV [1] theory, which are useful for the study of divergence-measure fields, with the aim of fixing also notation and making this exposition in some way self-contained.
In particular, we are going to focus on the properties of the space of Radon measures $\mathcal{M}(\Omega)$ as a dual space, on the notion of capacity and on the properties of sets of finite perimeter (and therefore, on the first generalizations of the Gauss-Green formula).
We will not provide all the proofs of the results we are going to exhibit, only those which contain techinques that we will use in the following chapters.

1.1 Radon measures and total variation

Definition 1.1.1. Let (X, Σ) be a measure space and μ be a function $\mu : \Sigma \to [0, +\infty]$.
μ is a *positive*[2] *measure* if $\mu(\emptyset) = 0$ and it is σ-additive, i.e. for any sequence of pairwise disjoint elements $\{E_k\} \subset \Sigma$

$$\mu\left(\bigcup_{k=0}^{+\infty} E_k\right) = \sum_{k=0}^{+\infty} \mu(E_k).$$

Moreover, μ is *finite* if $\mu(X) < \infty$ and it is *σ-finite* if X is the countable union of sets of finite measure.

Definition 1.1.2. Let (X, Σ) be a measure space and $m \in \mathbb{N}$.

1. $\mu : \Sigma \to \mathbb{R}^m$ is a measure if $\mu(\emptyset) = 0$ and it is σ-additive. If $m = 1$, μ is a *real signed measure*, if $m > 1$, μ is a *vector valued measure* (that is, a vector function whose components are real signed measures).

[1]BV is the space of functions of bounded variation, see 1.3.
[2]Many authors use the term nonnegative measure.

2. If μ is a measure, the *total variation* $||\mu||(E)$ for $E \in \Sigma$ is defined as follows:
$$||\mu||(E) := \sup \left\{ \sum_{k=0}^{+\infty} |\mu(E_k)| : E_k \in \Sigma \ \text{ pairwise } \ \text{disjoint}, E = \bigcup_{k=0}^{+\infty} E_k \right\}.$$

3. If μ is a real measure, we can define its *positive* and *negative parts* as
$$\mu^+ = \frac{||\mu|| + \mu}{2} \quad \text{and} \quad \mu^- = \frac{||\mu|| - \mu}{2}.$$
Obviously, we have $\mu = \mu^+ - \mu^-$ and $||\mu|| = \mu^+ + \mu^-$.

4. If μ is a positive measure we call the *support of* μ, denoted as $\text{supp}(\mu)$, the closed set of all points $x \in X$ such that $\mu(U) > 0$ for every neighbourhood U of x. If μ is a real signed or vector measure, we define $\text{supp}(\mu) := \text{supp}(||\mu||)$.

We now fix some notation.
For our purposes, X is an open subset of $\mathbb{R}^N$ and Σ is the σ-algebra of Lebesgue measurable sets, which contains the σ-algebra of Borel sets (that is, the σ-algebra generated by all open subsets of $\mathbb{R}^N$).
We shall indicate with $\mathcal{L}^N$ the Lebesgue N-dimensional measure and with $\mathcal{H}^\alpha$, for $\alpha \geq 0$, the α-dimensional Hausdorff measure (as is known, $\mathcal{L}^N = \mathcal{H}^N$).
Unless otherwise stated, a measurable set is a $\mathcal{L}^N$-measurable set.
For any measurable set $E \subset \mathbb{R}^N$, we denote by $|E|$ the $\mathcal{L}^N$-measure of E, while, when applied to a function with values in $\mathbb{R}^m$, $|.|$ is the euclidian norm.
$B(x, r)$ is the open ball with center in x and radius $r > 0$ and $\omega_N = |B(0, 1)|$, moreover, for $\alpha \geq 0$,
$$\omega_\alpha = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma\left(1 + \frac{\alpha}{2}\right)},$$
where Γ is Euler's gamma function.
We recall also the definition of α-dimensional spherical measure $\mathcal{S}^\alpha$ of a set A in $\mathbb{R}^N$:
$$\mathcal{S}^\alpha(A) := \lim_{\delta \to 0} \mathcal{S}^\alpha_\delta(A) = \sup_{\delta \to 0} \mathcal{S}^\alpha_\delta(A),$$
where
$$\mathcal{S}^\alpha_\delta := \inf \left\{ \sum_{j=1}^{\infty} \omega_\alpha r_j^\alpha : 2r_j < \delta, A \subset \bigcup_{j=1}^{\infty} B(x_j, r_j) \right\}.$$
This measure is strictly connected with the Hausdorff one, since we have just the additional condition that the sets in δ-cover have to be balls, and it also satisfies the inequalities
$$\mathcal{H}^\alpha \leq \mathcal{S}^\alpha \leq 2^\alpha \mathcal{H}^\alpha, \tag{1.1.1}$$
for which we refer to [Fe], pag. 171.
The symmetric difference of sets is denoted by

$$A\Delta B := (A \setminus B) \cup (B \setminus A).$$

Unless otherwise stated, $\Omega \subset \mathbb{R}^N$ is an open set, and $\subset$ is equivalent to $\subseteq$. We denote by $E \subset\subset \Omega$ a set E whose closure, $\bar{E}$, is compact and contained in Ω, by E° the interior of the set E and by ∂E its topological boundary.

For $k \in \mathbb{N}_0 \cup \{\infty\}$, $m \in \mathbb{N}$, $C_c^k(\Omega;\mathbb{R}^m) := \{\phi \in C^k(\Omega;\mathbb{R}^m),\ \mathrm{supp}(\phi) \subset\subset \Omega\}$ is the space of C^k functions compactly supported in Ω, endowed with the sup norm,

$$||\phi||_\infty = \sup_{x\in\Omega} |\phi(x)|.$$

Definition 1.1.3. (Borel and Radon measures)

1. A positive measure μ on Ω is called a *Borel measure* if every Borel set in Ω is μ-measurable.
2. A positive measure μ on Ω is a *positive Radon measure* if it is a Borel measure and it is finite on compact subsets of Ω.
3. A real signed (or vector valued) measure is called a *real signed (or vector valued) Radon measure* if it is defined on the Borel sigma algebra of any compact subset of Ω and $||\mu||(K) < \infty$, $\forall K \subset \Omega$ compact. The space of real Radon measures on Ω is denoted by $\mathcal{M}_{\mathrm{loc}}(\Omega)$ and the space of $\mathbb{R}^m$-vector valued Radon measures by $\mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$.
4. If $||\mu(\Omega)|| < \infty$, then μ is a *(real signed or vector valued) finite Radon measure.* The space of real finite Radon measures on Ω is denoted by $\mathcal{M}(\Omega)$ and the space of $\mathbb{R}^m$-vector valued finite Radon measures by $\mathcal{M}(\Omega;\mathbb{R}^m)$.

Remark 1.1.1. $\mathcal{M}(\Omega;\mathbb{R}^m)$, $m \geq 1$, endowed with the norm $|||\mu||| := ||\mu||(\Omega)$, is a Banach space.

Proposition 1.1.1. (Inner and outer regularity of Radon measures) *Let μ be a positive Radon measure on Ω, then, for any Borel set B,*

1. $\mu(B) = \sup\{\mu(K) : K \subset B, K$ *compact*$\}$,
2. $\mu(B) = \inf\{\mu(U) : B \subset U, U$ *open*$\}$.

Proof. Any open subset of $\mathbb{R}^N$ (with the induced euclidean topology) is a locally compact separable metric space and any positive Radon measure on Ω is Borel and σ-finite, since we can clearly cover Ω with the bounded open sets

$$\Omega_k := \left\{x \in \Omega : |x| < k, \mathrm{dist}(x,\partial\Omega) > \frac{1}{k}\right\},\ k \in \mathbb{N},$$

for which, since $\overline{\Omega}_k \subset\subset \Omega$, $\mu(\Omega_k) < \infty$ $\forall k$. Thus, the result follows from Proposition 1.43 in [AFP]. $\square$

Remark 1.1.2. Let μ be a positive Radon measure. If $\{A_t\}_{t\in\mathcal{I}}$, where $\mathcal{I}$ is uncountable, is a family of μ-measurable sets in Ω such that their boundaries are disjoint, $\bigcup_{t\in\mathcal{I}}\partial A_t = \Omega$ and for every compact K there exists an uncountable set of indices $\mathcal{J} \subset \mathcal{I}$ such that $K \cap \partial A_t \neq \emptyset$, $\forall t \in \mathcal{J}$, then there exists a countable set $\mathcal{N}$ such that

$$\mu(K \cap \partial A_t) = 0 \quad \forall t \notin \mathcal{N}.$$

We claim that, if such a set $\mathcal{N}$ did not exist, then there would be an uncountable set $\mathcal{Y}$ such that $\mu(K \cap \partial A_t) > \epsilon > 0$, $\forall t \in \mathcal{Y}$. Suppose to the contrary that for each $\epsilon > 0$ the set of t's which satisfy $\mu(K \cap \partial A_t) > \epsilon$ is countable.
We set $\epsilon_j = \frac{1}{j}$ and we have

$$\{t \in \mathcal{I} : \mu(K \cap \partial A_t) \neq 0\} = \bigcup_{j=1}^{+\infty} \left\{ t \in \mathcal{I} : \mu(K \cap \partial A_t) > \frac{1}{j} \right\},$$

so this set, being countable union of countable sets, is itself countable, contradicting our assumption. We extract now from $\mathcal{Y}$ a sequence $\{t_j\}$.
By the monotonicity and the σ-additivity, we have

$$\mu(K) \geq \sum_{j=1}^{+\infty} \mu(K \cap \partial A_{t_j}) = +\infty,$$

which is absurd, since μ is a Radon measure. Therefore, such a $\mathcal{Y}$ cannot exist and so $\mathcal{N}$ exists.
In the following chapters, the sets $\{A_t\}$ will usually be balls $B(x, r)$.

Proposition 1.1.2. *Let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$. Then, for every open set $A \subset \Omega$, we have*

$$||\mu||(A) = \sup \left\{ \int_\Omega \phi \cdot \, d\mu : \phi \in C_c(A), ||\phi||_\infty \leq 1 \right\}.$$

Proof. See [AFP], Proposition 1.47.

Remark 1.1.3. If $\mu \in \mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$, then clearly $\mu \in \mathcal{M}(W; \mathbb{R}^m)$ for any open $W \subset\subset \Omega$. Therefore Proposition 1.1.2 holds also for $\mu \in \mathcal{M}_{\text{loc}}(\Omega; \mathbb{R}^m)$ if we take open sets $A \subset\subset \Omega$.

It is possible to characterize $\mathcal{M}(\Omega; \mathbb{R}^m)$ as a dual space: this yields a weaker topology on it and therefore weak-star compactness of bounded sequences.
We denote by $C_0(\Omega; \mathbb{R}^m)$ the completion of $C_c(\Omega; \mathbb{R}^m)$ with respect to the sup norm. This is the space of continuous functions ϕ on Ω satisfying the property: for any $\epsilon > 0$ there exists a compact set $K \subset \Omega$ such that $|\phi(x)| < \epsilon$, $\forall x \in \Omega \setminus K$.

Theorem 1.1.1. (Riesz Representation Theorem)
Let $L : C_0(\Omega; \mathbb{R}^m) \to \mathbb{R}$ be a continuous linear functional; that is, L is linear and satisfies

$$\sup\{L(\phi) : \phi \in C_0(\Omega; \mathbb{R}^m), ||\phi||_\infty \leq 1\} < \infty.$$

Then there exists a unique $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ such that

$$L(\phi) = \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m).$$

Moreover,

$$|||\mu||| = ||\mu||(\Omega) = \sup\{L(\phi) : \phi \in C_c(\Omega;\mathbb{R}^m), ||\phi||_\infty \leq 1\} = ||L||.$$

Proof. See [AFP], Theorem 1.54.

The following corollary is a direct consequence of the global version of the Riesz Representation Theorem.

Corollary 1.1.1. *Let $L : C_c(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(\phi) : \phi \in C_c(\Omega;\mathbb{R}^m), ||\phi||_\infty \leq 1, \mathrm{supp}(\phi) \subset K\} < \infty,$$

for any compact set $K \subset \Omega$. Then there exists a unique $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ such that

$$L(\phi) = \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega;\mathbb{R}^m).$$

Thus we can identify any $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ with a continuous linear functional on $C_0(\Omega;\mathbb{R}^m)$, written as

$$L_\mu(\phi) := \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m),$$

and analogously $\mathcal{M}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ can be identified with the dual of $C_c(\Omega;\mathbb{R}^m)$.

This leads us to the following notion.

Definition 1.1.4. Given a sequence $\{\mu_k\}$ in $\mathcal{M}(\Omega)$, we say that μ_k *weak-star converges to* μ, if and only if

$$\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_0(\Omega;\mathbb{R}^m).$$

If $\{\mu_k\}$ and μ are in $\mathcal{M}_{\mathrm{loc}}(\Omega)$, we say that μ_k *locally weak-star converges to* μ, if and only if

$$\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega;\mathbb{R}^m).$$

Lemma 1.1.1. *Let $\{\mu_k\} \subset \mathcal{M}(\Omega;\mathbb{R}^m)$ be a weak-star convergent sequence, and let μ be its limit. Then we have*

$$\limsup_{k\to+\infty} |||\mu_k||| < \infty$$

and

$$|||\mu||| \leq \liminf_{k\to+\infty} |||\mu_k|||.$$

Proof. The first assertion follows from Uniform Boundedness Principle (Banach-Steinhaus Theorem), since $L_{\mu_k}(\phi) \to L_\mu(\phi)$ for each $\phi \in C_0(\Omega;\mathbb{R}^m)$ and therefore $\{L_{\mu_k}(\phi)\}$ is a bounded sequence in $\mathbb{R}$.
The second inequality comes from:

$$|L_{\mu_k}(\phi)| \leq ||\phi||_\infty |||\mu_k|||$$

then, passing to the limit we have $|L_\mu(\phi)| \leq \liminf_{k\to+\infty} ||\phi||_\infty |||\mu_k|||$ and taking supremum in ϕ yields the result. □

Remark 1.1.4. Weak-star convergence of finite Radon measures is equivalent to local weak-star convergence with the condition that $\sup ||\mu_k||(\Omega) = C < \infty$. We observe that, by Lemma 1.1.1, this condition implies $||\mu||(\Omega) \leq C$.
Clearly weak-star convergence always implies local weak-star convergence.
On the other hand, if we suppose that μ_k locally weak-star converges to μ, then, given $\psi \in C_0(\Omega;\mathbb{R}^m)$, for any $\epsilon > 0$ there exists $\phi \in C_c(\Omega;\mathbb{R}^m)$ such that $||\psi - \phi||_\infty < \epsilon$ and so

$$\begin{aligned}\left|\int_\Omega \psi \cdot d\mu_k - \int_\Omega \psi \cdot d\mu\right| &\leq \left|\int_\Omega (\psi-\phi)\cdot d\mu_k\right| + \left|\int_\Omega (\psi-\phi)\cdot d\mu\right| \\ &\quad + \left|\int_\Omega \phi \cdot d\mu_k - \int_\Omega \phi \cdot d\mu\right| \\ &\leq 2C\epsilon + \left|\int_\Omega \phi \cdot d\mu_k - \int_\Omega \phi \cdot d\mu\right|.\end{aligned}$$

Now, $\int_\Omega \phi \cdot d\mu_k \to \int_\Omega \phi \cdot d\mu$ and so, since ϵ is arbitrary, we obtain weak-star convergence.
Therefore, in what follows, we will always write $\mu_k \overset{*}{\rightharpoonup} \mu$ to denote local weak-star convergence, and, in the case of finite Radon measures, we will also check the condition $\sup ||\mu_k||(\Omega) < \infty$.

We quote now a useful result about weak-star convergence.

Lemma 1.1.2. *Let μ be a Radon measure on Ω, and let $\{\mu_k\}$ be a sequence of Radon measures.*
If μ_k and μ are positive, then the following are equivalent:

1. $\mu_k \overset{*}{\rightharpoonup} \mu$.
2. $\forall A \subset \Omega$ *open,*
$$\mu(A) \leq \liminf_{k\to+\infty} \mu_k(A)$$
and $\forall K \subset \Omega$ *compact,*
$$\mu(K) \geq \limsup_{k\to+\infty} \mu_k(K).$$

3. $\forall B \subset\subset \Omega$ Borel set with $\mu(\partial B) = 0$,

$$\lim_{k\to+\infty} \mu_k(B) = \mu(B).$$

If μ_k and μ are $\mathbb{R}^m$-vector valued Radon measures, $\mu_k \overset{*}{\rightharpoonup} \mu$ and $||\mu_k|| \overset{*}{\rightharpoonup} \nu$, then $||\mu|| \leq \nu$. Moreover, if a μ-measurable set $E \subset\subset \Omega$ satisfies $\nu(\partial E) = 0$, then

$$\mu(E) = \lim_{k\to+\infty} \mu_k(E).$$

More generally, if $f : \Omega \to \mathbb{R}^m$ is a bounded Borel function with compact support such that the set of its discontinuity points is ν-neglegible, then

$$\lim_{k\to+\infty} \int_\Omega f \cdot d\mu_k = \int_\Omega f \cdot d\mu.$$

Proof. For the second part of the statement and the implication $1 \to 2$ we refer to [AFP], Proposition 1.62. For the two remaining implications, we adapt the proof in [EG], Section 1.9, Theorem 1, where $\Omega = \mathbb{R}^N$.
In order to show that 2 implies 3, we take a Borel set B such that $\overline{B} \subset \Omega$ and $\mu(\partial B) = 0$. Then

$$\mu(B) = \mu(B^\circ) \leq \liminf_{k\to+\infty} \mu_k(B^\circ) \leq \limsup_{k\to+\infty} \mu_k(\overline{B}) \leq \mu(\overline{B}) = \mu(B).$$

Now we suppose that 3 holds and we observe that, since ϕ can be decomposed into its positive and negative parts, we need only to prove 1 for nonnegative functions. We fix $\epsilon > 0$ and $\phi \in C_c(\Omega)$ with $\phi \geq 0$. Let Ω_s be defined as in the proof of Proposition 1.1.1, but for $s \in (1, +\infty)$. By Remark 1.1.2, for all but countable s, we have $\mu(\partial\Omega_s) = 0$. Therefore, there exists s_0 such that $\mathrm{supp}(\phi) \subset \Omega_{s_0}$ and $\mu(\partial\Omega_{s_0}) = 0$. We can choose $0 = t_0 < t_1 < ... < t_N = 2||\phi||_\infty$ such that $0 < t_i - t_{i-1} < \epsilon$ and $\mu(\phi^{-1}(\{t_i\})) = 0$ for any $i = 1, ..., N$, by Remark 1.1.2. We set $B_i = \phi^{-1}((t_{i-1}, t_i])$, then $\mu(\partial B_i) = 0$ for $i \geq 2$. Now

$$\sum_{i=2}^N t_{i-1}\mu_k(B_i) \leq \int_\Omega \phi \, d\mu_k \leq \sum_{i=2}^N t_i\mu_k(B_i) + t_1\mu_k(\Omega_{s_0})$$

and

$$\sum_{i=2}^N t_{i-1}\mu(B_i) \leq \int_\Omega \phi \, d\mu \leq \sum_{i=2}^N t_i\mu(B_i) + t_1\mu(\Omega_{s_0});$$

and so 3 implies

$$\limsup_{k\to+\infty} \left| \int_\Omega \phi \, d\mu_k - \int_\Omega \phi \, d\mu \right| \leq 2\epsilon\mu(\Omega_{s_0}),$$

which gives 1. □

Remark 1.1.5. By Remark 1.1.2 and Lemma 1.1.2, we can assert that, if μ_k and μ are positive Radon measures in Ω, for any $x \in \Omega$ and almost every $r \in (0, R)$, with $R = R_x > 0$ such that $B(x, R_x) \subset\subset \Omega$, $\mu(\partial B(x, r)) = 0$ and so, if $\mu_k \overset{*}{\rightharpoonup} \mu$, $\mu_k(B(x, r)) \to \mu(B(x, r))$.
Moreover, if μ_k and μ are vector valued Radon measures, $\mu_k \overset{*}{\rightharpoonup} \mu$ and $||\mu_k|| \overset{*}{\rightharpoonup} \nu$, then for any $x \in \Omega$ and almost every $r \in (0, R)$, with $R = R_x > 0$ such that $B(x, R_x) \subset\subset \Omega$, $\nu(\partial B(x, r)) = 0$ and $\mu_k(B(x, r)) \to \mu(B(x, r))$.

Finally, we state a characterization of nonnegative linear functionals on $C_c^\infty(\Omega)$.

Lemma 1.1.3. *Let $L : C_c^\infty(\Omega) \to \mathbb{R}$ be linear and nonnegative; that is,*

$$L(\phi) \geq 0, \quad \forall \phi \in C_c^\infty(\Omega) \text{ with } \phi \geq 0.$$

Then there exists a positive Radon measure $\mu \in \mathcal{M}_{\text{loc}}(\Omega)$ such that

$$L(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_c^\infty(\Omega).$$

Proof. We choose a compact set $K \subset \Omega$ and we select a smooth function $\zeta \in C_c^\infty(\Omega)$ with $\zeta = 1$ on K and $0 \leq \zeta \leq 1$. Then, for any $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset K$, we set $\psi = ||\phi||_\infty \zeta - \phi \geq 0$. Therefore, since L is nonnegative, we have $0 \leq L(\psi) = ||\phi||_\infty L(\zeta) - L(\phi)$ and so $L(\phi) \leq C||\phi||_\infty$, with $C := L(\zeta)$.
L thus may be extended to a linear mapping $\hat{L} : C_c(\Omega) \to \mathbb{R}$ such that, for any compact $K \subset \Omega$,

$$\sup\{L(\phi) : \phi \in C_c(\Omega; \mathbb{R}^m), ||\phi||_\infty \leq 1, \text{supp}(\phi) \subset K\} < \infty.$$

Hence, Corollary 1.1.1 yields the existence of a real Radon measure μ such that

$$L(\phi) = \int_\Omega \phi \, d\mu, \quad \forall \phi \in C_c(\Omega).$$

By the polar decomposition of measures, $\mu = h||\mu||$, where $|h| = 1$ $||\mu||$-a.e. The fact that L is nonnegative implies that $h = 1$ $||\mu||$-a.e.; that is, μ is a positive Radon measure. □

We recall now the statement of Lebesgue-Besicovitch differentiation theorem and the definitions of approximate limit and precise representative.

Theorem 1.1.2. *Let μ be a positive Radon measure on $\mathbb{R}^N$ and $u \in L^1_{\text{loc}}(\mathbb{R}^N, \mu)$. Then*

$$\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} u \, d\mu = u(x)$$

for μ a.e. $x \in \mathbb{R}^N$.

Proof. See [EG] Section 1.7.1 Theorem 1.

Corollary 1.1.2. *Let* μ *be a positive Radon measure on* $\mathbb{R}^N$*,* $1 \leq p < \infty$*, and* $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N, \mu)$*. Then*

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |u - u(x)|^p \, d\mu = 0 \tag{1.1.2}$$

for μ *a.e.* x*.*

Proof. See [EG] Section 1.7.1 Corollary 1.

Definition 1.1.5. A point x for which (1.1.2) with $p = 1$ holds is called a *Lebesgue point* of u with respect to μ.

Definition 1.1.6. Assume $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then

$$u^*(x) := \begin{cases} \lim\limits_{r \to 0} \dfrac{1}{|B(x,r)|} \displaystyle\int_{B(x,r)} u(y) \, dy & \text{if this limit exists} \\ 0 & \text{otherwise} \end{cases}$$

is the *precise representative* of u.

Definition 1.1.7. Let $u : \mathbb{R}^N \to \mathbb{R}^M$.

1. $l \in \mathbb{R}^M$ is the *approximate limit* of u as $y \to x$, and is denoted by

   $$\operatorname{ap}\lim_{y \to x} u(y) = l$$

   if $\forall\, \epsilon > 0$,

   $$\lim_{r \to 0} \frac{|B(x,r) \cap \{|u - l| \geq \epsilon\}|}{|B(x,r)|} = 0;$$

2. u is *approximately continuous* at $x \in \mathbb{R}^N$ if

   $$\operatorname{ap}\lim_{y \to x} u(y) = u(x).$$

The following theorems assure the well posedness and the significance of the previous definitions.

Theorem 1.1.3. *The approximate limit is unique.*

Proof. See [EG] Section 1.7.2 Theorem 2.

Moreover, we have the following result on approximate continuity.

Theorem 1.1.4. *Let* $u : \mathbb{R}^N \to \mathbb{R}^M$ *be* $\mathcal{L}^N$*-measurable. Then* u *is approximately continuous* $\mathcal{L}^N$*-a.e.*

Proof. See [EG] Section 1.7.2 Theorem 3.

1.2 Sobolev functions and p-capacity

Definition 1.2.1. For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, we define the *Sobolev space*

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \ \forall \alpha \in \mathbb{N}_0^N, \ \ |\alpha| \leq k\}$$

where $D^\alpha u$ is the α^{th}-weak partial derivative of u, that is, an L^1_{loc} function wich satisfies

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u \phi \, dx \ \ \forall \phi \in C_c^\infty(\Omega)$$

and $D^\alpha \phi = \dfrac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_N}\phi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}}$.
The norm is given by

$$||u||_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p dx\right)^{\frac{1}{p}} \quad \text{for} \ \ 1 \leq p < \infty$$

$$||u||_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq k} \text{ess} \sup_\Omega |D^\alpha u| \ \ \text{for} \ \ p = +\infty.$$

We say that $u \in W^{k,p}_{\text{loc}}(\Omega)$ if $u \in W^{k,p}(W)$ for each open set $W \subset\subset \Omega$.

Definition 1.2.2. A function $\rho \in C_c^\infty(\mathbb{R}^N)$ is a *standard symmetric mollifying kernel* if it is a radial nonnegative function which satisfies $\text{supp}(\rho) \subset\subset B(0,1)$ and $||\rho||_{L^1(\mathbb{R}^N)} = 1$.
If $u \in L^1_{\text{loc}}(\Omega)$, we define, for $x \in \Omega_\epsilon := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}$,

$$u_\epsilon(x) := u * \rho_\epsilon(x) = \int_\Omega u(y)\rho_\epsilon(x-y)dy$$

the *mollification* of u, where $\rho_\epsilon(y) := \frac{1}{\epsilon^N}\rho(\frac{y}{\epsilon})$.

Theorem 1.2.1. (Properties of mollification)

1. *For each $\epsilon > 0$, $u_\epsilon \in C^\infty(\mathbb{R}^N)$ and $D^\alpha u_\epsilon = (D^\alpha \rho_\epsilon) * u$ for each multi-index α.*

2. *If $u \in C(\Omega)$, then $u_\epsilon \to u$ uniformly on compact subsets of Ω.*

3. *If $u \in L^p_{\text{loc}}(\Omega)$ for some $1 \leq p < \infty$, then $u_\epsilon \to u$ in $L^p_{loc}(\Omega)$.*

4. *$u_\epsilon(x) \to u(x)$ if x is a Lebesgue point of u, therefore $u_\epsilon \to u$ $\mathcal{L}^N$ a.e.*

5. *If $u \in W^{k,p}_{\text{loc}}(\Omega)$ for some $1 \leq p \leq \infty$, then $D^\alpha u_\epsilon = \rho_\epsilon * D^\alpha u$ in Ω_ϵ for each $|\alpha| \leq k$.*
   In particular, for $1 \leq p < \infty$, $u_\epsilon \to u$ in $W^{k,p}_{\text{loc}}(\Omega)$.

Proof. See [EG] Section 4.2.1 Theorem 1.

Theorem 1.2.2. (Meyers-Serrin Approximation Theorem)
Let $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$.
Then there exists a sequence $\{u_k\} \subset W^{k,p}(\Omega) \cap C^{\infty}(\Omega)$[3] such that $u_k \to u$ in $W^{k,p}(\Omega)$.

Proof. See [EG] Section 4.2.1 Theorem 2.

We present now the concept of capacity, which has been very useful in the study of fine properties of Sobolev functions and which we will need in order to prove results concerning absolute continuity of measures (Theorem 2.3.1).

Definition 1.2.3. For $1 \leq p \leq N$ and a compact subset K of the open set Ω in $\mathbb{R}^N$, we define the *p-capacity* of K relative to Ω as

$$\mathrm{Cap}_p(K,\Omega) := \inf\left\{\int_\Omega |\nabla\phi|^p dx : \phi \in C_c^\infty(\Omega), \phi \geq 1 \,\mathrm{on}\, K\right\}.$$

If $U \subset \Omega$ is open, we set

$$\mathrm{Cap}_p(U,\Omega) := \sup\{\mathrm{Cap}_p(K,\Omega) : K \subset U \text{ compact}\}$$

and, for an arbitrary set $A \subset \Omega$,

$$\mathrm{Cap}_p(A,\Omega) := \inf\{\mathrm{Cap}_p(U,\Omega) : A \subset U \subset \Omega,\, U \text{ open}\}.$$

If $\Omega = \mathbb{R}^N$, we write $\mathrm{Cap}_p(A,\mathbb{R}^N) = \mathrm{Cap}_p(A)$, for any set A.

Remark 1.2.1. If $1 \leq p < N$ and $\Omega = \mathbb{R}^N$, the p-capacity of a set A may also be defined as

$$\mathrm{Cap}_p(A) = \inf\left\{\int_{\mathbb{R}^N} |Df|^p dx : f \in K^p, \{f \geq 1\}^\circ \supset A\right\},$$

where

$$K^p := \{f : \mathbb{R}^N \to \mathbb{R} : f \geq 0, f \in L^{p*}(\mathbb{R}^N), Df \in L^p(\mathbb{R}^N;\mathbb{R}^N)\}.$$

For this definition we refer to [EG], Section 4.7.1.

Remark 1.2.2. It is possible to show that, for any compact subset K of Ω, Definition 1.2.3 is equivalent to

$$\mathrm{Cap}_p(K,\Omega) = \inf\left\{\int_\Omega |\nabla\phi|^p \, dx : \phi \in C_c^\infty(\Omega), 0 \leq \phi \leq 1, \{\phi = 1\}^\circ \supset K\right\},$$

[3]With an abuse of notation, we denote by $u \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$ the equivalence class of functions in $W^{k,p}(\Omega)$ which has a representative in $C_c^\infty(\Omega)$.

by the following approximation argument one finds in [Maz], §2.2.1, point (ii).
First, it is clear that for any $K \subset \Omega$

$$\inf\left\{\int_\Omega |\nabla\phi|^p dx : \phi \in C_c^\infty(\Omega), \phi \geq 1 \,\text{on}\, K\right\}$$
$$\leq \inf\left\{\int_\Omega |\nabla\phi|^p dx : \phi \in C_c^\infty(\Omega), 0 \leq \phi \leq 1, \{\phi = 1\}^\circ \supset K\right\},$$

since the second infimum is taken over a smaller set of functions.
Then we fix $\epsilon > 0$ and pick ψ from the functions competing in the first infimum such that $\int_\Omega |\nabla\psi|^p dx \leq \mathrm{Cap}_p(K,\Omega) + \epsilon$.
Let $\{\lambda_m\} \subset C_c^\infty(\mathbb{R})$ such that:

1. $0 \leq \lambda_m' \leq 1 + \frac{1}{m}$,
2. $\lambda_m(t) = 0$ in a neighborhood of $(-\infty, 0]$,
3. $\lambda_m(t) = 1$ in a neighborhood of $[1, +\infty)$,
4. $0 \leq \lambda_m \leq 1$.

Therefore, $0 \leq \lambda_m(\psi) \leq 1$ and this composition is clearly a smooth function equal to 1 in a neighborhood of K.
So

$$\inf\left\{\int_\Omega |\nabla\phi|^p dx : \phi \in C_c^\infty(\Omega), 0 \leq \phi \leq 1, \{\phi = 1\}^\circ \supset K\right\}$$
$$\leq \int_\Omega |\nabla\lambda_m(\psi)|^p dx = \int_\Omega |\lambda_m'(\psi)|^p |\nabla\psi|^p dx \leq \left(1 + \frac{1}{m}\right)^p (\mathrm{Cap}_p(K,\Omega) + \epsilon)$$

and sending $m \to +\infty$ yields the opposite inequality and so the desired result.

Proposition 1.2.1. (Properties of capacity)
Let $1 \leq p \leq N$.

1. *If $A_1 \subset A_2$, then $\mathrm{Cap}_p(A_1,\Omega) \leq \mathrm{Cap}_p(A_2,\Omega)$.*
2. *If $\Omega_1 \subset \Omega_2$ are open and $A \subset \Omega_1$, then $\mathrm{Cap}_p(A,\Omega_2) \leq \mathrm{Cap}_p(A,\Omega_1)$. In particular, if $\Omega_2 = \mathbb{R}^N$, $\mathrm{Cap}_p(A) \leq \mathrm{Cap}_p(A,\Omega)$ for any open set Ω and any set $A \subset \Omega$.*
3. *If K_1 and K_2 are compact subsets of Ω, then*
$$\mathrm{Cap}_p(K_1 \cup K_2,\Omega) + \mathrm{Cap}_p(K_1 \cap K_2,\Omega) \leq \mathrm{Cap}_p(K_1,\Omega) + \mathrm{Cap}_p(K_2,\Omega).$$
4. *If $\{K_j\}$ is a monotone decreasing sequence of compact subsets of Ω, then*
$$\lim_{j\to+\infty} \mathrm{Cap}_p(K_j,\Omega) = \mathrm{Cap}_p\left(\bigcap_{j=1}^{+\infty} K_j, \Omega\right).$$

5. *If $\{A_j\}$ is a monotone increasing sequence of subsets of Ω, then*

$$\lim_{j\to+\infty} \mathrm{Cap}_p(A_j,\Omega) = \mathrm{Cap}_p\left(\bigcup_{j=1}^{+\infty} A_j,\Omega\right).$$

6. *If $\{A_j\}$ is any sequence of subsets of Ω, then*

$$\mathrm{Cap}_p\left(\bigcup_{j=1}^{+\infty} A_j,\Omega\right) \le \sum_{j=1}^{+\infty} \mathrm{Cap}_p(A_j,\Omega).$$

7. *If A is a Borel subset of Ω, then*

$$\mathrm{Cap}_p(A,\Omega) = \sup\{\mathrm{Cap}_p(K,\Omega) : K \text{ compact}, K \subset A\}.$$

Proof. Properties 1 and 2 are immediate consequences of the definition of p-capacity.
For 3, we refer to [Maz], §2.2.1, point (v).
To prove 4, we notice that, for any $\epsilon > 0$, there exists a function $\phi \in C_c^\infty(\Omega)$, $\phi \ge 1$ on $\bigcap_{j=1}^{+\infty} K_j$ such that

$$\int_\Omega |\nabla\phi|^p\,dx \le \mathrm{Cap}_p\left(\bigcap_{j=1}^{+\infty} K_j,\Omega\right) + \epsilon.$$

Since the sequence of compact sets is decreasing, there exists $j_0 = j_0(\epsilon)$ such that, for any $j \ge j_0$, $K_j \subset \{\phi \ge 1-\epsilon\}$; therefore

$$\mathrm{Cap}_p\left(\bigcap_{j=1}^{+\infty} K_j,\Omega\right) \le \lim_{j\to+\infty} \mathrm{Cap}_p(K_j,\Omega) \le \mathrm{Cap}_p(\{\phi \ge 1-\epsilon\},\Omega)$$
$$\le (1-\epsilon)^{-p}\int_\Omega |\nabla\phi|^p\,dx \le (1-\epsilon)^{-p}(\mathrm{Cap}_p\left(\bigcap_{j=1}^{+\infty} K_j,\Omega\right) + \epsilon).$$

Since ϵ is arbitrary, property 4 follows.
For the other points, we refer to [HKM], Theorem 2.2 and 2.5. We observe that in [HKM], the authors assume $1 < p < \infty$, however, as we have shown, such an assumption may be dropped in the proof of point 4. Points 5 and 6 are consequences of point 1 and a lemma ([HKM], Lemma 2.3), whose proof relies only on topological facts and on properties 1 and 3, which Maz'ja proved for $1 \le p \le N$: therefore these points are proved also in the case $p = 1$.
Any set function which is defined in the family of all subsets of an open set Ω and which satisfies properties 1, 4 and 5 is called a *Choquet capacity* relative to Ω. Point 7 is a property of Choquet capacities, whose proof may be found in [H], p. 149. □

Remark 1.2.3. Property 7 in Proposition 1.2.1 is indeed true for a more general class of sets, the Suslin sets, which contains the Borel sets, as it is shown in [Fe], pag. 63-66.

Remark 1.2.4. Since clearly $\mathrm{Cap}_p(\emptyset, \Omega) = 0$, from property 6 in Proposition 1.2.1 we deduce that $\mathrm{Cap}_p(\cdot, \Omega)$ is an outer measure on Ω. However, the p-capacity relative to an open set Ω is not a Borel measure, since there are Borel sets of finite capacity which are not $\mathrm{Cap}_p(\cdot, \Omega)$-measurable.

Proposition 1.2.2. *Let $N \geq 2$, $1 \leq p \leq N$, $0 < r < R < \infty$ and $x \in \mathbb{R}^N$, then*

$$\mathrm{Cap}_p(B(x,r), B(x,R)) = \begin{cases} N\omega_N \left(\frac{N-p}{p-1}\right)^{p-1} |R^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}|^{1-p} & \textit{for} \;\; 1 < p < N \\ N\omega_N \left(\log \frac{R}{r}\right)^{1-N} & \textit{for} \;\; p = N \end{cases}$$

and

$$\mathrm{Cap}_1(B(x,r), B(x,R)) \leq N\omega_N r^{N-1}.$$

In particular,

$$\mathrm{Cap}_p(B(x,r), B(x,2r)) \leq C(N,p) r^{N-p} \;\; \textit{for} \;\; 1 \leq p < N.$$

Proof. For $1 < p \leq N$, see [HKM], Section 2.11, and [Maz], § 2.2.4.
For $p = 1$, we use the following lemma ([Maz], §2.2.5): for any compact set $K \subset \Omega$,

$$\mathrm{Cap}_1(K, \Omega) = \inf\{\mathcal{H}^{N-1}(\partial G) : K \subset G \subset\subset \Omega, \; G \text{ open}, \; \partial G \; C^\infty \text{ manifold}\}.$$

Hence, if $K = \overline{B(x,r)}$ and $\Omega = B(x,R)$, we see that

$$\mathrm{Cap}_1(B(x,r), B(x,R)) \leq \mathrm{Cap}_1(\overline{B(x,r)}, B(x,R)) \leq N\omega_N (r+\epsilon)^{N-1}$$

for any $0 < \epsilon < R - r$, and so the estimate follows. □

Theorem 1.2.3. (Relations between capacity and Hausdorff measure)
For any $1 \leq p < N$ and K compact subset of Ω, $\mathrm{Cap}_p(K, \Omega) \leq C(N,p)\mathcal{H}^{N-p}(K)$.
In particular, if $\mathcal{H}^{N-p}(K) = 0$, then $\mathrm{Cap}_p(K, \Omega) = 0$.
Moreover, if $\Omega = \mathbb{R}^N$ and $A \subset \mathbb{R}^N$, then

1. *if $1 < p < N$ and $\mathcal{H}^{N-p}(A) < \infty$, then $\mathrm{Cap}_p(A) = 0$;*
2. *if $1 < p < N$ and $\mathrm{Cap}_p(A) = 0$, then $\mathcal{H}^s(A) = 0$ for $s > N - p$;*
3. *$\mathrm{Cap}_1(A) = 0$ if and only if $\mathcal{H}^{N-1}(A) = 0$.*

Proof. Since $K \subset \Omega$ is compact, then $\mathrm{dist}(K, \partial\Omega) = d > 0$, and let $\{B(x_k, r_k)\}$ be a δ-covering of K; that is, $K \subset \bigcup_{k=1}^\infty B(x_k, r_k)$ with $2r_k < \delta$. We choose $\delta < \frac{d}{2}$. We observe that, among the δ-coverings of K, those which better approximate the $\mathcal{S}_\delta^{N-p}$-measure of K are those in which every ball has nonempty intersection with

K, since we can always throw away balls which do not intersect K and obtain a covering for which the sum $\sum_{k=1}^{\infty} \omega_{N-p} r_k^{N-p}$ is smaller. Thus, since $B(x_k, r_k) \cap K \neq \emptyset \ \forall k$, we have $\text{dist}(x_k, \partial\Omega) \geq d - r_k > d - \frac{\delta}{2} > \frac{3d}{4} > 0$, which implies that the balls $B(x_k, r_k)$ and $B(x_k, 2r_k)$ are inside Ω for each k. Therefore, $\text{Cap}_p(B(x_k, r_k), \Omega) \leq \text{Cap}_p(B(x_k, r_k), B(x_k, 2r_k))$ (property 2 in Proposition 1.2.1) and Proposition 1.2.2 states that

$$\text{Cap}_p(B(x_k, r_k), B(x_k, 2r_k)) \leq C(N, p) r_k^{N-p}.$$

Hence, by subadditivity (property 6 in Proposition 1.2.1), we have

$$\begin{aligned}\text{Cap}_p(K, \Omega) &\leq \sum_{k=1}^{\infty} \text{Cap}_p(B(x_k, r_k), \Omega) \\ &\leq \sum_{k=1}^{\infty} \text{Cap}_p(B(x_k, r_k), B(x_k, 2r_k)) \leq \sum_{k=1}^{\infty} C(N.p) r_k^{N-p}\end{aligned}$$

and so

$$\text{Cap}_p(K, \Omega) \leq C\mathcal{S}_\delta^{N-p}(K) \leq C\mathcal{S}^{N-p}(K) \leq C 2^{N-p} \mathcal{H}^{N-p}(K),$$

since we take the supremum over $0 < \delta < \frac{d}{2}$ and we use the estimate (1.1.1).

For the second part of the theorem, see [EG] Section 4.7.2 Theorems 3-4 and Section 5.6.3 Theorem 3. □

The following theorem will show one important result on fine properties of Sobolev functions in $\mathbb{R}^N$ obtained using capacity.

Theorem 1.2.4. *Let $u \in W^{1,p}(\mathbb{R}^N)$, $1 \leq p < N$.*

1. *There is a Borel set $B \subset \mathbb{R}^N$ such that $\text{Cap}_p(B) = 0$ and $\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dy$ exists for each $x \in \mathbb{R}^N \setminus B$.*
2. *For each $x \in \mathbb{R}^N \setminus B$*

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u - u^*(x)|^{p^*} dy = 0,$$

where u^* is the precise representative of u (Definition 1.1.6). Moreover, this precise representative is p-quasicontinuous; that is, for any $\epsilon > 0$, there exists an open set V such that $\text{Cap}_p(V) \leq \epsilon$ and $u|_{\mathbb{R}^N \setminus V}$ is continuous.

Proof. See [EG] Section 4.8 Theorem 1.

Remark 1.2.5. In particular, it follows that the mollification of u converges pointwise to u^* up to a set B of Hausdorff dimension at most $N-p$: if $x \in \mathbb{R}^N \setminus B$, then, by the definition of ρ and Jensen's inequality, we have

$$|u^*(x) - u_\epsilon(x)| = \left| \int_{\mathbb{R}^N} (u^*(x) - u(y))\rho_\epsilon(x-y)dy \right| \leq$$

$$||\rho||_\infty \omega_N \left(\frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} |u(y) - u^*(x)|^{p^*} dy \right)^{\frac{1}{p^*}} \to 0 \text{ as } \epsilon \to 0.$$

We state now a technical lemma which we will use in Chapter 2.

Lemma 1.2.1. *Let $1 \leq p < N$ and K be a compact subset of Ω. If $\mathrm{Cap}_p(K,\Omega) = 0$, then there exists a sequence of test functions $\phi_j \in C_c^\infty(\Omega)$ such that*

1. *$0 \leq \phi_j \leq 1$ and $\phi_j = 1$ on K,*
2. *$||\nabla \phi_j||_{L^p(\Omega;\mathbb{R}^N)} \to 0$,*
3. *for each j, $\mathrm{supp}(\phi_j)$ is contained in a compact set $C_j \subset \Omega$ such that*

$$C_1 \supset C_2 \supset ... \supset K \quad \textit{and} \quad \bigcap_{j=1}^{\infty} C_j = K,$$

4. *if $\Omega = \mathbb{R}^N$ and $1 < p < N$, $\phi_j(x) \to 0$ for all $x \in \mathbb{R}^N \setminus A$ for some set A with $\mathrm{Cap}_p(A) = 0$.*

Proof. By the definition of capacity, there exists of a sequence $\psi_j \in C_c^\infty(\Omega)$ such that $||\nabla \psi_j||_{L^p(\Omega;\mathbb{R}^N)} \to 0$, which is point 2.
Moreover, by Remark 1.2.2, we also have point 1.

We observe that, since $1 \leq p < N$ and ψ_j has compact support, the Gagliardo-Nirenberg-Sobolev inequality is valid, so $||\psi_j||_{L^{p^*}(\Omega)} \leq C||\nabla \psi_j||_{L^p(\Omega;\mathbb{R}^N)}$.
Thus, up to passing to a subsequence, $\psi_j \to 0$ $\mathcal{L}^N$-a.e. in Ω.
We also point out the fact that we cannot have $||\nabla \psi_j||_{L^p(\Omega;\mathbb{R}^N)} = 0$ for some j, otherwise ψ_j would be identically 0, which is in contradiction with the fact that $\psi_j = 1$ on K.
Therefore, without loss of generality, we can ask that

$$||\nabla \psi_j||_{L^p(\Omega;\mathbb{R}^N)} > ||\nabla \psi_{j+1}||_{L^p(\Omega;\mathbb{R}^N)}$$

for each j: indeed, $||\nabla \psi_j||_{L^p(\Omega;\mathbb{R}^N)} > 0$ for each j and goes to 0, so, up to choosing a subsequence, we have monotone decay.

Now, let $\{\delta_j\}_{j=0}^{+\infty}$ be a sequence which satisfies $\delta_j > 0$, $\delta_j > \delta_{j+1}$ and $\delta_j \to 0$, and define $K_{\delta_j} := \{x : \mathrm{dist}(x,K) \leq \delta_j\}$.

Then, by properties of convolution and mollification, if ρ is a standard symmetric mollifier, then $\chi_{K_{\delta_{j-1}}} * \rho_{\delta_j}$ is a smooth function whose compact support is contained in $K_{\delta_{j-1}+\delta_j} =: C_j$ for each $j \geq 1$.
We choose δ_0 and δ_1 in such a way that $C_1 \subset \Omega$.
Moreover, $0 \leq \chi_{K_{\delta_{j-1}}} * \rho_{\delta_j} \leq 1$ and it is identically equal to 1 in $K_{\delta_{j-1}-\delta_j}$.

So, if we define $\phi_j := \psi_j(\chi_{K_{\delta_{j-1}}} * \rho_{\delta_j})$, then we have property *1*, just by observing that

$$\{\phi_j = 1\}^\circ = \{\psi_j = 1\}^\circ \cap K^\circ_{\delta_{j-1}-\delta_j} \supset K \cap K_{\delta_{j-1}-\delta_j} \supset K.$$

Then, we have

$$\mathrm{supp}(\phi_j) \subset \mathrm{supp}(\psi_j) \cap K_{\delta_{j-1}+\delta_j} \subset C_j$$

and clearly $C_j \supset C_{j+1}$ by their definition, also $\bigcap_{j=1}^\infty C_j = \bigcap_{j=1}^\infty K_{\delta_{j-1}+\delta_j} = K$. So we have property 3.

Now we observe that

$$||\nabla \phi_j||_{L^p(\Omega;\mathbb{R}^N)} \leq ||\nabla \psi_j(\chi_{K_{\delta_{j-1}}} * \rho_{\delta_j})||_{L^p(\Omega;\mathbb{R}^N)} + ||\psi_j \nabla(\chi_{K_{\delta_{j-1}}} * \rho_{\delta_j})||_{L^p(\Omega;\mathbb{R}^N)}.$$

Clearly, $||\nabla \psi_j(\chi_{K_{\delta_{j-1}}} * \rho_{\delta_j})||_{L^p(\Omega;\mathbb{R}^N)} \leq ||\nabla \psi_j||_{L^p(\Omega;\mathbb{R}^N)}$. The Hölder inequality with $\hat{p} = \frac{N}{N-p}$, $\hat{p}' = \frac{N}{p}$ yields

$$||\psi_j \nabla(\chi_{K_{\delta_{j-1}}} * \rho_{\delta_j})||_{L^p(\Omega;\mathbb{R}^N)} \leq ||\psi_j||_{L^{p^*}(\Omega)} ||\nabla(\chi_{K_{\delta_{j-1}}} * \rho_{\delta_j})||_{L^N(\Omega;\mathbb{R}^N)}$$

and

$$|\nabla(\chi_{K_{\delta_{j-1}}} * \rho_{\delta_j})(x)| \leq \int_\Omega \chi_{K_{\delta_{j-1}}}(y) |\nabla \rho(\frac{x-y}{\delta_j})| \frac{dy}{\delta_j^{N+1}}$$
$$= \int_{B(0,1)} \chi_{K_{\delta_{j-1}}}(x - \delta_j y) |\nabla \rho(y)| \frac{dy}{\delta_j} \leq ||\nabla \rho||_{L^\infty(B(0,1);\mathbb{R}^N)} \frac{\omega_N}{\delta_j}.$$

Then, by the Gagliardo-Nirenberg-Sobolev inequality, we have

$$||\psi_j \nabla(\chi_{K_{\delta_{j-1}}} * \rho_{\delta_j})||_{L^p(\Omega;\mathbb{R}^N)} \leq C ||\nabla \psi_j||_{L^p(\Omega;\mathbb{R}^N)} |C_1|^{\frac{1}{N}} \frac{1}{\delta_j}.$$

So we need just to choose $\delta_j = \dfrac{\delta_1}{||\nabla \psi_1||^{\frac{1}{2}}_{L^p(\Omega;\mathbb{R}^N)}} ||\nabla \psi_j||^{\frac{1}{2}}_{L^p(\Omega;\mathbb{R}^N)}$ for $j \geq 1$ (the multiplicative constant is due to the fact that we already fixed δ_1 above) in order to obtain also property 2.

Finally, if $\Omega = \mathbb{R}^N$, in order to verify property 4, we notice that $\phi_j \to 0$ $\mathcal{L}^N$-a.e. in $\mathbb{R}^N$ and, by the Hölder inequality,

$$||\phi_j||_{L^p(\mathbb{R}^N)} \leq ||\phi_j||_{L^{p^*}(\mathbb{R}^N)} |C_1|^{\frac{1}{N}} \leq ||\psi_j||_{L^{p^*}(\mathbb{R}^N)} |C_1|^{\frac{1}{N}} \leq C ||\nabla \psi_j||_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \to 0,$$

and $||\nabla\phi_j||_{L^p(\mathbb{R}^N;\mathbb{R}^N)} \to 0$ as we showed.
So, $\phi_j \to 0$ in $W^{1,p}(\mathbb{R}^N)$ and, since $1 < p < N$, Theorem 4.3 in [HKM] implies that $\phi_j(x) \to 0$ for all $x \in \mathbb{R}^N \setminus A$, for some A with $\mathrm{Cap}_p(A) = 0$. □

1.3 Functions of Bounded Variation

Definition 1.3.1. A function $u \in L^1(\Omega)$ is called a *function of bounded variation* if

$$\sup\left\{\int_\Omega u\,\mathrm{div}\phi\,dx : \phi \in C_c^\infty(\Omega;\mathbb{R}^N), ||\phi||_\infty \le 1\right\} < \infty.$$

We denote by $BV(\Omega)$ the space of all functions of bounded variation on Ω.
We say that u is *locally of bounded variation*, and we write $u \in BV_{\mathrm{loc}}(\Omega)$, if $u \in L^1_{\mathrm{loc}}(\Omega)$ and if $\forall$ open set $W \subset\subset \Omega$,

$$\sup\left\{\int_W u\,\mathrm{div}\phi\,dx : \phi \in C_c^\infty(W;\mathbb{R}^d), ||\phi||_\infty \le 1\right\} < \infty.$$

Theorem 1.3.1. (Riesz) *Let $u \in BV_{\mathrm{loc}}(\Omega)$, then there exists a unique $\mathbb{R}^N$-vector valued Radon measure μ such that*

$$\int_\Omega u\,\mathrm{div}\phi\,dx = -\int_\Omega \phi\cdot d\mu \quad \forall\phi \in C_c^1(\Omega;\mathbb{R}^N).$$

Proof. We define the linear functional $L : C_c^1(\Omega;\mathbb{R}^N) \to \mathbb{R}$ by

$$L(\phi) := -\int_\Omega u\,\mathrm{div}\phi\,dx, \quad \text{for} \quad \phi \in C_c^1(\Omega;\mathbb{R}^N).$$

Since $u \in BV_{\mathrm{loc}}(\Omega)$, we have

$$\sup\left\{L(\phi) : \phi \in C_c^\infty(W;\mathbb{R}^N), ||\phi||_\infty \le 1\right\} = C(W) < \infty$$

for each open set $W \subset\subset \Omega$, and thus

$$|L(\phi)| \le C(W)||\phi||_\infty \quad \text{for} \quad \phi \in C_c^1(W;\mathbb{R}^N).$$

We fix any compact set $K \subset \Omega$ and then we choose an open set W such that $K \subset W \subset\subset \Omega$. For each $\phi \in C_c(\Omega;\mathbb{R}^N)$ with $\mathrm{supp}(\phi) \subset K$, we choose a sequence $\phi_k \in C_c^1(W;\mathbb{R}^N)$ such that $\phi_k \to \phi$ uniformly on W. Then we define

$$\bar{L}(\phi) := \lim_{k\to+\infty} L(\phi_k).$$

By the continuity of L on $C_c^1(\Omega;\mathbb{R}^N)$ we have that this limit exists and is independent of the choice of the sequence $\{\phi_k\}$ converging to ϕ. Thus $\bar{L}$ uniquely extends to a linear functional

$$\bar{L} : C_c(\Omega;\mathbb{R}^N) \to \mathbb{R}$$

and

$$\sup\left\{\bar{L}(\phi) : \phi \in C_c^\infty(\Omega;\mathbb{R}^N), ||\phi||_\infty \leq 1, \mathrm{supp}(\phi) \subset K\right\} < \infty$$

for each compact set $K \subset \Omega$. So, by the Riesz Representation Theorem (Corollary 1.1.1), there exists an $\mathbb{R}^N$-vector valued Radon measure μ satisfying

$$\bar{L}(\phi) = -\int_\Omega \phi \cdot d\mu, \quad \forall \phi \in C_c(\Omega, \mathbb{R}^N)$$

and so, since $\bar{L}(\phi) = L(\phi)$ for $\phi \in C_c^1(\Omega, \mathbb{R}^N)$, the result follows. □

This means that the distributional derivative Du of a BV function u is an $\mathbb{R}^N$-vector valued Radon measure.
We write $||Du||$ to indicate its total variation, which is a positive Radon measure on Ω.

Remark 1.3.1. $W^{1,1}(\Omega) \subset BV(\Omega)$ and $||Du||(\Omega) = ||Du||_{L^1(\Omega;\mathbb{R}^N)}$ for $u \in W^{1,1}(\Omega)$.

Theorem 1.3.2. *If $\{u_n\} \subset BV(\Omega)$ is such that $u_n \rightharpoonup u$ in $L^p(\Omega)$ for some $p \in [1, +\infty)$, or weak-star for $p = +\infty$, or in $L^p_{\mathrm{loc}}(\Omega)$. Then $\forall A \subseteq \Omega$ open*

$$||Du||(A) \leq \liminf_{n\to+\infty} ||Du_n||(A).$$

Proof. Indeed, we have $\forall \phi \in C_c^\infty(A;\mathbb{R}^N)$

$$\int_A u_n \,\mathrm{div}\phi \, dx \to \int_A u \,\mathrm{div}\phi \, dx$$

and so, by Proposition 1.1.2,

$$\int_A u \,\mathrm{div}\phi \, dx = \lim_{n\to+\infty} \int_A u_n \,\mathrm{div}\phi \, dx \leq \liminf_{n\to+\infty} ||Du_n||(A).$$

Taking the supremum over $\phi \in C_c^\infty(A;\mathbb{R}^N)$ with $||\phi||_\infty \leq 1$ on the left hand side, we have the claim. □

Remark 1.3.2. $||Du||(\Omega)$ is a seminorm in $BV(\Omega)$. Clearly it is positively homogeneous and we get subadditivity by observing that

$$\int_\Omega (u_1 + u_2)\mathrm{div}\phi \, dx \leq ||Du_1||(\Omega) + ||Du_2||(\Omega).$$

Theorem 1.3.3. *The space $BV(\Omega)$ endowed with the norm*

$$||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + ||Du||(\Omega)$$

is a Banach space.

Proof. Let $\{u_n\}$ be a Cauchy sequence in $BV(\Omega)$, then it is Cauchy in $L^1(\Omega)$ and so $\exists u \in L^1(\Omega)$ such that $u_n \to u$ in L^1.
By the lower semicontinuity (Theorem 1.3.2), $u \in BV(\Omega)$.
Moreover, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $||D(u_k - u_n)||(\Omega) < \epsilon, \forall k, n \geq N$.
So, again by lower semicontinuity, $||D(u_k - u)||(\Omega) \leq \liminf_n ||D(u_k - u_n)||(\Omega) < \epsilon$ and from this it follows u_n converges to u in BV norm. □

Theorem 1.3.4. (Meyers-Serrin Approximation theorem)
Let $u \in BV(\Omega)$, then $\exists \{u_n\} \subset BV(\Omega) \cap C^\infty(\Omega)$ such that

1. *$u_n \to u$ in $L^1(\Omega)$*
2. *$||Du_n||(\Omega) \to ||Du||(\Omega)$.*

Proof.
Fix $\epsilon > 0$. Given a positive integer m, we set $\Omega_0 = \emptyset$, define for each $k \in \mathbb{N}, k \geq 1$ the sets

$$\Omega_k = \left\{ x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\} \cap B(0, k+m)$$

and then we choose m such that $||Du||(\Omega \setminus \Omega_1) < \epsilon$.
We define now $\Sigma_k := \Omega_{k+1} \setminus \overline{\Omega}_{k-1}$. Since $\{\Sigma_k\}$ is an open cover of Ω, then there exists a partition of unity subordinate to that open cover; that is, a sequence of functions $\{\zeta_k\}$ such that:

1. $\zeta_k \in C_c^\infty(\Sigma_k)$;
2. $0 \leq \zeta_k \leq 1$;
3. $\sum_{k=1}^{+\infty} \zeta_k = 1$ on Ω.

Then we take a standard mollifier ρ and $\forall k$ we choose ϵ_k such that:

$$\mathrm{spt}(\rho_{\epsilon_k} * (u\zeta_k)) \subset \Sigma_k$$

$$||\rho_{\epsilon_k} * (u\zeta_k) - u\zeta_k||_{L^1(\Omega)} < \frac{\epsilon}{2^k}$$

$$||\rho_{\epsilon_k} * (u\nabla\zeta_k) - u\nabla\zeta_k||_{L^1(\Omega;\mathbb{R}^N)} < \frac{\epsilon}{2^k}$$

and we define $u_\epsilon = \sum_{k=1}^{+\infty} \rho_{\epsilon_k} * (u\zeta_k)$.
Then $u_\epsilon \in C^\infty$, since locally there are only a finite number of nonzero terms in the

sum.
Also, $u_\epsilon \to u$ in $L^1(\Omega)$ since

$$||u - u_\epsilon||_{L^1(\Omega)} \leq \sum_{k=1}^{+\infty} ||\rho_{\epsilon_k} * (u\zeta_k) - u\zeta_k||_{L^1(\Omega)} < \epsilon.$$

Now, since $u_\epsilon \in L^1(\Omega)$, Theorem 1.3.2 implies $||Du||(\Omega) \leq \liminf_{\epsilon \to 0} ||Du_\epsilon||(\Omega)$.

In order to obtain the reverse inequality, let $\phi \in C_c^\infty(\Omega; \mathbb{R}^N)$, $||\phi||_\infty \leq 1$. Then

$$\begin{aligned}\int_\Omega u_\epsilon \text{div}\phi dx &= \sum_{k=1}^{+\infty} \int_\Omega \rho_{\epsilon_k} * (u\zeta_k)\text{div}\phi dx = \sum_{k=1}^{+\infty} \int_\Omega u\zeta_k \text{div}(\rho_{\epsilon_k} * \phi) dx \\ &= \sum_{k=1}^{+\infty} \int_\Omega u\text{div}(\zeta_k(\rho_{\epsilon_k} * \phi)) dx - \sum_{k=1}^{+\infty} \int_\Omega u\nabla\zeta_k \cdot (\rho_{\epsilon_k} * \phi) dx.\end{aligned}$$

Using $\sum_{k=1}^{+\infty} \nabla\zeta_k = 0$ in Ω and the properties of the convolution, this last expression equals

$$\sum_{k=1}^{+\infty} \int_\Omega u\text{div}(\zeta_k(\rho_{\epsilon_k} * \phi)) dx - \sum_{k=1}^{+\infty} \int_\Omega \phi \cdot (\rho_{\epsilon_k} * (u\nabla\zeta_k) - u\nabla\zeta_k) dx =: I_1^\epsilon + I_2^\epsilon$$

Now, $|\zeta_k(\rho_{\epsilon_k} * \phi)| \leq 1$ and each point in Ω belongs to at most three of the sets $\{\Sigma_k\}$. Thus

$$|I_1^\epsilon| \leq \left| \int_\Omega u\text{div}(\zeta_1(\rho_{\epsilon_1} * \phi)) dx \quad + \quad \sum_{k=2}^{+\infty} \int_\Omega u\text{div}(\zeta_k(\rho_{\epsilon_k} * \phi)) dx \right| \leq$$

$$||Du||(\Omega) + \sum_{k=2}^{+\infty} ||Du||(\Sigma_k) \leq ||Du||(\Omega) + 3||Du||(\Omega \setminus \Omega_1) \leq ||Du||(\Omega) + 3\epsilon$$

For the second term, we have $|I_2^\epsilon| < \epsilon$ directly from our choice of ϵ_k.
Therefore, after passing to the supremum over ϕ, $||Du_\epsilon||(\Omega) \leq ||Du||(\Omega) + 4\epsilon$, which yields $u_\epsilon \in BV(\Omega)$ and point 2. □

Remark 1.3.3. If $u \in BV(\mathbb{R}^N)$; that is, if Ω is the entire space $\mathbb{R}^N$, then the approximating sequence satisfying properties 1) and 2) of Theorem 1.3.4 is much easier to construct. Indeed, we need just to take $u_\epsilon = u * \rho_\epsilon$, where ρ is a standard symmetric mollifier.
Indeed, $u_\epsilon \to u$ in $L^1(\mathbb{R}^N)$ since $u \in L^1(\mathbb{R}^N)$.

Secondly, we observe that

$$||\nabla u_\epsilon||_{L^1(\mathbb{R}^N;\mathbb{R}^N)} = \sup\left\{\int_{\mathbb{R}^N} u_\epsilon(x)\mathrm{div}\phi(x)\,dx : \phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), ||\phi|| \leq 1\right\}$$
$$= \sup\left\{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} u(y)\rho_\epsilon(x-y)\mathrm{div}\phi(x)\,dxdy : \phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), ||\phi|| \leq 1\right\}$$
$$= \sup\left\{\int_{\mathbb{R}^N} u(y)\mathrm{div}\phi_\epsilon(y)\,dx : \phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N), ||\phi|| \leq 1\right\} \leq ||Du||(\mathbb{R}^N)$$

and so, by lower semicontinuity of the total variation, $||\nabla u_\epsilon||_{L^1(\mathbb{R}^N;\mathbb{R}^N)} \to ||Du||(\mathbb{R}^N)$. We may fix a sequence $\epsilon_k \to 0$. Theorem 1.3.2 implies that for any open set A $||Du||(A) \leq \liminf_{k\to+\infty} ||Du_{\epsilon_k}||(A)$ and we observe that for any compact set K and $\phi \in C_c^\infty(K;\mathbb{R}^N), ||\phi||_\infty \leq 1$ we have

$$\int_{\mathbb{R}^N} u_{\epsilon_k}(x)\mathrm{div}\phi(x)\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \mathrm{div}\phi(x)u(y)\rho_{\epsilon_k}(x-y)\,dydx$$
$$= \int_{\mathbb{R}^N} u(y)\mathrm{div}\phi_{\epsilon_k}(y)\,dy \leq ||Du||(K+\overline{B(0,\epsilon_k)})$$

since $\mathrm{supp}(\phi_{\epsilon_k}) \subset K + \overline{B(0,\epsilon_k)}$. Thus we can take the supremum over ϕ in order to obtain $||Du_{\epsilon_k}||(K) \leq ||Du||(K+\overline{B(0,\epsilon_k)})$, which implies $\limsup_{k\to+\infty} ||Du_{\epsilon_k}||(K) \leq ||Du||(K)$ since K is compact.
Hence the sequence of Radon measures $||\nabla u_{\epsilon_k}||$ satisfies point 2 of Lemma 1.1.2 and so we have point 1 of the same lemma; that is, $||Du_{\epsilon_k}|| \overset{*}{\rightharpoonup} ||Du||$ in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, since we have shown above that $\sup_k ||Du_{\epsilon_k}||(\mathbb{R}^N) \leq ||Du||(\mathbb{R}^N) < \infty$, Remark 1.1.4 yields also weak-star convergence in $\mathcal{M}(\mathbb{R}^N)$.

This remark applies also to BV functions with compact support inside Ω, since these are trivially in $BV(\mathbb{R}^N)$. Given $u \in BV(\Omega)$ with compact support, we can indeed extend it to

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

It is clear that $\hat{u} \in L^1(\mathbb{R}^N)$. If we let $\xi \in C_c^\infty(\Omega)$, $||\xi||_\infty \leq 1$ and $\xi = 1$ in a neighborhood of the support of u, then, for any $\phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$, $||\phi||_\infty \leq 1$, we have

$$\int_{\mathbb{R}^N} \hat{u}\mathrm{div}\phi\,dx = \int_\Omega u\mathrm{div}\phi\,dx = \int_\Omega u\mathrm{div}(\xi\phi + (1-\xi)\phi)\,dx$$
$$= \int_\Omega u\mathrm{div}(\xi\phi)\,dx \leq ||Du||(\Omega),$$

since $\xi\phi \in C_c^\infty(\Omega;\mathbb{R}^N)$ and $||\xi\phi||_\infty \leq 1$.
Taking the supremum over ϕ we obtain $||D\hat{u}||(\mathbb{R}^N) \leq ||Du||(\Omega) < \infty$.

1.4 Sets of finite perimeter

Definition 1.4.1. A measurable set $E \subset \Omega$ is called a *finite perimeter set* in Ω (or a Caccioppoli set) if $\chi_E \in BV(\Omega)$.
A measurable set $E \subset \mathbb{R}^N$ is said to have *locally finite perimeter* in Ω if $\chi_E \in BV_{\text{loc}}(\Omega)$.

Consequently, $D\chi_E$ is an $\mathbb{R}^N$-vector valued Radon measure on Ω whose total variation is $||D\chi_E||$.
By the polar decomposition of measures, there exists a $||D\chi_E||$-measurable function with modulus 1 $||D\chi_E||$-a.e., which we denote by ν_E, such that $D\chi_E = \nu_E||D\chi_E||$.
Unless otherwise stated, from now on E will be a set of locally finite perimeter in Ω.

Example 1.4.1. Any open bounded set $E \subset \Omega$ with $\partial E \in C^2$ is a set of finite perimeter in Ω.
Indeed, $\forall \phi \in C_c^\infty(\Omega; \mathbb{R}^N)$ with $||\phi||_\infty \leq 1$, by the classical Gauss-Green formula we have

$$\begin{aligned}\int_{\Omega\cap E} \text{div}\phi \, dx &= -\int_{\partial(\Omega\cap E)} \phi \cdot \nu_E \, d\mathcal{H}^{N-1} = -\int_{\Omega\cap\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1} \\ &\leq \int_{\Omega\cap\partial E} |\phi||\nu_E| \, d\mathcal{H}^{N-1} \leq \mathcal{H}^{N-1}(\Omega \cap \partial E),\end{aligned}$$

where ν_E is the interior unit normal. Taking the supremum over ϕ yields $||D\chi_E||(\Omega) \leq \mathcal{H}^{N-1}(\Omega \cap \partial E)$.
Therefore, E has finite perimeter and so, for any $\phi \in C_c^\infty(\Omega; \mathbb{R}^N)$,

$$\int_\Omega \chi_E \text{div}\phi \, dx = -\int_\Omega \phi \cdot D\chi_E = -\int_{\Omega\cap\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}.$$

This implies that $D\chi_E = \nu_E \, \mathcal{H}^{N-1} \llcorner \partial E$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$, by Riesz Representation Theorem (Theorem 1.3.1), and so $||D\chi_E|| = \mathcal{H}^{N-1} \llcorner \partial E$, which in particular yields

$$||D\chi_E||(\Omega) = \mathcal{H}^{N-1}(\Omega \cap \partial E). \tag{1.4.1}$$

Remark 1.4.1. It can be also shown that every open bounded set with Lipschitz boundary is a set of finite perimeter, with equality (1.4.1) holding, since this is a consequence of the extension theorem for functions of bounded variation (see [EG], Section 5.4). Moreover, any bounded open set Ω satisfying $\mathcal{H}^{N-1}(\partial\Omega) < \infty$ has finite perimeter in $\mathbb{R}^N$ (see [AFP], Proposition 3.62).

Equality (1.4.1) is not valid in general for sets of finite perimeter, as the following example will show.

Example 1.4.2. Let $N \geq 2$, $\{x_i\} = \mathbb{Q}^N \cap [0,1]^N$, $E = \bigcup_{i=0}^{\infty} B(x_i, \epsilon 2^{-i})$, with $\epsilon > 0$ that shall be assigned, and $[0,1]^N \subset \Omega$. We have

$$|E| \leq \sum_{i=0}^{\infty} \omega_N \epsilon^N 2^{-iN} = \frac{\omega_N \epsilon^N}{1 - 2^{-N}}.$$

Since the rational points are dense in $[0,1]^N$, then $\overline{E} = [0,1]^N$ and $\partial E = \overline{E} \setminus E$, since E is open, which implies

$$|\partial E| \geq |\overline{E}| - |E| \geq 1 - \frac{\omega_N \epsilon^N}{1 - 2^{-N}} > 0$$

for ϵ small enough. This implies $\mathcal{H}^{N-1}(\partial E) = \infty$.
Observing that $\partial E \subset \bigcup_{i=0}^{\infty} \partial B(x_i, \epsilon 2^{-i})$, we have

$$\begin{aligned}\int_{\Omega \cap E} \mathrm{div}\phi \, dx = -\int_{\partial E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1} &\leq \sum_{i=0}^{\infty} \int_{\partial B(x_i, \epsilon 2^{-i})} |\phi||\nu_E| \, d\mathcal{H}^{N-1} \\ &\leq \sum_{i=0}^{\infty} \mathcal{H}^{N-1}(\partial B(x_i, \epsilon 2^{-i})) = \sum_{i=0}^{\infty} N \omega_N \epsilon^{N-1} 2^{-(N-1)i} = \frac{N \omega_N \epsilon^{N-1}}{1 - 2^{-(N-1)}}.\end{aligned}$$

Thus E is a set of finite perimeter for which $||D\chi_E||(\Omega) \neq \mathcal{H}^{N-1}(\Omega \cap \partial E)$.

This may suggest that for a set of finite perimeter is interesting to consider not the whole topological boundary, but subsets of ∂E instead.

Definition 1.4.2. Let $x \in \Omega$, then $x \in \partial^* E$, the *reduced boundary* of E, if

1. $||D\chi_E||(B(x,r)) > 0, \forall r > 0$;
2. $\lim_{r \to 0} \frac{1}{||D\chi_E||(B(x,r))} \int_{B(x,r)} \nu_E d||D\chi_E|| = \nu_E(x)$;
3. $|\nu_E(x)| = 1$.

It can be shown that this definition implies a geometrical characterisation of the reduced boundary, by using the blow-up of the set E around a point of $\partial^* E$.

Definition 1.4.3. For $x \in \partial^* E$ we define the hyperplane

$$H(x) = \{y \in \mathbb{R}^N : \ \nu(x) \cdot (y - x) = 0\}$$

and the half-spaces

$$H^+(x) = \{y \in \mathbb{R}^N : \ \nu(x) \cdot (y - x) \geq 0\},$$
$$H^-(x) = \{y \in \mathbb{R}^N : \ \nu(x) \cdot (y - x) \leq 0\}.$$

Also, for $r > 0$, we set

$$E_r(x) = \{y \in \mathbb{R}^N : \ x + r(y - x) \in E\}$$

Theorem 1.4.1. *If E is a set of finite perimeter in Ω, $x \in \partial^* E$ and $\nu(x) = -\nu_E(x)$, then*

$$\begin{aligned} \chi_{E_r} &\to \chi_{H^-(x)} \text{ in } L^1_{loc}(\Omega) \\ \chi_{\Omega \setminus E_r} &\to \chi_{H^-(x)} \text{ in } L^1_{loc}(\Omega) \end{aligned}$$

as $r \to 0$.

Proof. See [EG] Section 5.7.2 Theorem 1.

Formulated in another way, for $r > 0$ small enough, $E \cap B(x,r)$ is approximatively equal to the half ball $H^-(x) \cap B(x,r)$.

Corollary 1.4.1. *If $x \in \partial^* E$ and $\nu(x) = -\nu_E(x)$, then*

1. $\lim\limits_{r\to 0} \frac{1}{r^N} |B(x,r) \cap E \cap H^+(x)| = 0,$
2. $\lim\limits_{r\to 0} \frac{1}{r^N} |(B(x,r) \setminus E) \cap H^-(x)| = 0.$

Proof. We have

$$\frac{1}{r^N}|B(x,r)\cap E \cap H^+(x)| = |B(x,1)\cap E_r \cap H^+(x)| \to |B(x,1)\cap H^-(x)\cap H^+(x)| = 0,$$

by Theorem 1.4.1. Point 2 follows from the same theorem and

$$\begin{aligned} \frac{1}{r^N}|(B(x,r) \setminus E) \cap H^-(x)| &= \frac{1}{r^N}(|B(x,r) \cap H^-(x)| - |B(x,r) \cap E \cap H^-(x)|) \\ &= \frac{\omega_N}{2} - |B(x,1) \cap E_r \cap H^-(x)| \\ &\to \frac{\omega_N}{2} - |B(x,1) \cap H^-(x)| = 0. \end{aligned}$$

□

Using this result, we can give a generalization of the concept of unit interior normal (respectively, unit exterior normal, up to a sign).

Definition 1.4.4. A unit vector $\nu(x) = -\nu_E(x)$ for which property 1 of Corollary 1.4.1 holds is called the *measure theoretic unit exterior normal* to E at x, while, accordingly, $\nu_E(x)$ is called the *measure theoretic unit interior normal* to E at x.

It follows that the measure theoretic unit interior normal ν_E is well defined at least on the reduced boundary.
Moreover, the next theorem shows us that the reduced boundary can be written as a countable union of compact subset of C^1 manifolds, up to a set of Hausdorff dimension at most $N-1$.

Theorem 1.4.2. *Assume E is a set of locally finite perimeter in $\mathbb{R}^N$. Then*

1. *$\partial^* E$ is a $(N-1)$-rectifiable set; that is, there exist a countable family of C^1 manifolds S_k, a family of compact sets $K_k \subset S_k$ and set $\mathcal{N}$ of $\mathcal{H}^{N-1}$-measure zero such that*

$$\partial^* E = \bigcup_{k=1}^{+\infty} K_k \cup \mathcal{N};$$

2. *$\nu_E|_{S_k}$ is normal to S_k;*

3. *$||D\chi_E|| = \mathcal{H}^{N-1} \llcorner \partial^* E$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$,*

$$\lim_{r \to 0} \frac{||D\chi_E||(B(x,r))}{\omega_{N-1} r^{N-1}} = 1.$$

Proof. See [EG] Section 5.7.3 Theorem 2.

We introduce now the density of a set at a certain point, in order to select another useful subset of the topological boundary.

Definition 1.4.5. For every $\alpha \in [0,1]$ and every measurable set $E \subset \mathbb{R}^N$, we define

$$E^\alpha := \{x \in \mathbb{R}^N : D(E,x) = \alpha\},$$

where

$$D(E,x) := \lim_{r \to 0} \frac{|(B(x,r) \cap E)|}{|B(x,r)|}.$$

Definition 1.4.6. Referring to Definition 1.4.5,

1. E^1 is called the *measure theoretic interior* of E.
2. E^0 is called the *measure theoretic exterior* of E.
3. The *measure theoretic (or essential) boundary* of E is the set $\partial^m E := \mathbb{R}^N \setminus (E^0 \cup E^1)$.

Remark 1.4.2. It is clear that $E^\circ \subset E^1$ and $\mathbb{R}^N \setminus \overline{E} \subset E^0$. Hence one has

$$\partial^m E \subset \mathbb{R}^N \setminus (E^\circ \cup \mathbb{R}^N \setminus \overline{E}) = \overline{E} \setminus E^\circ = \partial E.$$

Moreover, by the Lebesgue-Besicovitch differentiation theorem (Theorem 1.1.2), $\partial^m E$ has $\mathcal{L}^N$-measure 0, since it is the set of non-Lebesgue points of χ_E.

We further observe that, as in [EG] Section 5.8, it is possibile to define the measure theoretic boundary without using the density of a set.
Indeed the previous definition is equivalent to the following:

Definition 1.4.6' Let $x \in \mathbb{R}^N$, then $x \in \partial^m E$, the meaure theoretic boundary of E, if the following two conditions hold:

1. $\limsup_{r\to 0} \frac{|B(x,r)\cap E|}{r^N} > 0$,
2. $\limsup_{r\to 0} \frac{|B(x,r)\setminus E|}{r^N} > 0$.

Theorem 1.4.3. *If $E \subset \Omega$ is a set of finite perimeter, then*

$$\partial^* E \subset E^{\frac{1}{2}} \subset \partial^m E, \quad \mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^* E \cup E^1) = 0.$$

In particular, E has density either 0, $\frac{1}{2}$ or 1 at $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$, and, even if E is only locally of finite perimeter, $\mathcal{H}^{N-1}$-a.e. $x \in \partial^m E$ belongs to $\partial^* E$; that is, $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$.

Proof. See [EG] Section 5.8 Lemma 1 and [AFP] Theorem 3.61.

Remark 1.4.3. Since the functions of bounded variations are elements of L^1, they are equivalence class of functions, so that changing the value of any such function on a set of $\mathcal{L}^N$-measure zero does not modify the BV class of the function.
Therefore, this is true also for sets of finite perimeter and we can choose any representative $\widetilde{E}$ for E, which differs only by a set of measure zero, without altering the measure theoretic boundary.
Throughout, we choose this preferred representative for E:

$$E := E^1 \cup \partial^m E.$$

One of the greatest achievements of BV theory is to establish a generalization of the Gauss-Green formula for every set of finite perimeter, though only for differentiable vector fields.

Theorem 1.4.4. (Gauss-Green formula on sets of finite perimeter)
Let $E \subset \mathbb{R}^N$ be a set of locally finite perimeter. Then for $\mathcal{H}^{N-1}$ a.e. $x \in \partial^m E$, there is a unique measure theoretic interior unit normal $\nu_E(x)$ such that $\forall \phi \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$ one has

$$\int_E \text{div}\phi \, dx = -\int_{\partial^m E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}.$$

Proof. Since E is a set of locally finite perimeter, $D\chi_E = \nu_E \mathcal{H}^{N-1} \llcorner \partial^* E$ (Theorem 1.4.2), where ν_E is the measure theoretic interior unit normal. Also, Theorem 1.4.3 implies $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$. Hence, for any $\phi \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$,

$$\int_\Omega \chi_E \text{div}\phi \, dx = -\int_\Omega \phi \cdot D\chi_E = -\int_{\partial^m E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}. \ \square$$

Remark 1.4.4. Since $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$ (Theorem 1.4.3), without change, we can integrate on the measure theoretic or on the reduced boundary with respect to the measure $\mathcal{H}^{N-1}$. Since in many practical cases $\partial^m E$ is easier to be determined, Theorem 1.4.4 is often stated in this way. However, since Theorem 1.4.2 states that $||D\chi_E|| \ll \mathcal{H}^{N-1} \llcorner \partial^* E$ and the precise representative of χ_E is well defined on $E^1 \cup \partial^* E \cup E^0$ (Lemma 1.4.1 below), in what follows we will always use the reduced boundary in the Gauss-Green formula.

Remark 1.4.5. We also observe that if E is a bounded set of finite perimeter in $\mathbb{R}^N$, then we can drop the assumption on the support of ϕ. Indeed, there exists $R > 0$ such that $\overline{E} \subset B(0, R)$, and so, given $\phi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$, we can take $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi = 1$ in $\overline{B(0, R)}$ (which in particular implies $\nabla\varphi = 0$ in E), in order to obtain

$$\begin{aligned}\int_E \text{div}\phi \, dx &= \int_E (\varphi \text{div}\phi + \phi \cdot \nabla\varphi) \, dx = \int_E \text{div}(\phi\varphi) \, dx \\ &= -\int_{\partial^* E} (\phi\varphi) \cdot \nu_E \, d\mathcal{H}^{N-1} = -\int_{\partial^* E} \phi \cdot \nu_E \, d\mathcal{H}^{N-1}.\end{aligned}$$

It is also easy to see that if $E \subset\subset \Omega \subset \mathbb{R}^N$, then we can take just $\phi \in C^1(\Omega; \mathbb{R}^N)$.

As in the case of Sobolev functions, it can be shown that for BV functions the precise representative is well defined and it is the limit of the mollified sequence.

Definition 1.4.7. Let $u \in L^1_{\text{loc}}(\Omega)$ and $a \in \mathbb{R}^N$.
We say that $u_a(x)$ is the approximate limit of u at $x \in \Omega$ restricted to $\Pi_a(x) := \{y \in \mathbb{R}^N : (y - x) \cdot a \geq 0\}$ if, for any $\epsilon > 0$,

$$\lim_{r \to 0} \frac{|\{y \in \mathbb{R}^N : |u(y) - u_a(x)| \geq \epsilon\} \cap B(x, r) \cap \Pi_a(x)|}{|B(x, r) \cap \Pi_a(x)|} = 0$$

Definition 1.4.8. We say that $x \in \Omega$ is a *regular point* of a function $u \in BV(\Omega)$ if there exists a vector $a \in \mathbb{R}^N$ such that the approximate limits $u_a(x)$ and $u_{-a}(x)$ exist. The vector a is called *defining vector*.

Example 1.4.3. Let E be a set of finite perimeter, for which we choose the representative $E^1 \cup \partial^m E$ (see Remark 1.4.3), and $u = \chi_E$, then each point in $E^1 \cup E^0 \cup \partial^* E$ is a regular point.
If $x \in E^1$, $\forall a \in \mathbb{R}^N$ $(\chi_E)_a(x) = 1$. $\forall \epsilon > 0$ we have

$$\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x, r) = E^0 \cap B(x, r).$$

So,

$$\begin{aligned}\lim_{r \to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x, r)|}{|B(x, r)|} &= \lim_{r \to 0} \frac{|E^0 \cap B(x, r)|}{|B(x, r)|} \\ &= 1 - D(E, x) = 0.\end{aligned}$$

Therefore, $\forall a \in \mathbb{R}^N$

$$\begin{aligned}
&\frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} \\
&\leq \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} \frac{|B(x,r)|}{|B(x,r) \cap \Pi_a(x)|} \\
&= 2\frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} \to 0 \quad \text{as} \quad r \to 0.
\end{aligned}$$

In an analogous way, we show that $\forall x \in E^0$ $(\chi_E)_a(x) = 0$ $\forall a \in \mathbb{R}^N$. $\forall \epsilon > 0$ we have

$$\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r) = E \cap B(x,r)$$

and so

$$\begin{aligned}
\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} &= \lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} \\
&= D(E,x) = 0.
\end{aligned}$$

Therefore, $\forall a \in \mathbb{R}^N$

$$\begin{aligned}
&\frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} \\
&\leq \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} \frac{|B(x,r)|}{|B(x,r) \cap \Pi_a(x)|} \\
&= 2\frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r)|}{|B(x,r)|} \to 0 \quad \text{as} \quad r \to 0.
\end{aligned}$$

Now let $x \in \partial^* E$ and a be the measure theoretic interior normal. Then $(\chi_E)_a(x) = 1$ and $(\chi_E)_{-a}(x) = 0$.
Referring to the notation of Corollary 1.4.1, we have $\Pi_a(x) = H^-(x)$ and $\Pi_{-a}(x) = H^+(x)$, hence $\forall \epsilon > 0$

$$\begin{aligned}
&\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y) - 1| \geq \epsilon\} \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} \\
&= \lim_{r\to 0} \frac{|E^0 \cap B(x,r) \cap \Pi_a(x)|}{|B(x,r) \cap \Pi_a(x)|} = \lim_{r\to 0} \frac{2}{\omega_N r^N} |(B(x,r) \setminus E) \cap H^-(x)| = 0
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{r\to 0} \frac{|\{y \in \mathbb{R}^N : |\chi_E(y)| \geq \epsilon\} \cap B(x,r) \cap \Pi_{-a}(x)|}{|B(x,r) \cap \Pi_{-a}(x)|} \\
&= \lim_{r\to 0} \frac{|E \cap B(x,r) \cap \Pi_{-a}(x)|}{|B(x,r) \cap \Pi_{-a}(x)|} = \lim_{r\to 0} \frac{2}{\omega_N r^N} |B(x,r) \cap E \cap H^+(x)| = 0.
\end{aligned}$$

This shows our claim.

Theorem 1.4.5. *Let $u \in BV(\Omega)$. The set of irregular points has $\mathcal{H}^{N-1}$-measure zero.*

Proof. See [VH] Chapter 4 §5.5, or [EG] Section 5.9 Theorem 3.

Theorem 1.4.6. *Let $u \in BV(\Omega)$ and x be a regular point of u. Then*

1. *If $u_a(x) = u_{-a}(x)$, any $b \in \mathbb{R}^N$ is a defining vector and $u_b(x) = u_a(x)$; that is, x is a point of approximate continuity.*
2. *If $u_a(x) \neq u_{-a}(x)$, then a is unique up to a sign.*
3. *The mollification of u converges to the precise representative u^* at each regular point and $u^*(x) = \frac{1}{2}(u_a(x) + u_{-a}(x))$.*

Proof. See [VH] Chapter 4 §4.4 and Chapter 4 §5.6 Theorem 1, or [EG] Section 5.9 Corollary 1.

Remark 1.4.6. By Theorem 1.1.4, we deduce that condition 1) in Theorem 1.4.6 holds $\mathcal{L}^N$-a.e.

We state now some standard results on the mollification of characteristic functions of sets of finite perimeter.

Remark 1.4.7. By Remark 1.3.3, if E be a set of finite perimeter and $\{\chi_{\delta_k}\}$ denotes the mollification of χ_E, then

$$||\nabla \chi_{\delta_k}||_{L^1(\mathbb{R}^N)} \leq ||D\chi_E||(\mathbb{R}^N)$$

and

$$||\nabla \chi_{\delta_k}||_{L^1(\mathbb{R}^N)} \to ||D\chi_E||(\mathbb{R}^N)$$

Lemma 1.4.1. *If χ_δ is the mollification of χ_E for a set of finite perimeter E, then the following hold:*

1. $\chi_\delta \in C^\infty(\mathbb{R}^N)$;
2. *There is a set $\mathcal{N}$ with $\mathcal{H}^{N-1}(\mathcal{N}) = 0$ such that, for all $x \notin \mathcal{N}$, $\chi_\delta(x) \to \chi_E^*(x)$ and*
$$\chi_E^*(x) = \begin{cases} 1 & \text{if } x \in E^1 \\ \frac{1}{2} & \text{if } x \in \partial^* E \\ 0 & \text{if } x \in E^0 \end{cases} \quad ;$$
3. $\nabla \chi_\delta \overset{*}{\rightharpoonup} D\chi_E^*$ *in* $\mathcal{M}(\mathbb{R}^N; \mathbb{R}^N)$;
4. $D\chi_E = D\chi_E^*$.
5. *If U is an open bounded set with $||D\chi_E||(\partial U) = 0$, then $||\nabla \chi_\delta||(U) \to ||D\chi_E||(U)$;*

Proof.

1. It follows from Theorem 1.2.1.

2. From Theorems 1.4.5 and 1.4.6, we know $\chi_\delta \to \chi_E^* \ \mathcal{H}^{N-1}$-a.e. and $\chi_E^*(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} \chi_E(y)dy = D(E,x)$, so if $x \in E^1$, $\chi_E^*(x) = 1$; if $x \in E^0$ then $\chi_E^*(x) = 0$ and, since $\partial^* E \subset E^{\frac{1}{2}}$, if $x \in \partial^* E$ then $\chi_E^*(x) = \frac{1}{2}$.

3. Since $\chi_\delta \to \chi_E^*$ in $L^1(\mathbb{R}^N)$, then
$$\int_{\mathbb{R}^N} \chi_\delta \psi \, dx \to \int_{\mathbb{R}^N} \chi_E^* \psi \, dx$$
for each $\psi \in C_c(\mathbb{R}^N)$; that is, they converge as distributions, which implies $\nabla \chi_\delta \overset{*}{\rightharpoonup} D\chi_E^*$ as distributions:
$$\int_{\mathbb{R}^N} \nabla \chi_\delta \cdot \phi \, dx = -\int_{\mathbb{R}^N} \chi_\delta \mathrm{div}\phi \, dx \to -\int_{\mathbb{R}^N} \chi_E^* \mathrm{div}\phi \, dx = \int_{\mathbb{R}^N} \phi \cdot dD\chi_E^*$$
for each $\phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$. Consequently they converge as $\mathbb{R}^N$-vector valued Radon measures, by the density of $C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$ in $C_c(\mathbb{R}^N;\mathbb{R}^N)$ with respect to the sup norm and by the uniform boundedness of total variation (see Remark 1.4.7).

4. It is immediate from the fact that $\chi_E^* = \chi_E \ \mathcal{L}^N$-a.e. and the definition of derivative of a BV function.

5. Let $\{\delta_k\}$ be a nonnegative sequence converging to 0. Since χ_E has compact support in Ω, Remark 1.3.3 yields that the sequence of Radon measures $||\nabla \chi_{\delta_k}||$ converges weak-star to $||D\chi_E||$ in $\mathcal{M}(\mathbb{R}^N)$ and thus Lemma 1.1.2 implies our assertion. □

We state now the co-area formula, which shows an important connection between BV functions and sets of finite perimeter.

Theorem 1.4.7. (Federer and Fleming co-area fromula)
If $u \in BV(\Omega)$, then for $\mathcal{L}^1$ a.e. $s \in \mathbb{R}$, the set $\{u > s\}$ has finite perimeter in Ω and
$$||Du||(\Omega) = \int_{-\infty}^{+\infty} ||D\chi_{\{u>s\}}||(\Omega) ds.$$

Conversely, if $u \in L^1(\Omega)$ and $\int_{-\infty}^{+\infty} ||D\chi_{\{u>s\}}||(\Omega)ds < \infty$, then $u \in BV(\Omega)$. Moreover, for any Borel set $B \subset \Omega$ we have
$$||Du||(B) = \int_{-\infty}^{+\infty} ||D\chi_{\{u>s\}}||(B) ds.$$

Proof. See [EG] Section 5.5 Theorem 1 and [AFP] Theorem 3.40.

Remark 1.4.8. A consequence of Theorem 1.4.7 is that, for any $u \in BV(\Omega)$, $||Du|| \ll \mathcal{H}^{N-1}$. Indeed, for any Borel set $B \subset \Omega$ such that $\mathcal{H}^{N-1}(B) = 0$, co-area formula implies $||Du||(B) = 0$.

Lemma 1.4.2. *Let $u : \Omega \to \mathbb{R}$ be a Lipschitz function, and let $A \subset \mathbb{R}^N$ be a set of measure zero.*
Then

$$\mathcal{H}^{N-1}(A \cap u^{-1}(s)) = 0 \quad for \quad \mathcal{L}^1\text{-}a.e.\, s \in \mathbb{R}.$$

Proof. It is an immediate consequence of the classical co-area formula for Lipschitz functions (see [EG], Section 3.4.2 Theorem 1); that is,

$$0 = \int_A |\nabla u(x)| dx = \int_{-\infty}^{+\infty} \mathcal{H}^{N-1}(A \cap u^{-1}(s)) ds. \square$$

1.5 Generalizations of the Gauss-Green formula

In this paragraph, we formulate three extensions of the Gauss-Green formula for sets of finite perimeter and fields with lower regularity, in order to compare them with those we will provide in the following chapters.
The first one is about Lipschitz fields.

Theorem 1.5.1. (De Giorgi and Federer) *If E is a bounded set of finite perimeter in $\mathbb{R}^N$ and $F : \mathbb{R}^N \to \mathbb{R}^N$ is locally Lipschitz, then*

$$\int_E \text{div} F \, dx = -\int_{\partial^* E} F \cdot \nu_E \, d\mathcal{H}^{N-1}, \tag{1.5.1}$$

where ν_E is the measure theoretic interior normal to E.

Proof. Let $F_\epsilon = F * \rho_\epsilon$ be a mollification of F, then, by Remark 1.4.5, we have

$$\int_E \text{div} F_\epsilon \, dx = -\int_{\partial^* E} F_\epsilon \cdot \nu_E \, d\mathcal{H}^{N-1}.$$

Since $F_\epsilon \to F$ uniformly on compact sets, by the continuity of F and Theorem 1.2.1, and $\partial^* E$ is bounded and has finite $\mathcal{H}^{N-1}$-measure, then we can apply Lebesgue's dominated convergence theorem to the right hand side and obtain

$$\lim_{\epsilon \to 0} \int_{\partial^* E} F_\epsilon \cdot \nu_E \, d\mathcal{H}^{N-1} = \int_{\partial^* E} F \cdot \nu_E \, d\mathcal{H}^{N-1}.$$

On the other hand, we can find R large enough such that $E \subset\subset B(0, R)$ and, clearly, F is Lipschitz continuous on $\overline{B(0, R)}$: hence, $F \in W^{1,\infty}(B(0, R), \mathbb{R}^N)$ (see [E], Section 5.8.2, Theorem 4) and, in particular, $F \in W^{1,1}(B(0, R), \mathbb{R}^N)$.
By Theorem 1.2.1, we have $F_\epsilon \to F$ in $W^{1,1}_{\text{loc}}(B(0, R), \mathbb{R}^N)$ and this yields

$\text{div}F_\epsilon \to \text{div}F$ in $L^1(E)$. Thus we obtain (1.5.1). □

A Gauss-Green formula for essentially bounded vector fields of bounded variations on bounded sets of finite perimeter was found by Vol'pert in the '60s ([VH]). The first ingredient is the product rule for essentially bounded BV functions, then we need to show that, broadly speaking, the gradient of compactly supported BV functions has zero mean value, as it happens for C^1_c functions.

Proposition 1.5.1. *Let $u, v \in BV(\Omega) \cap L^\infty(\Omega)$, then $uv \in BV(\Omega)$ and for each $i = 1, ..., N$*

$$D_i(uv) = u^* D_i v + v^* D_i u \,, \tag{1.5.2}$$

in the sense of Radon measures, where u^* is the precise representative of u (Definition 1.1.6).
In particular, for any set of finite perimeter $E \subset\subset \Omega$,

$$\chi_{\partial^* E} Du = (u_{\nu_E} - u_{-\nu_E}) D\chi_E \tag{1.5.3}$$

and

$$D(u\chi_E) = u_{\nu_E} D\chi_E + \chi_{E^1} Du, \tag{1.5.4}$$
$$D(u\chi_E) = u_{-\nu_E} D\chi_E + \chi_{E^1 \cup \partial^* E} Du, \tag{1.5.5}$$

where ν_E is the measure theoretic interior normal to E and $u_{\pm\nu_E}$ are the approximate limit of u restriced to $\Pi_{\pm\nu_E}$ (see Definition 1.4.7).

Proof. See [VH] Ch.4 §6.4. and Ch.5 §1.3. We also give a sketch of the proof of (1.5.4).
If we apply (1.5.2) to $v = \chi_E$, we obtain

$$D(u\chi_E) = u^* D\chi_E + \chi_E^* Du. \tag{1.5.6}$$

We observe that $D(u\chi_E) = D(u\chi_E^2)$: indeed, for any $\phi \in C^1_c(\Omega; \mathbb{R}^N)$, we have

$$\int_\Omega \phi \cdot dD(u\chi_E) = -\int_\Omega u\chi_E \text{div}\phi \, dx = -\int_\Omega u\chi_E^2 \text{div}\phi \, dx = \int_\Omega \phi \cdot dD(u\chi_E^2),$$

and the density of $C^1_c(\Omega; \mathbb{R}^N)$ in $C_c(\Omega; \mathbb{R}^N)$ yields the desired equality. Therefore we deduce

$$\begin{aligned} D(u\chi_E) &= u^* D\chi_E + \chi_E^* Du = D(u\chi_E^2) \\ &= (u\chi_E)^* D\chi_E + \chi_E^* D(u\chi_E) = (u\chi_E)^* D\chi_E + (\chi_E^*)^2 Du + \chi_E^* u^* D\chi_E. \end{aligned}$$

It is possible to show that $(u\chi_E)^*(x) = \frac{1}{2} u_{\nu_E}(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^m E$ (see [VH], Ch.5 §1.2 Theorem 1) and that the defining vector of u on $\partial^* E$ is the measure theoretic interior normal to E, ν_E. Moreover, since $\text{supp}(D\chi_E) \subset \partial^* E$, $\chi_E^* =$

$\chi_{E^1} + \frac{1}{2}\chi_{\partial^* E}$ $\mathcal{H}^{N-1}$-a.e. (Lemma 1.4.1 and Theorem 1.4.3) and $||Du|| \ll \mathcal{H}^{N-1}$ (Remark 1.4.8), we have

$$\begin{aligned}\frac{1}{2}u_{\nu_E}D\chi_E + (\chi_{E^1} + \frac{1}{4}\chi_{\partial^* E})Du + &\frac{1}{4}(u_{\nu_E} + u_{-\nu_E})D\chi_E \\ &= \frac{1}{2}(u_{\nu_E} + u_{-\nu_E})D\chi_E + (\chi_{E^1} + \frac{1}{2}\chi_{\partial^* E})Du,\end{aligned}$$

which implies

$$\frac{1}{4}(u_{\nu_E} - u_{-\nu_E})D\chi_E - \frac{1}{4}\chi_{\partial^* E}Du = 0; \tag{1.5.7}$$

that is, (1.5.3). Now, if we add twice (1.5.7) to (1.5.6) we get

$$\begin{aligned}D(u\chi_E) &= \frac{1}{2}(u_{\nu_E} + u_{-\nu_E})D\chi_E + (\chi_{E^1} + \frac{1}{2}\chi_{\partial^* E})Du - \frac{1}{2}\chi_{\partial^* E}Du + \frac{1}{2}(u_{\nu_E} - u_{-\nu_E})D\chi_E \\ &= u_{\nu_E}D\chi_E + \chi_{E^1}Du,\end{aligned}$$

which is (1.5.4). To obtain (1.5.5), we subtract twice equation (1.5.7) from (1.5.4) instead:

$$\begin{aligned}D(u\chi_E) &= \frac{1}{2}(u_{\nu_E} + u_{-\nu_E})D\chi_E + (\chi_{E^1} + \frac{1}{2}\chi_{\partial^* E})Du + \frac{1}{2}\chi_{\partial^* E}Du - \frac{1}{2}(u_{\nu_E} - u_{-\nu_E})D\chi_E \\ &= u_{-\nu_E}D\chi_E + \chi_{E^1\cup\partial^* E}Du.\end{aligned}$$

This ends the proof. □

Lemma 1.5.1. *If $u \in BV(\Omega)$ and has compact support, then*

$$\int_\Omega dDu = Du(\Omega) = 0.$$

Proof. Since u has compact support, we can extend it to

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

and, by Remark 1.3.3, $\hat{u} \in BV(\mathbb{R}^N)$. With a little abuse of notation, we will denote this extension again by u.

So, $u = 0$ on $\mathbb{R}^N \setminus \Omega$. In particular, this implies that $||Du||(A) = 0$ for each open set $A \subset \mathbb{R}^N \setminus \Omega$: indeed

$$0 = \int_{\mathbb{R}^N} u\mathrm{div}\phi\, dx = -\int_{\mathbb{R}^N} \phi \cdot dDu \quad \forall \phi \in C_c^\infty(A; \mathbb{R}^N)$$

and $||Du||(A)$ is the supremum of these integrals over $\phi \in C_c^\infty(A;\mathbb{R}^N)$ with $||\phi||_\infty \leq 1$, by Proposition 1.1.2. By the properties of positive Radon measures (Proposition

1.1.1), this implies $||Du||(B) = 0$ for any Borel set $B \subset \mathbb{R}^N \setminus \Omega$.
We set $\Omega_k := \{x \in \mathbb{R}^N : k > \text{dist}(x, \overline{\Omega}) \geq k-1\}$ for $k \geq 2$ and $\Omega_1 := \{x \in \mathbb{R}^N : 1 > \text{dist}(x, \overline{\Omega}) \geq 0\} \setminus \Omega$. Then, $||Du||(\mathbb{R}^N \setminus \Omega) = 0$ since $\mathbb{R}^N \setminus \Omega = \bigcup_{k=1}^{+\infty} \Omega_k$ and each one of these sets has $||Du||$-measure zero.
Now let $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi = 1$ on $\Omega_1 \cup \Omega$. Then it is clear that

$$\int_{\mathbb{R}^N} \varphi \, dDu = \int_{\mathbb{R}^N} dDu$$

and, by the definition of the distributional derivative,

$$\int_{\mathbb{R}^N} \varphi \, dDu = -\int_{\mathbb{R}^N} \nabla\varphi \, u \, dx = -\int_{\mathbb{R}^N \setminus (\Omega_1 \cup \Omega)} \nabla\varphi \, u \, dx = 0$$

since u has support inside Ω, thus $Du(\mathbb{R}^N) = 0$, which implies $Du(\Omega) = 0$. $\square$

Now we need only to exploit the product rule for the function $\chi_E u$ in order to obtain the following version of the Gauss-Green formula.

Theorem 1.5.2. (Integration by Parts and Gauss-Green Formula for BV Fields)
Let $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $E \subset\subset \Omega$ be a set of finite perimeter. Then $u_{\pm\nu_E} \in L^\infty(\partial^* E; \mathcal{H}^{N-1})$ with the estimates

$$||u_{\nu_E}||_{L^\infty(\partial^* E; \mathcal{H}^{N-1})} \leq ||u||_{L^\infty(E)},$$
$$||u_{-\nu_E}||_{L^\infty(\partial^* E; \mathcal{H}^{N-1})} \leq ||u||_{L^\infty(\Omega \setminus E)}.$$

In addition, we have

$$\int_{E^1} dDu = Du(E^1) = -\int_{\partial^* E} u_{\nu_E} \, \nu_E \, d\mathcal{H}^{N-1}, \tag{1.5.8}$$
$$\int_{E} dDu = Du(E) = -\int_{\partial^* E} u_{-\nu_E} \, \nu_E \, d\mathcal{H}^{N-1}, \tag{1.5.9}$$
$$\int_{B} dDu = Du(B) = \int_{B} (u_{\nu_E} - u_{-\nu_E}) \, \nu_E \, d\mathcal{H}^{N-1} \tag{1.5.10}$$

for any Borel set $B \subset \partial^* E$.
If $u \in BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$,

$$\int_{E^1} d\,\text{div}(u) = \text{div}u(E^1) = -\int_{\partial^* E} u_{\nu_E} \cdot \nu_E \, d\mathcal{H}^{N-1}, \tag{1.5.11}$$
$$\int_{E} d\,\text{div}(u) = \text{div}u(E) = -\int_{\partial^* E} u_{-\nu_E} \cdot \nu_E \, d\mathcal{H}^{N-1}, \tag{1.5.12}$$
$$\int_{B} d\,\text{div}u = \text{div}u(B) = \int_{B} (u_{\nu_E} - u_{-\nu_E}) \cdot \nu_E \, d\mathcal{H}^{N-1} \tag{1.5.13}$$

for any Borel set $B \subset \partial^* E$.

Proof. For $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$ we have that $(u\chi_E)^*(x) = \frac{1}{2} u_{\nu_E}(x)$ (see the proof of Proposition 1.5.1) and that the precise representative is the limit of the mollification (Theorems 1.4.5 and 1.4.6). Hence, it follows that

$$|u_{\nu_E}(x)| \leq 2 \lim_{\epsilon \to 0} |(u\chi_E) * \rho_\epsilon(x)| \leq 2||u||_{L^\infty(E)} \lim_{\epsilon \to 0} \int_E \rho_\epsilon(x-y)\, dy$$
$$= 2||u||_{L^\infty(E)} \lim_{\epsilon \to 0} \int_{(E-x)/\epsilon} \rho(z)\, dz = ||u||_{L^\infty(E)}$$

by Theorem (1.4.1), for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$.
Then, we notice that $u^*(x) = \frac{1}{2}(u_{\nu_E}(x) + u_{-\nu_E}(x))$, and so $(u\chi_{\Omega \setminus E})^*(x) = \frac{1}{2} u_{-\nu_E}(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$, which implies

$$|u_{-\nu_E}(x)| \leq 2 \lim_{\epsilon \to 0} |(u\chi_{\Omega \setminus E}) * \rho_\epsilon(x)| \leq 2||u||_{L^\infty(\Omega \setminus E)} \lim_{\epsilon \to 0} \int_{\Omega \setminus E} \rho_\epsilon(x-y)\, dy$$
$$= 2||u||_{L^\infty(\Omega \setminus E)} \lim_{\epsilon \to 0} \int_{(\Omega \setminus E - x)/\epsilon} \rho(z)\, dz = ||u||_{L^\infty(\Omega \setminus E)}$$

by Theorem (1.4.1), for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$.
Equations (1.5.8) and (1.5.11) follow from Proposition 1.5.1 and Lemma 1.5.1. Indeed, $u\chi_E \in BV(\Omega)$ and has compact support, so

$$\int_\Omega dD(u\chi_E) = 0$$

which implies

$$\int_\Omega u_{\nu_E} dD\chi_E + \chi_{E^1} dDu = 0.$$

$D\chi_E = \nu_E \mathcal{H}^{N-1} \llcorner \partial^* E$ by Theorem 1.4.2, and, in particular, $u_{\nu_E} \in L^1(\partial^* E; \mathcal{H}^{N-1})$; thus we have

$$\int_{E^1} dDu = -\int_{\partial^* E} u_{\nu_E} \nu_E \, d\mathcal{H}^{N-1},$$

which is an Integration by Parts formula. In a similar way, we deduce (1.5.9).
On the other hand, equation (1.5.10) follows from the evaluation of (1.5.3) over B and again from $D\chi_E = \nu_E \mathcal{H}^{N-1} \llcorner \partial^* E$.
Now, in order to prove the Gauss-Green formulas, as in the classical case, we need just to apply Integration by Parts formulas to each component of $u \in BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$, then for each $i = 1, ..., N$

$$\int_{E^1} dD_i u^i = -\int_{\partial^* E} u^i_{\nu_E} \nu^i_E \, d\mathcal{H}^{N-1}.$$

Summing over i yields (1.5.11). Arguing in this way, we obtain also (1.5.12) and (1.5.13). □

Remark 1.5.1. Proposition 1.5.1 and Theorem 1.5.2 can be formulated without the hypothesis of the essential boundedness, adding the requirements that $u^* \in L^1_{\text{loc}}(\mathbb{R}^N; ||Dv||)$ and $v^* \in L^1_{\text{loc}}(\mathbb{R}^N; ||Du||)$ in the proposition and therefore that $u_{\pm\nu_E} \in L^1(\partial^* E; \mathcal{H}^{N-1})$ in the theorem. For a good exposition of such variants of Theorem 1.5.2 we refer the reader to the classical treatice of Maz'ya ([Maz]).

Finally, we quote a result by Fuglede, which concerns L^p vector fields and is related to De Giorgi and Federer theorem (Theorem 1.5.1). It employs however totally different techniques and concepts, starting from the definition of the module of order p of a family of measures.

Definition 1.5.1. Let $\mathcal{S}$ be a family of Radon measures in $\mathbb{R}^N$. We associate to such a family the set of nonnegative Lebesgue-measurable functions f on $\mathbb{R}^N$ such that $\int_{\mathbb{R}^N} f(x)\, d\mu(x) \geq 1\ \forall \mu \in \mathcal{S}$. If f enjoys these properties, we write $f \wedge \mathcal{S}$.
We define the *module of order* $p \in (0, +\infty)$ of $\mathcal{S}$ as

$$\mathrm{M}_p(\mathcal{S}) := \inf_{f \wedge \mathcal{S}} \int_{\mathbb{R}^N} f^p(x)\, dx.$$

Definition 1.5.2. A family of Radon measures $\mathcal{S}$ is said to be *exceptional of order* p (p-exc) if $\mathrm{M}_p(\mathcal{S}) = 0$.
We shall say that a property holds p-a.e. if it holds for all $\sigma \in \mathcal{M}(\mathbb{R}^N) \setminus \mathcal{S}$, with $\mathrm{M}_p(\mathcal{S}) = 0$.

Theorem 1.5.3. *$\mathcal{S} \subset \mathcal{M}(\mathbb{R}^N)$ is p-exc if and only if $\exists f \in L^p(\mathbb{R}^N)$, $f \geq 0$: $\int_{\mathbb{R}^N} f\, d\mu = +\infty\ \forall \mu \in \mathcal{S}$.*
Moreover, if $f \in L^p(\mathbb{R}^N)$, then $f \in L^1(\sigma)$ for p-a.e. $\sigma \in \mathcal{M}(\mathbb{R}^N)$.

Proof. See [Fu1] Theorem 2 and [Fu2] Ch. 1 Theorem 2.

Since to every set of finite perimeter E is possibile to associate the Radon measure $||D\chi_E|| = \mathcal{H}^{N-1} \llcorner \partial^* E$, we can import these definitions and results into the context of sets finite perimeter. Thus we have the following notion.

Definition 1.5.3. Let $1 \leq p < \infty$. A collection $\mathcal{E}$ of sets of finite perimeter $E \subset \mathbb{R}^N$ is called *exceptional of order* p (abbreviated as p-exc) if there exists $g \in L^p(\mathbb{R}^N)$, $g \geq 0$ such that

$$\int_{\partial^* E} g(x) d\mathcal{H}^{N-1}(x) = +\infty \quad \forall E \in \mathcal{E}.$$

Theorem 1.5.4. (Fuglede)
Let $F \in L^p(\mathbb{R}^N; \mathbb{R}^N)$, $1 \leq p < \infty$, with $\mathrm{div} F \in L^p(\mathbb{R}^N)$. Then

$$\int_E \mathrm{div} F dx = -\int_{\partial^* E} F(x) \cdot \nu_E(x) d\mathcal{H}^{N-1}(x) \tag{1.5.14}$$

for each set E of finite perimeter except those in a p-exc collection $\mathcal{E}$.

Proof. Since we have excluded the p-exc collection $\mathcal{E}$, we have that $F \in L^1(\partial^* E; \mathcal{H}^{N-1})$, by Theorem 1.5.3. In order to show (1.5.14), we prove that the weak and the flux extensions of the differential operator div coincides, see [Fu2], pag. 27-34. □

In the next chapter, we will introduce a new space of vector fields in order to show that it is possible to extend Fuglede's theorem to all sets of finite perimeter, not just to p-almost all in the sense written above. However, we will require $p = \infty$, though we will relax the assumptions on div$F$.

Chapter 2

Divergence-measure fields

2.1 Definition and first properties

In this chapter we introduce the main object of our study; that is, the function spaces $\mathcal{DM}^p(\Omega;\mathbb{R}^N)$, and we present their relevant properties. Most of these are strictly connected with the theory of BV functions previously described. This exposition is largely based on the initial paragraphs of the articles [CF1], [CF2], [CF3], [CT] and [CTZ1].

Definition 2.1.1. A vector field $F \in L^p(\Omega;\mathbb{R}^N), 1 \le p \le \infty$ is called a *divergence-measure field*, and we write $F \in \mathcal{DM}^p(\Omega;\mathbb{R}^N)$, if

$$||\text{div}F||(\Omega) := \sup\left\{\int_\Omega F\cdot\nabla\phi\,dx : \phi \in C_c^\infty(\Omega), ||\phi||_\infty \le 1\right\} < \infty.$$

A vector field F is a *locally divergence-measure field*, and we write $F \in \mathcal{DM}^p_{\text{loc}}(\Omega;\mathbb{R}^N)$, if $F \in \mathcal{DM}^p(W;\mathbb{R}^N)$ for any $W \subset\subset \Omega$ open.

As a consequence of Riesz Representation Theorem for Radon measures (Theorem 1.3.1), we have the following Riesz theorem for divergence-measure fields.

Theorem 2.1.1. *If $F \in \mathcal{DM}^p_{\text{loc}}(\Omega;\mathbb{R}^N)$, then* div$F$ *is a (real) Radon measure on Ω and $\forall\ \phi \in C_c^1(\Omega)$ we have*

$$\int_\Omega \phi\,d\text{div}F = -\int_\Omega F\cdot\nabla\phi\,dx.$$

Proof. One merely repeats the proof of the Riesz theorem for BV functions (Theorem 1.3.1), where one just needs to define $L(\phi) := -\int_\Omega F\cdot\nabla\phi\,dx$ for $\phi \in C_c^1(\Omega)$ and proceed in the same way. □

In the following chapters, we are going to consider the case $p = \infty$; that is, the space of essentially bounded divergence-measure fields. However, many basic facts can be proved for any $1 \leq p \leq \infty$ and hence we give a general description of these spaces.

Proposition 2.1.1. *Let* $\{F_j\} \subset \mathcal{DM}^p(\Omega;\mathbb{R}^N)$ *a sequence such that* $F_j \rightharpoonup F$ *in* $L^q(\Omega;\mathbb{R}^N)$ *for some* $q \in [1,+\infty)$ *or weak-star for* $q = +\infty$, *or in* $L^q_{\text{loc}}(\Omega;\mathbb{R}^N)$. *Then* $\forall A \subseteq \Omega$ *open,*

$$||\text{div}F||(A) \leq \liminf_{j\to\infty} ||\text{div}F_j||(A)$$

Proof. Let $\phi \in C_c^\infty(A)$ with $||\phi||_\infty \leq 1$, then Proposition 1.1.2 implies

$$\int_\Omega F \cdot \nabla\phi\, dx = \lim_{j\to+\infty} \int_\Omega F_j \cdot \nabla\phi\, dx \leq \liminf_{j\to\infty} ||\text{div}F_j||(A)$$

and so, by taking the supremum over ϕ on the left hand side, we have the claim. □

Theorem 2.1.2. $\mathcal{DM}^p(\Omega;\mathbb{R}^N)$ *endowed with the norm*

$$||F||_{\mathcal{DM}^p(\Omega;\mathbb{R}^N)} := ||F||_{L^p(\Omega;\mathbb{R}^N)} + ||\text{div}F||(\Omega)$$

is a Banach space.

Proof. Let $\{F_j\}$ be a Cauchy sequence Cauchy in $\mathcal{DM}^p(\Omega;\mathbb{R}^N)$, then it is Cauchy also in L^p and therefore there exists $F \in L^p(\Omega;\mathbb{R}^N)$ such that $F_j \to F$ in L^p. So, in particular, $F_j \to F$ in $L^1_{\text{loc}}(\Omega;\mathbb{R}^N)$, and, by Proposition 2.1.1,

$$||\text{div}F||(\Omega) \leq \liminf_{j\to\infty} ||\text{div}F_j||(\Omega),$$

which implies $F \in \mathcal{DM}^p(\Omega;\mathbb{R}^N)$.
Moreover, $\forall\epsilon > 0\ \exists j_0$ such that $\forall j,k \geq j_0$ $||\text{div}(F_j - F_k)||(\Omega) < \epsilon$ and, by lower semicontinuity,

$$||\text{div}(F_j - F)||(\Omega) \leq \liminf_{k\to\infty} ||\text{div}(F_j - F_k)||(\Omega) < \epsilon$$

for $j \geq j_0$ and therefore $F_j \to F$ in $\mathcal{DM}^p(\Omega;\mathbb{R}^N)$. □

Theorem 2.1.3. (Approximation by smooth function)
Let $F \in \mathcal{DM}^p(\Omega;\mathbb{R}^N)$, *then* $\exists\{F_n\} \subset \mathcal{DM}^p(\Omega;\mathbb{R}^N) \cap C^\infty(\Omega;\mathbb{R}^N)$ *such that*

1. $\int_\Omega |F_n - F|dx \to 0$;
2. $||\text{div}F_n||(\Omega) \to ||\text{div}F||(\Omega)$.

Proof.
Fix $\epsilon > 0$. Given a positive integer m, we set $\Omega_0 = \emptyset$, define for each $k \in \mathbb{N}, k \geq 1$ the sets

$$\Omega_k = \left\{x \in \Omega :\ \text{dist}(x,\partial\Omega) > \frac{1}{m+k}\right\} \cap B(0,k+m)$$

and then we choose m such that $||\text{div}F||(\Omega \setminus \Omega_1) < \epsilon$.
We define now $\Sigma_k := \Omega_{k+1} \setminus \overline{\Omega}_{k-1}$. Since $\{\Sigma_k\}$ is an open cover of Ω, then there exists a partition of unity subordinate to that open cover; that is, a sequence of functions $\{\zeta_k\}$ such that:

1) $\zeta_k \in C_c^\infty(\Sigma_k)$;

2) $0 \leq \zeta_k \leq 1$;

3) $\sum_{k=1}^{+\infty} \zeta_k = 1$ on Ω.

Then we take a standard mollifier ρ and $\forall k$ we choose ϵ_k such that:

$$\text{spt}(\rho_{\epsilon_k} * (F\zeta_k)) \subset \Sigma_k$$

$$\int_\Omega |\rho_{\epsilon_k} * (F\zeta_k) - F\zeta_k| dx < \frac{\epsilon}{2^k}$$

$$\int_\Omega |\rho_{\epsilon_k} * (F \cdot \nabla\zeta_k) - F \cdot \nabla\zeta_k| dx < \frac{\epsilon}{2^k}$$

and we define $F_\epsilon = \sum_{k=1}^{+\infty} \rho_{\epsilon_k} * (F\zeta_k)$.
Then $F_\epsilon \in C^\infty$, since locally there are only a finite number of nonzero terms in the sum. Moreover, $F_\epsilon \in L^p(\Omega; \mathbb{R}^N)$. If $1 \leq p < \infty$, by the properties of convolution,

$$||F_\epsilon||^p_{L^p(\Omega;\mathbb{R}^N)} \leq \sum_{k=1}^{+\infty} ||F\zeta_k||^p_{L^p(\Omega;\mathbb{R}^N)} = \int_\Omega |F(x)|^p \sum_{k=1}^{+\infty} \zeta_k(x)^p \, dx \leq ||F||^p_{L^p(\Omega;\mathbb{R}^N)},$$

since the series of ζ_k converges uniformly (being locally a finite sum of bounded functions) and $0 \leq \zeta_k \leq 1$, which implies $\zeta_k^p \leq \zeta_k \ \forall p \geq 1$. If $p = \infty$, we observe that $\forall x \in \Omega$, x belongs to at most 3 sets of the open cover $\{\Sigma_k\}$, hence we have $|F_\epsilon(x)| \leq 3||F||_{L^\infty(\Omega;\mathbb{R}^N)}$, which implies

$$||F_\epsilon||_{L^\infty(\Omega;\mathbb{R}^N)} \leq 3||F||_{L^\infty(\Omega;\mathbb{R}^N)}. \tag{2.1.1}$$

Also,

$$\int_\Omega |F - F_\epsilon| dx \leq \sum_{k=1}^{+\infty} \int_\Omega |\rho_{\epsilon_k} * (F\zeta_k) - F\zeta_k| dx < \epsilon.$$

Now, by Proposition 2.1.1, $||\text{div}F||(\Omega) \leq \liminf_{\epsilon \to 0} ||\text{div}F_\epsilon||(\Omega)$.

In order to obtain the reverse inequality, let $\phi \in C_c^\infty(\Omega)$, $||\phi||_\infty \leq 1$. Then

$$\int_\Omega F_\epsilon \cdot \nabla\phi dx = \sum_{k=1}^{+\infty} \int_\Omega \rho_{\epsilon_k} * (\zeta_k F) \cdot \nabla\phi dx = \sum_{k=1}^{+\infty} \int_\Omega \zeta_k F \cdot \nabla(\rho_{\epsilon_k} * \phi) dx$$
$$= \sum_{k=1}^{+\infty} \int_\Omega F \cdot \nabla(\zeta_k(\rho_{\epsilon_k} * \phi)) dx - \sum_{k=1}^{+\infty} \int_\Omega F \cdot \nabla\zeta_k(\rho_{\epsilon_k} * \phi) dx.$$

Using $\sum_{k=1}^{+\infty} \nabla\zeta_k = 0$ in Ω and the properties of the convolution, this last expression equals

$$\sum_{k=1}^{+\infty} \int_\Omega F \cdot \nabla(\zeta_k(\rho_{\epsilon_k} * \phi)) dx - \sum_{k=1}^{+\infty} \int_\Omega \phi(\rho_{\epsilon_k} * (F \cdot \nabla\zeta_k) - F \cdot \nabla\zeta_k) dx =: I_1^\epsilon + I_2^\epsilon.$$

Now, $|\zeta_k(\rho_{\epsilon_k} * \phi)| \leq 1$ and each point in Ω belongs to at most three of the sets $\{\Sigma_k\}$. Thus

$$|I_1^\epsilon| \leq \left| \int_\Omega F \cdot \nabla(\zeta_1(\rho_{\epsilon_1} * \phi)) dx \quad + \quad \sum_{k=2}^{+\infty} \int_\Omega F \cdot \nabla(\zeta_k(\rho_{\epsilon_k} * \phi)) dx \right| \leq$$

$$||\mathrm{div} F||(\Omega) + \sum_{k=2}^{+\infty} ||\mathrm{div} F||(\Sigma_k) \leq ||\mathrm{div} F||(\Omega) + 3||\mathrm{div} F||(\Omega \setminus \Omega_1) \leq ||\mathrm{div} F||(\Omega) + 3\epsilon.$$

For the second term, we have $|I_2^\epsilon| < \epsilon$ directly from our choice of ϵ_k. Therefore, after passing to the supremum over ϕ, $||\mathrm{div} F_\epsilon||(\Omega) \leq ||\mathrm{div} F||(\Omega) + 4\epsilon$, which yields $F_\epsilon \in \mathcal{DM}^p(\Omega; \mathbb{R}^N)$ and point 2. □

We give now a few useful results which will allow us to establish when a sequence of smooth functions $\{F_j\}$ approximating $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^N)$ is such that $\mathrm{div} F_j \overset{*}{\rightharpoonup} \mathrm{div} F$.

Proposition 2.1.2. *Let $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^N)$ and $\{F_j\}$ be a sequence in $\mathcal{DM}^p(\Omega; \mathbb{R}^N)$ such that $F_j \to F$ in $L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and $||\mathrm{div} F_j||(\Omega) \to ||\mathrm{div} F||(\Omega)$.*
Then, for every open set $A \subset \Omega$,

$$||\mathrm{div} F||(\bar{A} \cap \Omega) \geq \limsup_{j \to +\infty} ||\mathrm{div} F_j||(\bar{A} \cap \Omega). \tag{2.1.2}$$

In particular, if $||\mathrm{div} F||(\partial A \cap \Omega) = 0$, then

$$||\mathrm{div} F||(A) = \lim_{j \to +\infty} ||\mathrm{div} F_j||(A). \tag{2.1.3}$$

Proof. Let $B = \Omega \setminus \bar{A}$. Then, by Proposition 2.1.1 (lower semicontinuity),

$$||\mathrm{div}F||(B) \le \liminf_{j\to\infty} ||\mathrm{div}F_j||(B).$$

On the other hand, we have

$$\begin{aligned}
||\mathrm{div}F||(\bar{A}\cap\Omega) + ||\mathrm{div}F||(B) &= ||\mathrm{div}F||(\Omega) = \lim_{j\to+\infty} ||\mathrm{div}F_j||(\Omega)\\
&= \limsup_{j\to+\infty} ||\mathrm{div}F_j||(\bar{A}\cap\Omega) + ||\mathrm{div}F_j||(B)\\
&\ge \limsup_{j\to+\infty} ||\mathrm{div}F_j||(\bar{A}\cap\Omega) + \liminf_{j\to\infty} ||\mathrm{div}F_j||(B)\\
&\ge \limsup_{j\to+\infty} ||\mathrm{div}F_j||(\bar{A}\cap\Omega) + ||\mathrm{div}F||(B)
\end{aligned}$$

and then (2.1.2) follows. Now, if $||\mathrm{div}F||(\partial A\cap\Omega) = 0$, $||\mathrm{div}F||(\bar{A}\cap\Omega) = ||\mathrm{div}F||(A)$ and so

$$\begin{aligned}
||\mathrm{div}F||(A) \le \liminf_{j\to+\infty} ||\mathrm{div}F_j||(A) &\le \limsup_{j\to+\infty} ||\mathrm{div}F_j||(\bar{A}\cap\Omega)\\
&\le ||\mathrm{div}F||(\bar{A}\cap\Omega) = ||\mathrm{div}F||(A),
\end{aligned}$$

which implies (2.1.3). □

Corollary 2.1.1. *Let $F \in \mathcal{DM}^p(\Omega;\mathbb{R}^N)$ and $\{F_j\}$ be a sequence in $\mathcal{DM}^p(\Omega)$ such that $F_j \to F$ in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $||\mathrm{div}F_j||(\Omega) \to ||\mathrm{div}F||(\Omega)$.*
Then we have $||\mathrm{div}F_j|| \overset{*}{\rightharpoonup} ||\mathrm{div}F||$ in $\mathcal{M}(\Omega)$.

Proof. By Proposition 2.1.1, for each open set $A \subset \Omega$,

$$||\mathrm{div}F||(A) \le \liminf_{j\to\infty} ||\mathrm{div}F_j||(A).$$

By Proposition 2.1.2, for each compact $K \subset \Omega$,

$$||\mathrm{div}F||(K) \ge \limsup_{j\to+\infty} ||\mathrm{div}F_j||(K).$$

So, by Lemma 1.1.2, $||\mathrm{div}F_j|| \overset{*}{\rightharpoonup} ||\mathrm{div}F||$, and, since $||\mathrm{div}F_j||(\Omega) \to ||\mathrm{div}F||(\Omega)$ implies $\sup ||\mathrm{div}F_j||(\Omega) < \infty$, Remark 1.1.4 yields the weak-star convergence in $\mathcal{M}(\Omega)$. □

Remark 2.1.1. Under the same hypotheses of Corollary 2.1.1 it is easy to see that $\mathrm{div}F_j \overset{*}{\rightharpoonup} \mathrm{div}F$ in $\mathcal{M}(\Omega)$. Indeed, for any $\phi \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \phi \mathrm{div}F_j\,dx = -\int_\Omega \nabla\phi\cdot F_j\,dx \to -\int_\Omega \nabla\phi\cdot F\,dx = \int_\Omega \phi\, d\mathrm{div}F,$$

by Lebesgue's dominated convergence theorem. Now, if $\phi \in C_c(\Omega)$, $\forall \epsilon > 0$ there exists $\phi_\epsilon \in C_c^\infty(\Omega)$ such that $||\phi - \phi_\epsilon||_\infty < \epsilon$ and so

$$\begin{aligned}\left|\int_\Omega \phi \mathrm{div} F_j \, dx - \int_\Omega \phi \, d\mathrm{div} F\right| &\le \left|\int_\Omega \phi_\epsilon \mathrm{div} F_j \, dx - \int_\Omega \phi_\epsilon \, d\mathrm{div} F\right| \\ &+ \left|\int_\Omega (\phi - \phi_\epsilon)\mathrm{div} F_j \, dx\right| + \left|\int_\Omega (\phi - \phi_\epsilon) \, d\mathrm{div} F\right| \\ &\le \left|\int_\Omega \phi_\epsilon \mathrm{div} F_j \, dx - \int_\Omega \phi_\epsilon \, d\mathrm{div} F\right| \\ &+ \epsilon(||\mathrm{div} F||(\Omega) + ||\mathrm{div} F_j||(\Omega)).\end{aligned}$$

By Theorem 2.1.3, we have

$$\lim_{j \to +\infty} \left|\int_\Omega \phi \mathrm{div} F_j \, dx - \int_\Omega \phi \, d\mathrm{div} F\right| \le 2\epsilon ||\mathrm{div} F||(\Omega).$$

Thus, the arbitrariety of ϵ yields the desired result. Finally, as shown in the proof of Corollary 2.1.1, the sequence $||\mathrm{div} F_j||(\Omega)$ is uniformly bounded and so Remark 1.1.4 gives the weak-star convergence in the sense of Radon measures.

Remark 2.1.2. In a way similar to the case of BV functions, if $F \in \mathcal{DM}^p(\mathbb{R}^N; \mathbb{R}^N)$, $1 \le p \le \infty$, then $F_\epsilon = F * \rho_\epsilon$ satisfies point 2 of Theorem 2.1.3. Indeed, for any $\phi \in C_c^\infty(\mathbb{R}^N)$ with $||\phi||_\infty \le 1$,

$$\int_{\mathbb{R}^N} F_\epsilon(x) \cdot \nabla\phi(x) \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \nabla\phi(x) \cdot F(y) \rho_\epsilon(x - y) \, dy dx =$$

$$\int_{\mathbb{R}^N} F(y) \cdot \nabla\phi_\epsilon(y) \, dy \le ||\mathrm{div} F||(\mathbb{R}^N).$$

So, if we take the supremum over $\phi \in C_c^\infty(\mathbb{R}^N)$ with $||\phi||_\infty \le 1$, we gain $||\mathrm{div} F_\epsilon||(\mathbb{R}^N) \le ||\mathrm{div} F||(\mathbb{R}^N)$, and this, combined with lower semicontinuity, yields

$$||\mathrm{div} F_\epsilon||(\mathbb{R}^N) \to ||\mathrm{div} F||(\mathbb{R}^N).$$

Moreover, we know by the property of mollification (Theorem 1.2.1) that $F_\epsilon \to F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$, therefore, by Corollary 2.1.1 and Remark 2.1.1, $||\mathrm{div} F_\epsilon|| \overset{*}{\rightharpoonup} ||\mathrm{div} F||$ and $\mathrm{div} F_\epsilon \overset{*}{\rightharpoonup} \mathrm{div} F$ in $\mathcal{M}(\mathbb{R}^N)$.
It is clear that this remark applies also to $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^N)$, with compact support inside Ω, since it can be extended to zero on $\mathbb{R}^N \setminus \Omega$.
Indeed, let its extension be

$$\hat{F}(x) = \begin{cases} F(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then trivially $\hat{F} \in L^p(\mathbb{R}^N;\mathbb{R}^N)$. If we let $\xi \in C_c^\infty(\Omega)$, $||\xi||_\infty \leq 1$ and $\xi = 1$ in a neighborhood of the support of F, then, for any $\phi \in C_c^\infty(\mathbb{R}^N)$, $||\phi||_\infty \leq 1$, we have

$$\int_{\mathbb{R}^N} \hat{F} \cdot \nabla\phi \, dx = \int_\Omega F \cdot \nabla\phi \, dx = \int_\Omega F \cdot \nabla(\xi\phi + (1-\xi)\phi) \, dx$$
$$= \int_\Omega F \cdot \nabla(\xi\phi) \, dx \leq ||\mathrm{div}F||(\Omega),$$

since $\xi\phi \in C_c^\infty(\Omega)$.
Taking the supremum over ϕ we obtain $||\mathrm{div}\hat{F}||(\mathbb{R}^N) \leq ||\mathrm{div}F||(\Omega) < \infty$, which implies $\hat{F} \in \mathcal{DM}^p(\mathbb{R}^N;\mathbb{R}^N)$.

If we have just $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$, then we would still have $F_\epsilon \to F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ and for each compact K and $\phi \in C_c^\infty(K)$ with $||\phi||_\infty \leq 1$,

$$\int_{\mathbb{R}^N} F_\epsilon(x) \cdot \nabla\phi(x) \, dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \nabla\phi(x) \cdot F(y)\rho_\epsilon(x-y) \, dydx =$$
$$\int_{\mathbb{R}^N} F(y) \cdot \nabla\phi_\epsilon(y) \, dy \leq ||\mathrm{div}F||(K + \overline{B(0,\epsilon)}),$$

since $\mathrm{supp}(\phi_\epsilon) \subset K + \overline{B(0,\epsilon)}$. Now we take the supremum as before and we obtain $||\mathrm{div}F_\epsilon||(K) \leq ||\mathrm{div}F||(K + \overline{B(0,\epsilon)})$. Hence we have

$$\limsup_{\epsilon\to 0} ||\mathrm{div}F_\epsilon||(K) \leq \limsup_{\epsilon\to 0} ||\mathrm{div}F||(K + \overline{B(0,\epsilon)}) = ||\mathrm{div}F||(K).$$

Since for any bounded open set W we have $F \in \mathcal{DM}^p(W;\mathbb{R}^N)$, we have also the lower semicontinuity on open subsets of W. Thus, if B is a bounded Borel set with $||\mathrm{div}F||(\partial B) = 0$, we see that

$$||\mathrm{div}F||(B) = ||\mathrm{div}F||(B^\circ) \leq \liminf_{\epsilon\to 0} ||\mathrm{div}F_\epsilon||(B^\circ) \leq \limsup_{\epsilon\to 0} ||\mathrm{div}F_\epsilon||(\bar{B})$$
$$\leq ||\mathrm{div}F||(\bar{B}) = ||\mathrm{div}F||(B),$$

which implies $||\mathrm{div}F_\epsilon||(B) \to ||\mathrm{div}F||(B)$.
Then Lemma 1.1.2 gives us $||\mathrm{div}F_\epsilon|| \overset{*}{\rightharpoonup} ||\mathrm{div}F||$ in $\mathcal{M}(W)$.
Moreover, a slight modification of the argument of Remark 2.1.1 yields $\mathrm{div}F_\epsilon \overset{*}{\rightharpoonup} \mathrm{div}F$ in $\mathcal{M}(W)$: indeed, we do not have anymore $||\mathrm{div}F_\epsilon||(W) \to ||\mathrm{div}F||(W)$, but, by the previous calculations, $\limsup_{\epsilon\to 0} ||\mathrm{div}F_\epsilon||(W) \leq ||\mathrm{div}F||(\overline{W}) < \infty$ since $\overline{W}$ is compact and $\mathrm{div}F$ is a Radon measure.
Therefore, we can conclude that $||\mathrm{div}F_\epsilon|| \overset{*}{\rightharpoonup} ||\mathrm{div}F||$ and $\mathrm{div}F_\epsilon \overset{*}{\rightharpoonup} \mathrm{div}F$ in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^N)$.

2.2 Comparison with $BV(\Omega;\mathbb{R}^N)$ and Examples

From theorems in section 2.1, it is reasonable to ask if the spaces of divergence-measure vector fields actually provide example of functions which are not in BV (or, for $p>1$, $BV\cap L^p$). Indeed, we can show that $L^p(\Omega;\mathbb{R}^N)\cap BV(\Omega;\mathbb{R}^N)\subset \mathcal{DM}^p(\Omega;\mathbb{R}^N)\cap L^1(\Omega;\mathbb{R}^N)$ (this two spaces obviously coincide if $N=1$).

If $u\in L^p(\Omega;\mathbb{R}^N)\cap BV(\Omega;\mathbb{R}^N)$, then each component of u is a BV function; so for each $i=1,...,N$ and for any $\phi^i\in C_c^\infty(\Omega;\mathbb{R}^N)$ we have

$$-\int_\Omega u_i\,\mathrm{div}\phi^i\,dx=\int_\Omega \phi^i\cdot dDu_i.$$

We choose $\phi^i=\psi e_i$, where e_i is the i-th element of the canonical basis of $\mathbb{R}^N$ and $\psi\in C_c^\infty(\Omega)$, thus

$$-\int_\Omega u_i\,\frac{\partial\psi}{\partial x_i}\,dx=\int_\Omega \psi\,dD_iu_i \quad i=1,...,N.$$

Summing over i we obtain

$$-\int_\Omega u\cdot\nabla\psi\,dx=\int_\Omega \psi\,d\left(\sum_{i=1}^N D_iu_i\right)$$

and, since $\sum_{i=1}^N D_iu_i$ is a finite Radon measure by BV theory, Definition 2.1.1 yields $u\in\mathcal{DM}^p(\Omega;\mathbb{R}^N)\cap L^1(\Omega;\mathbb{R}^N)$.

However, the following example shows that the inclusion above is strict.

Example 2.2.1. The field

$$F(x)=\frac{x}{|x|^N}$$

belongs to $\mathcal{DM}^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)\setminus BV_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$.
Indeed, $F\in L^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ since, for every $R>0$, passing to polar coordinates in $\mathbb{R}^N$,

$$\int_{B(0,R)}|F(x)|dx=\int_U\int_0^R\frac{1}{\rho^{N-1}}\rho^{N-1}|\det(J\Phi)|d\rho d\underline{\theta}=RN\omega_N$$

where $U=(0,2\pi)\times(0,\pi)^{N-2}$, $\underline{\theta}$ is the vector angular variable on U and $\det(J\Phi)$ is the Jacobian of the change of variables divided by the factor ρ^{N-1}.
Moreover, we will show that

$$\mathrm{div}(F)=N\omega_N\delta,$$

where δ is the Dirac delta measure centered in the origin.
Since $F \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R}^N)$ we have

$$\operatorname{div}\left(\frac{x}{|x|^N}\right) = \sum_{i=1}^{N} \frac{1}{|x|^N} - \frac{N x_i^2}{|x|^{N+2}} = \frac{N}{|x|^N}\left(1 - \frac{|x|^2}{|x|^2}\right) = 0 \ \forall x \neq 0.$$

We can therefore apply classical divergence theorem in $\mathbb{R}^N \setminus B(0,\epsilon)\ \ \forall \epsilon > 0$, so for each open A and $\forall\ \phi \in C_c^\infty(A)$ we have

$$\int_A \frac{x}{|x|^N} \cdot \nabla\phi(x) dx = 0$$

if $0 \notin A$, whereas, if $0 \in A$,

$$\begin{aligned}\int_{A\setminus B(0,\epsilon)} \frac{x}{|x|^N} \cdot \nabla\phi(x) dx &= -\int_{\partial B(0,\epsilon)} \phi(x)\frac{x}{|x|^N} \cdot \frac{x}{|x|} d\mathcal{H}^{N-1} \\ &\quad - \int_{A\setminus B(0,\epsilon)} \phi(x) \operatorname{div}\left(\frac{x}{|x|^N}\right) dx \\ &= -\int_{\partial B(0,\epsilon)} \phi(x)\frac{x}{|x|^N} \cdot \frac{x}{|x|} d\mathcal{H}^{N-1} \quad \forall \epsilon > 0.\end{aligned}$$

Now we let $\epsilon \to 0^+$ and, since $F \cdot \nabla\phi \in L^1(A)$, by Lebesgue's dominated convergence theorem we have

$$\begin{aligned}\lim_{\epsilon\to 0^+} \int_{A\setminus B(0,\epsilon)} \frac{x}{|x|^N} \cdot \nabla\phi(x) dx &= \int_A \frac{x}{|x|^N} \cdot \nabla\phi(x)\, dx \\ &= \lim_{\epsilon\to 0^+} - \int_{\partial B(0,\epsilon)} \phi(x)\, \frac{1}{|x|^{N-1}}\, d\mathcal{H}^{N-1}.\end{aligned}$$

By smoothness, $\phi(x) = \phi(0) + |x| R(x)$, with $R(x)$ bounded for $|x| \leq \epsilon$, so

$$\lim_{\epsilon\to 0^+} \int_{\partial B(0,\epsilon)} \phi(x) \frac{1}{|x|^{N-1}}\, d\mathcal{H}^{N-1} = \lim_{\epsilon\to 0^+} \int_U \left(\phi(0) + \epsilon R(\epsilon,\theta)\right) \frac{1}{\epsilon^{N-1}} \epsilon^{N-1} \left|\det(J\Phi)\right| d\theta.$$

Thus,

$$\int_A \frac{x}{|x|^N} \cdot \nabla\phi(x)\, dx = -\phi(0) N \omega_N,$$

from which our claim follows.
On the other hand, $F \notin BV_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$, since for example $\frac{\partial F_1}{\partial x_2} = -N \frac{x_1 x_2}{|x|^{N+2}} \in L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and so, for every $R > \epsilon > 0$,

$$\begin{aligned}&\sup\left\{\int_{B(0,R)} \frac{\partial\phi}{\partial x_2} F_1\, dx : \phi \in C_c^\infty(B(0,R) \setminus B(0,\epsilon)), ||\phi||_\infty \leq 1\right\} \\ &= \int_{B(0,R)\setminus B(0,\epsilon)} \left|\frac{\partial F_1}{\partial x_2}\right| dx \geq C_N \int_\epsilon^R \frac{\rho^2}{\rho^{N+2}} \rho^{N-1}\, d\rho = C_N \log\frac{R}{\epsilon}.\end{aligned}$$

Since ϵ is arbitrary, it follows that the total variation of $\frac{\partial F_1}{\partial x_2}$ is unbounded on compact sets containing the origin.

Remark 2.2.1. The Gauss-Green formula fails for the field of Example 2.2.1. Let $E = B(0,1) \cap \{x \in \mathbb{R}^N : x_N > 0\}$, then $\partial^* E = (\partial B(0,1) \cap \{x \in \mathbb{R}^N : x_N > 0\}) \cup (B(0,1) \cap \{x \in \mathbb{R}^N : x_N = 0\})$ and the interior unit normal is $\nu_E = e_N$ on $B(0,1) \cap \{x \in \mathbb{R}^N : x_N = 0\}$ and $\nu_E = -\frac{x}{|x|}$ on $\partial B(0,1) \cap \{x \in \mathbb{R}^N : x_N > 0\}$.
So, since $0 \notin E$ (not even in E^1),

$$\int_E d\mathrm{div} F = 0$$

but

$$\int_{\partial^* E} F \cdot \nu_E \, d\mathcal{H}^{N-1} = \int_{\partial B(0,1) \cap \{x_N > 0\}} -\frac{1}{|x|^{N-1}} d\mathcal{H}^{N-1} = -\frac{N\omega_N}{2}.$$

As we shall see, high summability plays an important role in the possibility of establishing a Gauss-Green formula.

We now provide a general example of an essentially bounded divergence measure field, the main topic of our exposition.

Example 2.2.2. Let $v \in L^\infty(\mathbb{R})$ and define the field $F(x,y) := (v(x-y), v(x-y)) \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$. For each $\phi \in C_c^\infty(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} F \cdot \nabla\phi \, dxdy = \int_{\mathbb{R}^2} v(x-y) \left(\frac{\partial \phi(x,y)}{\partial x} + \frac{\partial \phi(x,y)}{\partial y} \right) dxdy.$$

We perform a change of variables: $(x,y) = \Phi(t,u) := (\frac{t+u}{2}, \frac{u-t}{2})$, whose Jacobian is $J\Phi = |\det(D\Phi)| = \frac{1}{2}$.
We write $\phi(\frac{t+u}{2}, \frac{u-t}{2}) = \varphi(t,u)$ and so, by Fubini's theorem, the previous integral becomes

$$\int_{\mathbb{R}^2} v(t) \frac{\partial \varphi(t,u)}{\partial u} \, dudt = 0,$$

since $\int_{\mathbb{R}} \frac{\partial \varphi(t,u)}{\partial u} du = 0$ for each $\varphi \in C_c^\infty(\mathbb{R}^2)$.
Thus we can conclude that $\mathrm{div} F = 0$ and so $F \in \mathcal{DM}^\infty(\mathbb{R}^2;\mathbb{R}^2)$.
In a similar way, we can also construct examples of fields $F \in \mathcal{DM}^\infty(\mathbb{R}^N;\mathbb{R}^N)$, just by considering functions $v_i \in L^\infty(\mathbb{R})$, $i = 1, 2, ..., N$ if N is even, and then defining F as

$$F(x) := (v_1(x_1 - x_2), v_2(x_1 - x_2), ..., v_{N-1}(x_{N-1} - x_N), v_N(x_{N-1} - x_N)).$$

If N is odd, then we take $i = 1, 2, ..., N-1$ and we set a constant as the N-th component of F.

Also in this case the inclusion is strict. If, for example, $v(t) = \sin(\frac{1}{t})$, then $F \in \mathcal{DM}^\infty(\mathbb{R}^2;\mathbb{R}^2) \setminus BV_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$.
Indeed, let $\phi \in C_c^\infty(B(0,R);\mathbb{R}^2)$ for some $R > 0$ and, for each $\epsilon > 0$, $L_\epsilon := \{(x,y) \in \mathbb{R}^2 : |y-x| \le \epsilon\}$. Then, by Lebesgue's dominated convergence theorem,

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^2 \setminus L_\epsilon} \sin\left(\frac{1}{x-y}\right) \mathrm{div}\phi(x,y)\,dxdy = \int_{\mathbb{R}^2} \sin\left(\frac{1}{x-y}\right) \mathrm{div}\phi(x,y)\,dxdy.$$

Now, by classical integration by parts,

$$\int_{\mathbb{R}^2 \setminus L_\epsilon} \sin\left(\frac{1}{x-y}\right) \mathrm{div}\phi(x,y)\,dxdy = -\int_{\mathbb{R}} \sin\left(\frac{1}{\epsilon}\right)(\phi^1(t,t-\epsilon) - \phi^1(t,t+\epsilon))\frac{1}{\sqrt{2}}\,dt$$
$$-\int_{\mathbb{R}} \sin\left(\frac{1}{\epsilon}\right)(\phi^2(t,t+\epsilon) - \phi^2(t,t-\epsilon))\frac{1}{\sqrt{2}}\,dt$$
$$-\int_{\mathbb{R}^2 \setminus L_\epsilon} \cos\left(\frac{1}{x-y}\right)\frac{1}{(x-y)^2}(-\phi^1(x,y) + \phi^2(x,y))\,dxdy$$
$$= I_1 + I_2 + I_3.$$

If $||\phi||_\infty \le 1$, then $|I_1 + I_2| \le 8R$ for each $\epsilon > 0$, since $\mathcal{H}^1(\mathrm{supp}(\phi^i) \cap \partial L_\epsilon) \le 2\mathcal{H}^1(B(0,R) \cap L_0) = 4R$.
We can choose a sequence ϕ_j in $C_c^\infty(B(0,R);\mathbb{R}^2)$ in such a way that $\phi_j^1 \to \cos\left(\frac{1}{x-y}\right)$ and $\phi_j^2 \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $||\phi_j||_\infty \le 1$ for each j.
Then there exists a j_0 such that, for each $j \ge j_0$,

$$||\phi_j - (\cos\left(\frac{1}{x-y}\right), 0)||_{L^1(B(0,R))} < \epsilon^3.$$

Therefore, since the supremum of $\cos\left(\frac{1}{x-y}\right)\frac{1}{(x-y)^2}$ over $B(0,R) \setminus L_\epsilon$ is less than $\frac{1}{\epsilon^2}$, we obtain

$$\int_{\mathbb{R}^2 \setminus L_\epsilon} \sin\left(\frac{1}{x-y}\right) \mathrm{div}\phi(x,y)\,dxdy$$
$$\ge \int_{B(0,R) \setminus L_\epsilon} \left(\cos\left(\frac{1}{x-y}\right)\right)^2 \frac{1}{(x-y)^2}\,dxdy - 8R - \epsilon^3\frac{1}{\epsilon^2}$$

and the integral diverges as $\epsilon \to 0$.
Indeed, with the same change of variables as above; that is, $(x,y) = \Phi(t,u) :=$

$(\frac{t+u}{2}, \frac{u-t}{2})$, we have $\Phi^{-1}(B(0,R) \setminus L_\epsilon) = B(0, \sqrt{2}R) \setminus \{|t| \leq \epsilon\}$ and so

$$\int_{B(0,R)\setminus L_\epsilon} \left(\cos\left(\frac{1}{x-y}\right)\right)^2 \frac{1}{(x-y)^2}\, dxdy = \int_{B(0,\sqrt{2}R)\setminus\{|t|\leq\epsilon\}} \left(\cos\left(\frac{1}{t}\right)\right)^2 \frac{1}{t^2}\frac{1}{2} dtdu$$

$$\geq \int_{[-R,R]^2\setminus\{|t|\leq\epsilon\}} \left(\cos\left(\frac{1}{t}\right)\right)^2 \frac{1}{2t^2}\, dtdu = \int_{-R}^{R}\int_{\{\epsilon\leq|t|\leq R\}} \left(\cos\left(\frac{1}{t}\right)\right)^2 \frac{1}{2t^2}\, dtdu$$

$$= \left[t = \frac{1}{\tau}\right] = 2R\int_{\frac{1}{R}}^{\frac{1}{\epsilon}} (\cos\tau)^2\tau^2 \frac{1}{\tau^2} d\tau = 2R[\frac{1}{2}(\tau + \cos\tau\sin\tau)]_{\frac{1}{R}}^{\frac{1}{\epsilon}} \geq \frac{R}{\epsilon} - 1 - 2R$$

which proves our claim.
Therefore the total variation of $\sin\left(\frac{1}{x-y}\right)$ is unbounded on any compact set containing a segment of the line L_0 and so this function does not belong to $BV_{\text{loc}}(\mathbb{R}^2)$.

Remark 2.2.2. Moreover, since $F \in C^\infty(\mathbb{R}^2 \setminus L_\epsilon; \mathbb{R}^2)$ for each $\epsilon > 0$, then Gauss-Green formula is valid for each $E \subset\subset \mathbb{R}^2 \setminus L_\epsilon$ of finite perimeter by Remark 1.4.5. However, it is clearly impossible to define any reasonable notion of the trace on the line $y = x$ for F in a classical sense, since on that line the components of the field have essential singularities. Nevertheless, the unit normal to the line $x - y = \epsilon$ is (up to a sign) $\nu_\epsilon = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ so that the scalar product is meaningful and satisfies $F(x, x-\epsilon) \cdot \nu_\epsilon = 0$.
Then, by classical Gauss-Green formula, for any $\phi \in C_c^1(\mathbb{R}^2)$,

$$0 = \int_{\{x>y+\epsilon\}} \text{div}F\, \phi\, dxdy = -\int_{\{x>y+\epsilon\}} F \cdot \nabla\phi\, dxdy$$

and, by the dominated convergence theorem, this identity remains valid for $\epsilon \to 0$. This allows us to conclude that if we define the normal trace of F over $y = x$ as $F \cdot \nu = 0$, it would be coherent with the limit of the classical results.
As we shall see later, F has a weak normal trace over the boundary of any bounded set of finite perimeter that is sufficient for the Gauss-Green formula to hold and which can be shown to be an L^∞ function identically 0 on the line $y = x$.

Example 2.2.3. Another archetypical example of a divergence measure field are the so called transversal fields.
Let $f \in L^p_{\text{loc}}(\mathbb{R}^{N-1})$ for some $1 \leq p \leq \infty$, and define $F(x) = (0, 0, ..., f(\hat{x}_N))$, where $\hat{x}_N = (x_1, ..., x_{N-1})$.
It is clear that $F \in L^p_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ and for each $\phi \in C_c^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} F \cdot \nabla\phi\, dx = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} f(\hat{x}_N) \frac{\partial\phi}{\partial x_N}\, dx_N d\hat{x}_N = 0$$

by Fubini's theorem, since $\int_{\mathbb{R}} \frac{\partial\phi}{\partial x_N}\, dx_N = 0$.
Therefore, $\text{div}F = 0$ and $F \in \mathcal{DM}^p_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$.

More generally, we can take N functions $f_i \in L^p_{\text{loc}}(\mathbb{R}^{N-1})$ and define F to be the vector field whose components are $F_i(x) = f_i(\hat{x}_i)$. In the same way, we can show $\text{div}F = 0$ and $F \in \mathcal{DM}^p_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$.

2.3 Normal trace and absolute continuity

Now we will introduce a first generalization of the notion of normal trace for a divergence-measure field F on the boundary of a set E in a distributional sense and prove that, in order to deal with it, we need just to consider ∂E.

Definition 2.3.1. Let $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^N)$ with $1 \leq p \leq \infty$. For a measurable set $E \subset\subset \Omega$, the *trace of the normal component* of F on ∂E is the linear functional defined by

$$(TF)_{\partial E}(\phi) := \int_E \nabla\phi \cdot F \, dx + \int_E \phi \, d\text{div}F \quad \text{for any} \quad \phi \in C^\infty_c(\Omega).$$

Remark 2.3.1. It is clear that $(TF)_{\partial E}$ is a distribution. Moreover, if $F \in C^1_c(\mathbb{R}^N; \mathbb{R}^N)$ and E is a regular or admissible set, then, by the classical divergence theorem,

$$(TF)_{\partial E}(\phi) = -\int_{\partial E} \phi F \cdot \nu \, d\mathcal{H}^{N-1},$$

where ν is the unit interior normal to ∂E.

Proposition 2.3.1. *Let $E \subset\subset \Omega$ be an open set. Then* $\text{supp}((TF)_{\partial E}) \subset \partial E$*; that is, $\forall\ \phi, \psi \in C^\infty_c(\Omega)$ with $\psi = \phi$ on ∂E, then $(TF)_{\partial E}(\psi) = (TF)_{\partial E}(\phi)$.*

Proof. Obviously, $\text{supp}((TF)_{\partial E}) \subset \bar{E}$.
By contradiction, suppose that there exists a point $x_0 \notin \partial E$ with $x_0 \in \text{supp}((TF)_{\partial E}) \cap E$. This means that for each open set U containing x_0 there exists a test function $\phi \in C^\infty_c(U \cap E)$ such that

$$(TF)_{\partial E}(\phi) \neq 0. \tag{2.3.1}$$

We choose $U \subset \mathbb{R}^N \setminus \partial E$. Let F_ϵ be the mollification of F, then, since $\text{supp}(F_\epsilon \phi) \subset\subset E$ and $F_\epsilon \phi$ is a smooth function, one has

$$(TF_\epsilon)_{\partial E}(\phi) = \int_E \nabla\phi \cdot F_\epsilon \, dx + \int_E \phi \text{div} F_\epsilon \, dx = \int_E \text{div}(F_\epsilon \phi) \, dx = 0.$$

Now, by the Lebesgue's dominated convergence theorem, one finds

$$\int_E \nabla\phi \cdot F_\epsilon \, dx \to \int_E \nabla\phi \cdot F \, dx.$$

Since $\int_E \phi \text{div} F_\epsilon \, dx = -\int_E \nabla\phi \cdot F_\epsilon \, dx \to -\int_E \nabla\phi \cdot F \, dx = \int_E \phi \, d\text{div}F$, by sending $\epsilon \to 0$ we have

$$0 = (TF)_{\partial E}(\phi),$$

which contradicts (2.3.1) above. □

The following theorem is a really interesting result, since it provides a way to find the sets of $\text{div}F$-measure zero in $\mathbb{R}^N$ with $N > 1$.

Theorem 2.3.1. (Absolute continuity of $\text{div}F$ with respect to capacity) *If $F \in \mathcal{DM}^p_{\text{loc}}(\Omega; \mathbb{R}^N)$ with $\frac{N}{N-1} < p \le \infty$, then $||\text{div}F|| \ll \text{Cap}_q(\cdot, \Omega)$ ($q := \frac{p}{p-1}$); that is, for each Borel set $B \subset \Omega$ such that $\text{Cap}_q(B, \Omega) = 0$, $||\text{div}F||(B) = 0$.*

Proof.
Since $\text{div}F$ is a Radon measure on Ω, then its positive and negative parts $\text{div}F^+$ and $\text{div}F^-$ are well defined.
Let $B \subset \Omega$ be a Borel set with $\text{Cap}_q(B, \Omega) = 0$.
By the Hahn decomposition theorem, there exist Borel sets $B_\pm \subset B$ with $B_+ \cup B_- = B$ and $B_+ \cap B_- = \emptyset$ such that $\pm\text{div}F \llcorner B_\pm \ge 0$; that is, $\text{div}F^+ \llcorner B = \text{div}F \llcorner B_+$ and $\text{div}F^- \llcorner B = -\text{div}F \llcorner B_-$.
Hence, it suffices to prove that $\text{div}F(B_\pm) = 0$, and, in order to do so, it suffices to prove $\text{div}F(K) = 0$ for any compact subset K of $B_\pm$, by Proposition 1.1.1.
We show only the case $K \subset B_+$, as the case of B_- is analogous.
By monotonicity (Proposition 1.2.1, property 1), $\text{Cap}_q(K, \Omega) = 0$ for any $K \subset B$ if $\text{Cap}_q(B, \Omega) = 0$.
Since $\text{Cap}_q(K, \Omega) = 0$ and $1 \le q < N$, we can apply Lemma 1.2.1 in order to find a sequence of test functions $\phi_j \in C^\infty_c(\Omega)$ such that

1. $0 \le \phi_j \le 1$ and $\phi_j = 1$ on K,
2. $||\nabla\phi_j||_{L^p(\Omega;\mathbb{R}^N)} \to 0$,
3. for each j, $\text{supp}(\phi_j)$ is contained in a compact set $C_j \subset \Omega$ such that

$$C_1 \supset C_2 \supset ... \supset K \quad \text{and} \quad \bigcap_{j=1}^{\infty} C_j = K,$$

Then, property 1 and the Hölder inequality yield

$$\begin{aligned}\int_\Omega \phi_j \, d\text{div}F &= \text{div}F(K) + \int_{\Omega\setminus K} \phi_j \, d\text{div}F \\ &= -\int_\Omega F \cdot \nabla\phi_j \, dx \le ||F||_{L^p(C_1;\mathbb{R}^N)} ||\nabla\phi_j||_{L^q(\Omega;\mathbb{R}^N)}\end{aligned}$$

and so, by properties 2 and 3,

$$\text{div}F(K) \le ||\text{div}F||(C_j \setminus K) + ||F||_{L^p(C_1;\mathbb{R}^N)} ||\nabla\phi_j||_{L^q(\Omega;\mathbb{R}^N)} \to 0 \quad \text{as} \quad j \to +\infty.$$

□

Corollary 2.3.1. *If* $F \in \mathcal{DM}^p_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $\frac{N}{N-1} \leq p \leq \infty$ *and* $q = \frac{p}{p-1}$, *then* $||\mathrm{div}F|| \ll \mathcal{H}^{N-q}$.
Moreover, if $\Omega = \mathbb{R}^N$ *and* $\frac{N}{N-1} < p < \infty$, *we we have also that, if* $\mathcal{H}^{N-q}(B) < \infty$ *for a Borel set* B, *then* $||\mathrm{div}F||(B) = 0$.

Proof. If $p = \frac{N}{N-1}$, $q = N$, so if a Borel set $B \subset \Omega$ satisfies $\mathcal{H}^0(B) = 0$, then $B = \emptyset$ and trivially $||\mathrm{div}F||(B) = 0$.
If $\frac{N}{N-1} < p \leq \infty$, it is enough to apply Theorems 1.2.3 and 2.3.1. Indeed, we need to show that if we have a Borel set $B \subset \Omega$ such that $\mathcal{H}^{N-q}(B) = 0$, then $||\mathrm{div}F||(B) = 0$. Since for every compact $K \subset B$ we have $\mathcal{H}^{N-q}(K) = 0$, then $\mathrm{Cap}_q(K,\Omega) = 0$ and so $||\mathrm{div}F||(K) = 0$, but this is a Radon measure in Ω, thus, by inner regularity, $||\mathrm{div}F||(B) = 0$.
Then second part of the statement follows again immediately from Theorems 1.2.3 and 2.3.1 if $p > \frac{N}{N-1}$. □

Remark 2.3.2. If $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$, $\frac{N}{N-1} < p \leq \infty$, then, since by Theorem 1.2.5, each $\phi \in W^{1,q}(\mathbb{R}^N)$ is defined up to a set of Cap_q-measure zero, and therefore, by Theorem 2.1.4, $||\mathrm{div}F||$-a.e., it follows that the integral

$$\int_{\mathbb{R}^N} \phi \, d\mathrm{div}F$$

is well defined.

Remark 2.3.3. We observe that there is a parallelism between a field $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^N)$ and a function $u \in BV(\Omega)$ since, by Remark 1.4.8, we have $||Du|| \ll \mathcal{H}^{N-1}$ and, by Corollary 2.3.1, $||\mathrm{div}F|| \ll \mathcal{H}^{N-1}$. As we shall see in Chapter 3, this will be of great relevance in order to obtain the Gauss-Green formula.

The result of Corollary 2.3.1 is optimal, as it is shown in [S], Example 3.3. Indeed we have the following result.

Proposition 2.3.2. *If* $1 \leq p < \frac{N}{N-1}$, *then for an arbitrary signed Radon measure with compact support* μ *there exists* $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ *such that* $\mathrm{div}F = \mu$. *This means that* μ *may be not absolutely continuous with respect to any Hausdorff measure or capacity.*
On the other hand, if $\frac{N}{N-1} \leq p \leq \infty$, *then for any* $s > N - q$ *there exists a field* $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ *such that* $||\mathrm{div}F||$ *is not* $\mathcal{H}^s$ *absolutely continuous.*

In order to prove the Proposition 2.3.2, we need the following result.

Proposition 2.3.3. *Let* μ *be a signed Radon measure on* $\mathbb{R}^N$ *with compact support and set*

$$F(x) := \frac{1}{N\omega_N}\int_{\mathbb{R}^N} \frac{(x-y)}{|x-y|^N}\,d\mu(y) \quad \textit{for a.e.} \quad x. \tag{2.3.2}$$

Then

1. $F \in \mathcal{DM}^1_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ *and* $\operatorname{div}F = \mu$;
2. *if* $1 \leq p < \frac{N}{N-1}$, *then* $F \in L^p_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$;
3. *if* $\frac{N}{N-1} \leq p \leq \infty$, *then* $F \in L^p_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ *provided* $||\mu||(B(x,r)) \leq cr^m$ $\forall x \in \mathbb{R}^N$ *and* $r \in (0,a)$, *where* $a > 0$, $c > 0$, $m > d$ *are constants and*

$$d := \begin{cases} N - \frac{p}{p-1} & \textit{if} \;\; p < \infty \\ N-1 & \textit{if} \;\; p = \infty. \end{cases}$$

Proof. For each $x \in \mathbb{R}^N$, let

$$G(x) := \int_{\mathbb{R}^N} |x-y|^{1-N}\,d||\mu||(y).$$

By Fubini's theorem for abstract measures, G is Lebesgue-measurable, since $f(x,y) = |x-y|^{1-N}$ is nonnegative and $||\mu|| \otimes \mathcal{L}^N$-measurable.
We prove that $G \in L^p(\mathbb{R}^N)$ for every $1 \leq p < \frac{N}{N-1}$. By Hölder's inequality with $q := \frac{p}{p-1}$, one has

$$G(x)^p \leq ||\mu||(\mathbb{R}^N)^{\frac{p}{q}} \int_{\mathbb{R}^N} |x-y|^{-p(N-1)}\,d||\mu||(y).$$

Therefore, if $z \in \mathbb{R}^N$ and $r > 0$, by Fubini's theorem,

$$\int_{B(z,r)} G(x)^p\,dx \leq ||\mu||(\mathbb{R}^N)^{\frac{p}{q}} \int_{\mathbb{R}^N}\int_{B(z,r)} |x-y|^{-p(N-1)}\,dxd||\mu||(y).$$

For any $y \in \mathbb{R}^N$ we have $B(z,r) \subset B(y, |z-y|+r)$ and so

$$\begin{aligned} \int_{B(z,r)} |x-y|^{-p(N-1)}\,dx &\leq \int_{B(y,|z-y|+r)} |x-y|^{-p(N-1)}\,dx \\ &= \int_U \int_0^{|z-y|+r} \rho^{N-1-p(N-1)} |\det(J\Phi)|\,d\rho d\underline{\theta} \\ &= \frac{N\omega_N}{N-p(N-1)}(|z-y|+r)^{N-p(N-1)}, \end{aligned}$$

where we passed to polar coordinates and used the fact that $N-1-p(N-1) > -1$; that is $p < \frac{N}{N-1}$. Hence we have

$$\int_{B(z,r)} G(x)^p \, dx \leq \frac{N\omega_N}{N - p(N-1)} ||\mu||(\mathbb{R}^N)^{\frac{p}{q}} \int_{\mathbb{R}^N} (|z-y| + r)^{N-p(N-1)} \, d||\mu||(y).$$

The integrand on the right hand side is a bounded function of y on the compact support of $||\mu||$ and thus the integral is finite. Hence $G \in L^p_{\text{loc}}(\mathbb{R}^N)$, which implies that is finite and well defined for a.e. x.
Now, since $|F(x)| \leq \frac{1}{N\omega_N} G(x)$, we see that $F \in L^p_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N) \subset L^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ for $1 \leq p < \frac{N}{N-1}$, which implies 2.
In order to prove 1, we recall that $\text{div}(\frac{x}{|x|^N}) = N\omega_N \delta$ in the sense of distributions (as shown in Example 2.2.1). So, $\forall \phi \in C^\infty_c(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} F(x) \cdot \nabla \phi(x) \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{N\omega_N} \frac{(x-y)}{|x-y|^N} \cdot \nabla \phi(x) \, d\mu(y) \, dx,$$

which equals, by Fubini's theorem,

$$\int_{\mathbb{R}^N} -\frac{1}{N\omega_N} \left(\frac{(\cdot)}{|\cdot|^N} * \nabla \phi \right)(y) \, d\mu(y) = \int_{\mathbb{R}^N} -(\delta * \phi)(y) \, d\mu(y) = -\int_{\mathbb{R}^N} \phi(y) \, d\mu(y),$$

which implies that $\text{div} F = \mu$.

Now let μ satisfy the hypothesis of point 3, and let $s \in (d, m)$.
Assume first that $p < \infty$. Writing $|x-y|^{1-N} = |x-y|^{-\frac{s}{q}} |x-y|^{\frac{s}{q}+1-N}$ and using Hölder's inequality we obtain

$$G(x)^p \leq \left(\int_{\mathbb{R}^N} |x-y|^{-s} \, d||\mu||(y) \right)^{\frac{p}{q}} \int_{\mathbb{R}^N} |x-y|^{p(\frac{s}{q}+1-N)} \, d||\mu||(y).$$

We now prove that $\int_{\mathbb{R}^N} |x-y|^{-s} \, d||\mu||(y)$ is bounded for every $x \in \mathbb{R}^N$. By the layer cake representation formula, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |x-y|^{-s} \, d||\mu||(y) &\leq \int_{B(x,a)} |x-y|^{-s} \, d||\mu||(y) + \int_{\mathbb{R}^N \setminus B(x,a)} |x-y|^{-s} \, d||\mu||(y) \\
&\leq \int_0^{+\infty} ||\mu||(\{y \in B(x,a) : |x-y|^{-s} > t\}) \, dt + \int_{\mathbb{R}^N \setminus B(x,a)} a^{-s} \, d||\mu||(y) \\
&\leq \int_0^{a^{-s}} ||\mu||(B(x,a)) \, dt + \int_{a^{-s}}^{+\infty} ||\mu||(B(x, t^{-\frac{1}{s}})) \, dt + a^{-s} ||\mu||(\mathbb{R}^N) \\
&\leq \frac{cma^{m-s}}{m-s} + a^{-s} ||\mu||(\mathbb{R}^N) =: C < \infty \quad \forall x \in \mathbb{R}^N.
\end{aligned}$$

Hence,

$$G(x)^p \leq C^{\frac{p}{q}} \int_{\mathbb{R}^N} |x-y|^{p(\frac{s}{q}+1-N)} \, d||\mu||(y).$$

Thus if $z \in \mathbb{R}^N$ and $r > 0$, we have, using $B(z,r) \subset B(y, |z-y|+r)$,

$$\int_{B(z,r)} G(x)^p \, dx \leq C^{\frac{p}{q}} \int_{\mathbb{R}^N} \int_{B(y,|z-y|+r)} |x-y|^{p(\frac{s}{q}+1-N)} \, dx \, d||\mu||(y).$$

By $s > d$, we have $N + p(\frac{s}{q} - N + 1) > 0$; thus the inner integral is finite and, as above,

$$\int_{B(y,|z-y|+r)} |x-y|^{p(\frac{s}{q}+1-N)} \, dx = \frac{N\omega_N}{N + p(\frac{s}{q} - N + 1)} (|z-y| + r)^{N+p(\frac{s}{q}-N+1)},$$

which is a bounded function of y on the compact support of $||\mu||$. Hence $G \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$.
If $p = \infty$, for $s \in (N-1, m)$ we have

$$\begin{aligned} G(x) &\leq \int_{\mathbb{R}^N} |x-y|^{-s} \, d||\mu||(y) \ \operatorname*{ess\,sup}_{y \in \mathrm{supp}(\mu)} |x-y|^{s-N+1} \\ &\leq C \operatorname*{ess\,sup}_{y \in \mathrm{supp}(\mu)} |x-y|^{s-N+1}, \end{aligned}$$

by the above steps. Now, $s - N + 1 > 0$ and hence $|x-y|^{s-N+1}$ is bounded on every compact of $\mathbb{R}^N \times \mathbb{R}^N$, which implies $G \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$. Since $|F(x)| \leq \frac{1}{N\omega_N} G(x)$, we have proved point 3. □

Proof of Proposition 2.3.2: In the case $1 \leq p < \frac{N}{N-1}$ it is an immediate consequence of Proposition 2.3.3.
If $\frac{N}{N-1} \leq p \leq \infty$, let $m \in [0, N]$, then (See [Fa], Corollary 4.12) there exists a compact set K such that $0 < \mathcal{H}^m(K) < \infty$ and, for some constant c,

$$\mathcal{H}^m(K \cap B(x,r)) \leq c r^m \quad \forall x \in \mathbb{R}^N, \, \forall r > 0.$$

Choose any $m \in (d, s)$ and let $\mu := \mathcal{H}^m \llcorner K$ and F as in (2.3.2). By Proposition 2.3.3, $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ and, since $m < s$ and $\mathcal{H}^m(K) < \infty$, we have $\mathcal{H}^s(K) = 0$ and thus $\mathrm{div} F$ is not $\mathcal{H}^s$-absolutely continuous. □

Remark 2.3.4. The vector field F introduced in Example 2.2.1 can be constructed as in Proposition 2.3.3 with $\mu = N\omega_N \delta$ and, since clearly we do not have $\delta(B(0,r)) \leq c r^m$ for $m > 0$, we can conclude that $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ $\forall \, 1 \leq p < \frac{N}{N-1}$.
Actually, this could also be checked just by calculating its L^p-norm over balls.

Another interesting consequence of Proposition 2.3.2 is that we cannot extend the Theorem 3.2.1 to vector fields in $\mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ for $1 \leq p < \frac{N}{N-1}$ in a trivial way; that is, just by substituting ∞ with p.

Indeed, suppose otherwise that for any bounded set of finite perimeter E there exists $\mathcal{F}_i \cdot \nu_E \in L^p(\partial^* E; \mathcal{H}^{N-1})$ which satisfies

$$\mathrm{div} F(E^1) = -\int_{\partial^* E} (\mathcal{F}_i \cdot \nu_E)(x)\, d\mathcal{H}^{N-1}(x)$$

and, for some constant C independent of E and F,

$$||\mathcal{F}_i \cdot \nu_E||_{L^p(\partial^* E; \mathcal{H}^{N-1})} \leq C ||F||_{L^p(E;\mathbb{R}^N)}. \tag{2.3.3}$$

We set $E = B(0, r)$ with $r < 1$.
Now, we choose again $F(x) = \frac{x}{|x|^N}$ and, since $E^1 = B(0, r)$ and $\partial^* E = \partial B(0, r)$, we have

$$\mathrm{div} F(B(0, r)) = N\omega_N = -\int_{\partial B(0,r)} (\mathcal{F}_i \cdot \nu_E)(x)\, d\mathcal{H}^{N-1}(x) \quad \forall r < 1.$$

Hence the Hölder inequality and (2.3.3) yield

$$\begin{aligned} N\omega_N &\leq ||\mathcal{F}_i \cdot \nu_E||_{L^p(\partial B(0,r); \mathcal{H}^{N-1})} (N\omega_N r^{N-1})^{1-\frac{1}{p}} \\ &\leq C ||F||_{L^p(B(0,r);\mathbb{R}^N)} (N\omega_N r^{N-1})^{1-\frac{1}{p}} \\ &= C \left(\int_{B(0,r)} \frac{1}{|x|^{(N-1)p}}\, dx \right)^{\frac{1}{p}} (N\omega_N r^{N-1})^{1-\frac{1}{p}} \\ &= C N\omega_N \left(\frac{1}{N - (N-1)p} \right)^{\frac{1}{p}} r^{\frac{N}{p} - (N-1) + (N-1)\left(1 - \frac{1}{p}\right)} \\ &= C_1 r^{\frac{1}{p}} \quad \forall r < 1. \end{aligned}$$

Thus we can send $r \to 0$ and this leads to a contradiction.

We further observe that it is actually possible to define a consistent interior normal trace for F on $\partial B(0, r)$ for any $r > 0$. Since F is continuous in $\mathbb{R}^N \setminus \{0\}$, we can define it pointwise as

$$(\mathcal{F}_i \cdot \nu_{B(0,r)})(x) := F(x) \cdot -\frac{x}{|x|} = -\frac{1}{|x|^{N-1}}$$

and it is imediate to see that it satisfies Gauss-Green formula on $B(0, r)$, since

$$\mathrm{div} F(B(0, r)) = N\omega_N$$

and

$$-\int_{\partial B(0,r)} (\mathcal{F}_i \cdot \nu_{B(0,r)})(x)\, d\mathcal{H}^{N-1}(x) = \int_{\partial B(0,r)} \frac{1}{r^{N-1}}\, d\mathcal{H}^{N-1}(x) = N\omega_N.$$

On the other hand, condition (2.3.3) fails to be satisfied: we would have

$$||\mathcal{F}_i \cdot \nu_{B(0,r)}||_{L^p(\partial B(0,r);\mathcal{H}^{N-1})} = (N\omega_N)^{\frac{1}{p}} \leq C(N,p) \left(\frac{N\omega_N}{N-(N-1)p} \right)^{\frac{1}{p}} r^{\frac{N}{p}-N+1}$$

which is false for r small enough, since $p < \frac{N}{N-1}$.

Remark 2.3.5. More generally, we can show that if $F \in \mathcal{DM}^p_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ an analogue to Theorems 3.2.1 and 3.2.2 for $p < \infty$ fails in general.
Indeed, we assert that we can always find a bounded set of finite perimeter E such that there do not exist interior and exterior normal traces $(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^1(\partial^* E; \mathcal{H}^{N-1})$ satisfying respectively

$$\int_{E^1} \phi \, d\text{div}F = -\int_{\partial^* E} \phi \mathcal{F}_i \cdot \nu_E \, d\mathcal{H}^{N-1} - \int_{E^1} \nabla\phi \cdot F \, dx \quad \forall \phi \in C_c^\infty(\mathbb{R}^N) \qquad (2.3.4)$$

and, recalling that $E = E^1 \cup \partial^m E$,

$$\int_{E} \phi \, d\text{div}F = -\int_{\partial^* E} \phi \mathcal{F}_e \cdot \nu_E \, d\mathcal{H}^{N-1} - \int_{E} \nabla\phi \cdot F \, dx \quad \forall \phi \in C_c^\infty(\mathbb{R}^N). \qquad (2.3.5)$$

Choose F such that $\text{div}F = \mathcal{H}^m \llcorner K$, where $0 < m < N-1$ and $K \subset B(0, \frac{1}{2}) \cap \{x_N = 0\}$, with the same property as in the proof of Proposition 2.3.2.
This implies that for any Borel set in A we have

$$\text{div}F(A) = \text{div}F(A \cap K) = \text{div}F(A \cap \{x_N = 0\}).$$

Then we take $E = B(0,1) \cap \{x_N > 0\}$ and we subtract (2.3.4) from (2.3.5), obtaining

$$\int_{\partial^*(B(0,1)\cap\{x_N>0\})} \phi \, d\text{div}F = -\int_{\partial^*(B(0,1)\cap\{x_N>0\})} \phi(\mathcal{F}_e \cdot \nu_E - \mathcal{F}_i \cdot \nu_E) \, d\mathcal{H}^{N-1} \quad (2.3.6)$$

since $|E \Delta E^1| = 0$.
We observe that

$$\partial^*(B(0,1) \cap \{x_N > 0\}) = (B(0,1) \cap \{x_N = 0\}) \cup (\partial B(0,1) \cap \{x_N > 0\}).$$

Since $\mathcal{H}^m(K) < \infty$, $\text{Cap}_{N-m}(K) = 0$ (see Theorem 1.2.3) and so, by Lemma 1.2.1, there exists a sequence $\phi_j \in C_c^\infty(\mathbb{R}^N)$ which satisfies $0 \leq \phi_j \leq 1$, $\phi_j = 1$ on K and $\phi_j(x) \to 0$ for all $x \in \mathbb{R}^N \setminus A$ for some set A with $\text{Cap}_{N-m}(A) = 0$.
We can write equation (2.3.6) for any ϕ_j and, since the measure $\text{div}F$ is supported in K, we have

$$\int_{\partial^*(B(0,1)\cap\{x_N>0\})} \phi_j \, d\text{div}F = \int_K d\text{div}F = \mathcal{H}^m(K) > 0.$$

On the other hand, we know that $\phi_j \to 0$ $\mathcal{H}^{N-1}$-a.e. since $\mathrm{Cap}_{N-m}(A) = 0$ implies $\mathcal{H}^s(A) = 0$ for any $s > m$ (see Theorem 1.2.3), hence in particular for $s = N - 1$. Thus we may apply Lebesgue's dominated convergence theorem to the right hand side of (2.3.6), since $0 \le \phi_j \le 1$ and $(\mathcal{F}_e \cdot \nu_E - \mathcal{F}_i \cdot \nu_E) \in L^1(\partial^*(B(0,1) \cap \{x_N > 0\}); \mathcal{H}^{N-1})$. So we obtain

$$\lim_{j \to +\infty} \int_{\partial^*(B(0,1)\cap\{x_N>0\})} \phi_j \, d\mathrm{div}F = \mathcal{H}^m(K)$$
$$= \lim_{j \to +\infty} - \int_{\partial^*(B(0,1)\cap\{x_N>0\})} \phi_j(\mathcal{F}_e \cdot \nu_E - \mathcal{F}_i \cdot \nu_E) \, d\mathcal{H}^{N-1} = 0,$$

which is absurd.

It is interesting to notice that what makes the Gauss-Green formula fail in this case was the possibility of having $\mathrm{div}F$ supported on a set of Hausdorff dimension strictly smaller than $N - 1$ which lies on the reduced boundary of a set of finite perimeter.
Indeed, it can been shown that it is possible to recover such formula also in the case $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N) \setminus \mathcal{DM}^\infty_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$ on bounded sets of finite perimeter E such that $||\mathrm{div}F||(\partial^m E) = 0$ (we actually need also another summability hypothesis on F, see [S], Theorem 4.6).

Remark 2.3.6. We observe that in the case $N = 1$ we have trivially $||\mathrm{div}F|| \ll \mathcal{H}^0$ and we cannot improve this result since if $F(x) = \chi_{(0,+\infty)}(x)$, then $F \in \mathcal{DM}^p_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$ for every $1 \le p \le \infty$ (actually $F \in BV_{\mathrm{loc}}(\mathbb{R})$) and $\mathrm{div}F = DF = \delta$, which is not absolutely continuous with respect to $\mathcal{H}^\alpha$ for any $\alpha > 0$.

2.4 Product Rules

Finally, we establish the following useful product rules.

Theorem 2.4.1. *Let $F \in \mathcal{DM}^p(\Omega; \mathbb{R}^N)$, $1 \le p < \infty$, and $g \in C(\Omega) \cap L^\infty(\Omega)$. Suppose also that the distributional derivatives of g satisfy: for each $j = 1, ..., N$, $\frac{\partial g}{\partial x_j} F_j \in L^1(\Omega)$ and the complement of the Lebesgue set of $\frac{\partial g}{\partial x_j}$ has measure zero with respect to the measure $|F_j|dx$.*
Then $gF \in \mathcal{DM}^p(\Omega; \mathbb{R}^N)$ and

$$\mathrm{div}(gF) = g\mathrm{div}F + \nabla g \cdot F \tag{2.4.1}$$

Proof. First, we notice that, by the boundedness of g, we have immediately that $gF \in L^p(\Omega; \mathbb{R}^N)$.
Then, for any $\phi \in C^\infty_c(\Omega)$, we have, by definition of the distributional derivative,

$$< \mathrm{div}(gF), \phi > = - < F, g\nabla(\phi) > = - < F, \nabla(g\phi) > + < F, \phi\nabla g > .$$

Therefore, it suffices to show that

$$< F, \nabla(g\phi) > = \ - < \mathrm{div}F, g\phi > .$$

We set $g_\delta = g * \rho_\delta$, where ρ is a standard symmetric mollifier, then we have

$$\begin{aligned} &- < \mathrm{div}F, g_\delta\phi > = < F, \nabla(g_\delta\phi) > \\ &= \sum_{j=1}^{N} < F_j, \frac{\partial(g_\delta\phi)}{\partial x_j} > \\ &= \sum_{j=1}^{N} < F_j, \phi\frac{\partial g_\delta}{\partial x_j} > + < F_j, g_\delta\frac{\partial\phi}{\partial x_j} > \end{aligned}$$

Now we let $\delta \to 0$, so, by Lebesgue's dominated convergence theorem,

$$< \mathrm{div}F, g_\delta\phi > \ \to \ < \mathrm{div}F, g\phi >$$

and

$$< F_j, g_\delta\frac{\partial\phi}{\partial x_j} > \ \to \ < F_j, g\frac{\partial\phi}{\partial x_j} >$$

for each j, while, by the assumption on the set of non-Lebesgue points of $\frac{\partial g}{\partial x_j}$,

$$< F_j, \phi\frac{\partial g_\delta}{\partial x_j} > \ \to \ < F_j, \phi\frac{\partial g}{\partial x_j} >$$

for each j. Thus, by Leibniz rule, we have (2.4.1) in the sense of distributions, and hence, by density, in the sense of Radon measures.
Since for each measurable set $A \subset \Omega$

$$||\mathrm{div}(gF)||(A) \le ||g||_\infty||\mathrm{div}F||(A) + \int_A \sum_{j=1}^{N} \left|\frac{\partial g}{\partial x_j}\right| |F_j| dx < \infty,$$

by the summability assumptions on $\frac{\partial g}{\partial x_j}$, $gF \in \mathcal{DM}^p(\Omega; \mathbb{R}^N)$, which gives the claim.
□

Now we provide a product rule for the case $p = \infty$, which we will need in order to establish generalizations of the Gauss-Green formula.

Theorem 2.4.2. *Let $g \in BV(\Omega) \cap L^{\infty}(\Omega)$ and $F \in \mathcal{DM}^{\infty}(\Omega; \mathbb{R}^N)$. Then $gF \in \mathcal{DM}^{\infty}(\Omega; \mathbb{R}^N)$.*

Moreover, if g is also locally Lipschitz, then, in the sense of Radon measures on Ω,

$$\mathrm{div}(gF) = g\mathrm{div}F + F \cdot \nabla g. \tag{2.4.2}$$

If g is not locally Lipschitz but with compact support, then we have

$$\mathrm{div}(gF) = g^{*}\mathrm{div}F + \overline{F \cdot Dg}\,,$$

where g^{*} is the precise representative of g (therefore, the limit of the mollified sequence g_{δ}) and $\overline{F \cdot Dg}$ is a Radon measure, which is the weak-star limit of $F \cdot \nabla g_{\delta}$ and is absolutely continuous with respect to $||Dg||$.

Proof. Let F_j be the sequence of smooth functions associated to F as in Theorem 2.1.3 and g_j the sequence of smooth functions associated to g as in Theorem 1.3.4.
We have

$$\int_{\Omega} |\mathrm{div}(g_j F_j)| dx = \sup \left\{ \int_{\Omega} g_j F_j \cdot \nabla \phi \, dx : \phi \in C_c^{\infty}(\Omega), ||\phi||_{\infty} \leq 1 \right\}$$

and $\int_{\Omega} g_j F_j \cdot \nabla \phi \, dx = \int_{\Omega} F_j \cdot \nabla (g_j \phi) \, dx - \int_{\Omega} \phi F_j \cdot \nabla g_j \, dx$. Now, by their definitions, $||F_j||_{\infty} \leq 3||F||_{\infty}$ and $||g_j||_{\infty} \leq 3||g||_{\infty}$ (see also (2.1.1)), therefore

$$\begin{aligned}\int_{\Omega} |\mathrm{div}(g_j F_j)| dx &\leq 3||g||_{\infty} \sup \left\{ \int_{\Omega} F_j \cdot \nabla \phi \, dx : \phi \in C_c^{\infty}(\Omega), ||\phi||_{\infty} \leq 1 \right\} + \\ &\quad + 3||F||_{\infty} \sup \left\{ \int_{\Omega} \nabla g_j \cdot \phi \, dx : \phi \in C_c^{\infty}(\Omega; \mathbb{R}^N), ||\phi||_{\infty} \leq 1 \right\} \\ &\leq 3(||g||_{\infty} ||\mathrm{div} F_j||(\Omega) + ||F||_{\infty} ||\nabla g_j||_{L^1(\Omega)})\end{aligned}$$

Hence, for any $\phi \in C_c^{\infty}(\Omega)$, $||\phi||_{\infty} \leq 1$, one has

$$\left| \int_{\Omega} gF \cdot \nabla \phi \, dx \right| = \lim_{j \to +\infty} \left| \int_{\Omega} g_j F_j \cdot \nabla \phi \, dx \right| \leq 3(||g||_{\infty} ||\mathrm{div} F||(\Omega) + ||F||_{\infty} ||\nabla g||_{L^1(\Omega)})$$

since, by property 1 in Theorems 1.3.4 and 2.1.3, $g_j \to g$ and $F_j \to F$ in $L^1(\Omega)$ (resp. $L^1(\Omega; \mathbb{R}^N)$) and so

$$\begin{aligned}&\left| \int_{\Omega} g_j F_j \cdot \nabla \phi \, dx - \int_{\Omega} gF \cdot \nabla \phi \, dx + \int_{\Omega} g_j F \cdot \nabla \phi \, dx - \int_{\Omega} g_j F \cdot \nabla \phi \, dx \right| \\ &\leq 3||\nabla \phi||_{L^{\infty}(\Omega;\mathbb{R}^N)} (||g||_{L^{\infty}(\Omega)} ||F - F_j||_{L^1(\Omega;\mathbb{R}^N)} + ||F||_{L^{\infty}(\Omega;\mathbb{R}^N)} ||g - g_j||_{L^1(\Omega)}),\end{aligned}$$

which gives us the desired convergence result.
Now, $gF \in L^{\infty}(\Omega; \mathbb{R}^N; \mathbb{R}^N)$, therefore $gF \in \mathcal{DM}^{\infty}(\Omega; \mathbb{R}^N)$.

By Remark 2.1.1, we have also $\mathrm{div} F_j \overset{*}{\rightharpoonup} \mathrm{div} F$ in $\mathcal{M}(\Omega)$. Hence, if g is locally Lipschitz, then, also by point 1 in Theorem 2.1.3,

$$g \mathrm{div} F_j + F_j \cdot \nabla g \overset{*}{\rightharpoonup} g \mathrm{div} F + F \cdot \nabla g \ \text{ in } \ \mathcal{M}(\Omega).$$

On the other hand, clearly $\mathrm{div}(gF_j) \overset{*}{\rightharpoonup} \mathrm{div}(gF)$ in the sense of distributions. Taking the limit in the identity

$$\mathrm{div}(gF_j) = g \mathrm{div} F_j + F_j \cdot \nabla g$$

in the sense of distributions and using the fact that $C_c^\infty(\Omega)$ is dense in $C_c(\Omega)$ with respect to the norm $||.||_\infty$, we obtain (2.4.2).

If g is not locally Lipschitz, but with compact support, let $g_\delta = g * \rho_\delta$ be the mollification of g, then formula (2.4.2) holds.
Now, it follows from Theorem 1.4.6 that $g_\delta \to g^* \ \mathcal{H}^{N-1}$-a.e. in Ω. Then, since by Corollary 2.3.1, $\mathrm{div} F \ll \mathcal{H}^{N-1}$,

$$g_\delta \mathrm{div} F \overset{*}{\rightharpoonup} g^* \mathrm{div} F \ \text{ in } \ \mathcal{M}(\Omega)$$

as a consequence of the dominated convergence theorem applied to the measure $||\mathrm{div} F||$.
Now we show that $\{\mathrm{div}(g_\delta F)\}$ is uniformly bounded in $\mathcal{M}(\Omega)$: by (2.4.2),

$$||\mathrm{div}(g_\delta F)||(\Omega) \leq ||g||_\infty ||\mathrm{div} F||(\Omega) + ||F||_\infty \sup_{\delta > 0} ||\nabla g_\delta||_{L^1(\Omega)} \tag{2.4.3}$$

and the supremum is bounded by $||Dg||(\Omega)$, by Remark 1.3.3.
By uniqueness of weak-star limits, $\mathrm{div}(g_\delta F) \overset{*}{\rightharpoonup} \mathrm{div}(gF)$, since this latter is the actual limit in the sense of distributions, and again we can argue with the density of $C_c^\infty(\Omega)$ in $C_c(\Omega)$. Then Remark 1.1.4 and the uniform boundedness of the sequence, (2.4.3), imply the weak-star convergence in $\mathcal{M}(\Omega)$.
Hence, $F \cdot \nabla g_\delta$ is weakly-star convergent and, by (2.4.2),

$$F \cdot \nabla g_\delta \overset{*}{\rightharpoonup} \overline{F \cdot Dg} = \mathrm{div}(gF) - g^* \mathrm{div} F.$$

Finally we treat the claim concerning $\overline{F \cdot Dg}$. Let $A \subset \Omega$ be a measurable set with $||Dg||(A) = 0$, we are going to show that $||\overline{F \cdot Dg}||(A) = 0$.
Since $\overline{F \cdot Dg}$ is a Radon measure on Ω, then its positive and negative parts $(\overline{F \cdot Dg})^+$ and $(\overline{F \cdot Dg})^-$ are well defined.
By the Hahn decomposition theorem, there exist Borel sets $A_\pm \subset A$ with $A_+ \cup A_- = A$ and $A_+ \cap A_- = \emptyset$ such that $\pm \overline{F \cdot Dg} \llcorner A_\pm \geq 0$; that is, $(\overline{F \cdot Dg})^+ \llcorner A = \overline{F \cdot Dg} \llcorner A_+$ and $(\overline{F \cdot Dg})^- \llcorner A = -\overline{F \cdot Dg} \llcorner A_-$.
Hence, it suffices to prove that $\overline{F \cdot Dg}(A_\pm) = 0$, and, in order to do so, it suffices to prove $\overline{F \cdot Dg}(K) = 0$ for any compact subset K of $A_\pm$, by Proposition 1.1.1.

We show only the case $K \subset A_+$, as the case of A_- is analogous.
Since $||Dg||(K) = 0$, for any $\epsilon > 0$ there exists an open set $V \supset K$ such that $||Dg||(V) < \epsilon$ (Proposition 1.1.1). Clearly, $\{B(x, r_x)\}_{x \in V}$ is an open cover of K, where we can choose $0 < r_x < \epsilon$ such that $B(x, r_x) \subset V$ for each $x \in V$ and $||Dg||(\partial B(x, r_x)) = 0$, since $B(x, r_x) \subset V$ for any $r_x < \rho_x$, for some $\rho_x > 0$, and therefore we apply Remark 1.1.2 to the family $\{B(x,t)\}_{0<t<\rho_x}$.
Since K is compact, we can extract a finite subcover of J of balls such that

$$K \subset \bigcup_{j=1}^{J} B(x_j, r_j), \quad r_j < \epsilon, \quad ||Dg|| \left(\bigcup_{j=1}^{J} B(x_j, r_j) \right) < \epsilon,$$

and $||Dg||(\partial B(x_j, r_j)) = 0$ for each j.
Let $\phi \in C_c(\bigcup_{j=1}^{J} B(x_j, r_j))$. Since, by Remark 1.3.3, $||\nabla g_\delta|| \stackrel{*}{\rightharpoonup} ||Dg||$, then

$$\begin{aligned}
| < \overline{F \cdot Dg}, \phi > | &= \lim_{\delta \to 0} \left| \int_\Omega \phi(x) F(x) \cdot \nabla g_\delta(x) \, dx \right| \\
&\leq ||\phi||_\infty ||F||_\infty \lim_{\delta \to 0} ||\nabla g_\delta||_{L^1(\bigcup_{j=1}^{J} B(x_j, r_j))} \\
&= ||\phi||_\infty ||F||_\infty ||Dg|| \left(\bigcup_{j=1}^{J} B(x_j, r_j) \right) < \epsilon ||\phi||_\infty ||F||_\infty
\end{aligned}$$

by point 2 of Lemma 1.1.2. We can choose $0 \leq \phi \leq 1$ such that $\phi = 1$ on K and $\left| \int_{\bigcup_{j=1}^{J} B(x_j, r_j) \setminus K} \phi \, d\overline{F \cdot Dg} \right| \leq C\epsilon$: for example, we can take $\phi = \chi_{K + \overline{B(0,\delta)}} * \rho_\delta$, where ρ is a standard symmetric mollifier and $\delta := \delta(\epsilon) > 0$ is small enough, in order to have $\phi \in C_c(\bigcup_{j=1}^{J} B(x_j, r_j))$. In this way

$$\left| \int_{\bigcup_{j=1}^{J} B(x_j, r_j) \setminus K} \phi \, d\overline{F \cdot Dg} \right| \leq ||\overline{F \cdot Dg}||((K + \overline{B(0, 2\delta)}) \setminus K) < C\epsilon$$

since $||\overline{F \cdot Dg}||((K + \overline{B(0, 2\delta)}) \setminus K) \to 0$ as $\delta \to 0$. Thus we obtain

$$\overline{F \cdot Dg}(K) = \left| \int_{\bigcup_{j=1}^{J} B(x_j, r_j)} \phi \, d\overline{F \cdot Dg} - \int_{\bigcup_{j=1}^{J} B(x_j, r_j) \setminus K} \phi \, d\overline{F \cdot Dg} \right| \leq \epsilon(||F||_\infty + C).$$

Since ϵ is arbitrary, this yields the desired result. □

Remark 2.4.1. In particular, this theorem is valid for $g = \chi_E$ for any $E \subset\subset \Omega$ of finite perimeter.

Remark 2.4.2. If $F \in BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$, then clearly $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$ and Theorem 2.4.2 is consistent with Proposition 1.5.1.

Indeed, for any $g \in BV(\Omega) \cap L^\infty(\Omega)$ with compact support, we have $gF \in BV(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ and, for any $j = 1, ..., N$,

$$D_j(gF_j) = g^* D_j F_j + F_j^* D_j g,$$

which implies

$$\mathrm{div}(gF) = g^* \mathrm{div} F + F^* \cdot Dg. \tag{2.4.4}$$

Now, we recall that $\overline{F \cdot Dg}$ is the weak-star limit of $F \cdot \nabla g_\delta$ as $\delta \to 0$, where g_δ is a mollification of g. For any $\phi \in C_c^1(\Omega)$, we see that

$$\begin{aligned}\int_\Omega \phi F \cdot \nabla g_\delta \, dx &= \int_\Omega F \cdot \nabla(\phi g_\delta) \, dx - \int_\Omega g_\delta F \cdot \nabla \phi \, dx \\ &= -\int_\Omega \phi g_\delta \, d\mathrm{div} F - \int_\Omega g_\delta F \cdot \nabla \phi \, dx.\end{aligned}$$

Since $\mathrm{div} F \ll \mathcal{H}^{N-1}$ (Corollary 2.3.1), $g_\delta \to g^*$ $\mathcal{H}^{N-1}$-a.e. (Theorems 1.4.5 and 1.4.6) and $g^* = g$ $\mathcal{L}^N$-a.e. (Theorem 1.1.2), we send $\delta \to 0$ in order to obtain

$$\begin{aligned}\int_\Omega \phi \, d\overline{F \cdot Dg} &= -\int_\Omega \phi g^* \, d\mathrm{div} F - \int_\Omega g F \cdot \nabla \phi \, dx \\ &= -\int_\Omega \phi g^* \, d\mathrm{div} F + \int_\Omega \phi \, d\mathrm{div}(Fg).\end{aligned}$$

Equation (2.4.4) yields

$$\int_\Omega \phi \, d\overline{F \cdot Dg} = -\int_\Omega \phi g^* \, d\mathrm{div} F + \int_\Omega \phi g^* \, d\mathrm{div} F + \int_\Omega \phi F^* \cdot \, dDg;$$

that is,

$$\int_\Omega \phi \, d\overline{F \cdot Dg} = \int_\Omega \phi F^* \cdot \, dDg \quad \forall \phi \in C_c^1(\Omega).$$

The density of $C_c^1(\Omega)$ in $C_c(\Omega)$ implies the identity $\overline{F \cdot Dg} = F^* \cdot Dg$ in $\mathcal{M}(\Omega)$, and hence the consistency of the two product rules.

Chapter 3

The Gauss-Green formula for $\mathcal{DM}^\infty$ fields

In this chapter we will provide two versions of the Gauss-Green formula for essentially bounded divergence-measure fields, obtained through different methods: the first one depends on a more geometrical approach, while the second on a measure-theoretical one. Indeed, the former relies on a property of C^1 compact manifolds which allows us to apply an approximation argument from an interior neighborhood. On the other hand, the latter is based on exploiting Leibniz rules (Thereom 2.4.2) and thus on finding identities between Radon measures.

3.1 Gauss-Green formula on bounded sets with regular boundary

In this section, we will prove the existence of the normal trace and the corresponding Gauss-Green formula for an essentially bounded divergence-measure field over any bounded set with C^1 boundary.
The method of proof of this theorem from [CTZ1] consists, roughly speaking, in approximating the boundary of the given set by a family of suitable surfaces for which the Gauss-Green theorem holds and then obtaining the desired trace as the density of the weak-star limit of measures concentrated over these approximating surfaces.
In this way, once defined a suitable notion of interior and exterior of a C^1 orientable manifold M, one can emphasize the fact that the normal trace related to Gauss-Green formula over M is indeed an interior normal trace in the sense that it is determined by the behavior of F in the interior of M.
As a first result, we prove the following lemma, which is already a version of the general form of divergence formula, but it requires two more hypothesis which can be removed.

Remark 3.1.1. We note that, since $F \in L^\infty_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$, then $F \in L^1_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$, so the precise representative F^* (Definition 1.1.6) of F is well defined on $\mathbb{R}^N$, equal to F $\mathcal{L}^N$-a.e. and $|F^*(x)| \leq ||F||_\infty$ $\forall x$. Therefore, we always are going to choose F^* as representative of the equivalence class of F in what follows, and we will denote this representative simply by F.

Lemma 3.1.1. *Let $F \in \mathcal{DM}^\infty_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ and F_ϵ be a mollification of F. Then the classical divergence theorem holds for F_ϵ on any bounded set of finite perimeter E; that is,*

$$\int_E \text{div} F_\epsilon(x)dx = -\int_{\partial^* E} F_\epsilon(x) \cdot \nu_E(x)\, d\mathcal{H}^{N-1}(x), \tag{3.1.1}$$

where ν_E is the measure theoretic unit interior normal. If in addition we assume

1. *$F_\epsilon \to F$ $\mathcal{H}^{N-1}$-a.e. on $\partial^* E$,*
2. *$||\text{div}F||(\partial E) = 0$,*

then

$$\text{div}F(E) = -\int_{\partial^* E} F(x) \cdot \nu_E(x)\, d\mathcal{H}^{N-1}(x). \tag{3.1.2}$$

Proof. By Theorem 1.5.1, we have that (3.1.1) holds, since F_ϵ is smooth, and so, in particular, locally Lipschitz.
Let W be a bounded open set such that $E \subset\subset W$. Since $\text{div}F_\epsilon \overset{*}{\rightharpoonup} \text{div}F$ and $||\text{div}F_\epsilon|| \overset{*}{\rightharpoonup} ||\text{div}F||$ in $\mathcal{M}(W)$ by Remark 2.1.2, then, by assumption 2 and Lemma 1.1.2, we can conclude that $\text{div}F_\epsilon(E) \to \text{div}F(E)$.
By the properties of the mollification, $|F_\epsilon(x)| \leq ||F||_{L^\infty(W)}$ $\forall x \in \overline{E}$ and ϵ small enough, so, by assumption 1 and the fact that $\mathcal{H}^{N-1}(\partial^* E) < \infty$, we can apply Lebesgue's dominated convergence theorem with respect to the measure $\mathcal{H}^{N-1}$ to find

$$\int_{\partial^* E} F_\epsilon(x) \cdot \nu_E(x)\, d\mathcal{H}^{N-1}(x) \to \int_{\partial^* E} F(x) \cdot \nu_E(x)\, d\mathcal{H}^{N-1}(x).$$

Thus, we pass to the limit for $\epsilon \to 0$ in identity (3.1.1) and we obtain (3.1.2). □

Conditions 1 and 2 are those we are going to get rid of, by showing that they are always satisfied on almost every C^1 surface approximating the manifold M. Now we give a definition of what shall be called interior or exterior determined by a compact C^1 manifold.

Definition 3.1.1. Let M be a compact C^1 manifold of dimension $N-1$.

1. We define the *exterior determined by M* to be the connected component $\mathcal{U}$ of $\mathbb{R}^N \setminus M$ that is unbounded.

The *interior I determined by M* is defined to be everything else in the complement of M; namely,

$$I = \bigcup_{k=1}^{\infty} B_k \quad \text{with} \quad B_k \subset \mathbb{R}^N \setminus M \quad \text{a bounded component.}$$

Thus,

$$\mathbb{R}^N \setminus M = \mathcal{U} \cup \left(\bigcup_{k=1}^{\infty} B_k \right) = \mathcal{U} \cup I.$$

2. We say that M is *orientable* with respect to $\mathcal{U}$ if on every conncected component there is a unique interior normal well defined; that is, exactly one unit vector normal to the manifold such that its opposite points towards $\mathcal{U}$.

We observe that in such a way we allow compact manifolds which are not connected, and at same time we discard coonected component of M which are compact manifolds with boundary. Indeed, since the interior determined by the components with boundary will be empty, they cannot be orientable with respect to $\mathcal{U}$.
As an example we can consider the cylinder $M = \{x \in \mathbb{R}^{N-1} : |x| = 1\} \times [0,1]$: $\mathcal{U} = \mathbb{R}^N \setminus M$ and $I = \emptyset$.

Theorem 3.1.1. (Gauss-Green formula)
Let $I \subset \mathbb{R}^N$ be the interior determined by a compact C^1 manifold M of dimension $N-1$ with $\mathcal{H}^{N-1}(M) < \infty$. Then, for any $F \in \mathcal{DM}^\infty_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N;\mathbb{R}^N)$, there exists a signed Radon measure σ on $\partial I = M$ with $\sigma \ll \mathcal{H}^{N-1} \llcorner I$ and a function $\mathcal{F}_i \cdot \nu : \partial I \to \mathbb{R}$, which we shall denote as interior normal trace of F on ∂I, such that, for any $\phi \in C^1_c(\mathbb{R}^N)$,

$$\int_I \operatorname{ddiv}(\phi F) = \int_I \phi \, \operatorname{ddiv} F + \int_I F \cdot \nabla\phi \, dx = -\int_{\partial I} \phi \, d\sigma = -\int_{\partial I} \phi(\mathcal{F}_i \cdot \nu) \, d\mathcal{H}^{N-1} \tag{3.1.3}$$

and

$$||\mathcal{F}_i \cdot \nu||_{L^\infty(\partial I;\mathcal{H}^{N-1})} \leq C||F||_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)}, \tag{3.1.4}$$

where $C = C(N, I)$.

Before we prove this theorem, we will state a version of a theorem due to Whitney adapted to our situation.

Theorem 3.1.2. (Whitney)
Let M and I be as in Definition 3.1.1. Let ν be the interior normal and $\alpha > 0$. Then there exists a unit C^1 vector field $\Lambda^* : M \to \mathbb{R}^N$ and a number $\delta = \delta(\alpha) \in (0,1)$ with the following properties.

1. *If $\pi_p : \mathbb{R}^N \to T_p(M)$ is the orthogonal projection onto the tangent space of M at p, then $|\pi_p(\Lambda^*(p))| \leq \alpha$. Thus, $\Lambda^*(p)$ is close to $\nu(p)$ when α is small, and*

$$S_p^* := \{q \in \mathbb{R}^N : q = p + t\Lambda^*(p), 0 < t < \delta\} \subset I.$$

2. *As p ranges over M, the segments S_p^* fill up an open interior neighborhood U_δ^* of M in a one-to-one way; that is,*

$$U_\delta^* = \bigcup_{p \in M} S_p^*,$$

where $S_{p_1}^* \cap S_{p_2}^* = \emptyset$ whenever $p_1, p_2 \in M$ with $p_1 \neq p_2$.

3. *The mapping $\pi^* : U_\delta^* \to M$ defined by*

$$\pi^*(q) := p \quad \text{if} \quad q \in S_p^*$$

is of class C^1 and has the property that

$$\psi^*(q) := |\pi^*(q) - q| \leq 2\,\mathrm{dist}(q, M) \quad \text{for} \quad q \in U_\delta^*.$$

Proof. See [W], Theorem 10A, p. 121.

Remark 3.1.2. This theorem is needed to produce an inward pointing vector at each point of M such that the vectors corresponding to different points in M do not intersect. When M is of class C^2, the interior normal themselves satisfy this property in a sufficiently small open interior neighborhood of M, so in this case we may take ν in place of Λ^*.

Proof of Theorem 3.1.1
We divide the proof into ten steps.

1. *Preliminaries*
   The number $\psi^*(q)$ is the distance from q to M, measured along S_p^*, where $\pi^*(q) = p$.
   The open sets $I_t := I \setminus \{q \in U_\delta^* : \psi^*(q) < t\}$ for $0 < t < \delta$ are nested, contained in I and $\bigcup_{t>0} I_t = I$.
   Since ψ^* is continuous, we have $\partial I_t \subset (\psi^*)^{-1}(t)$ for all $t \in (0, \delta)$, with equality holding whenever t is not a critical value of ψ^*.

2. $\forall t \in (0, \delta)$, *$\partial I_t$ is a C^1 compact manifold, there exists a constant $C(\partial I, N)$ independent of t such that*

$$\mathcal{H}^{N-1}(\partial I_t) \leq C(\partial I, N)\mathcal{H}^{N-1}(\partial I) \tag{3.1.5}$$

and I_t is a set of finite perimeter.

The set ∂I_t may be considered as a deformation of $M = \partial I$ along the vector field Λ^*. In order to show this, we consider the mapping $h_t : \partial I \to \partial I_t$ defined for $t \in (0, \delta)$ as

$$q = h_t(p) := p + t\Lambda^*(p)$$

with $h_0(p) = p$, so that $\pi^*(q) = p$ with $|h_t(p) - p| = t$ and $h_t(\partial I) = \partial I_t$.
By Theorem 3.1.2, $h_t \in C^1(M)$, with the Jacobian Jh_t depending only on t and $||D\Lambda^*||_{L^\infty(M)}$. Therefore, since h is injective, we may use co-area formula to conclude that, for any $A \subset M$ which is $\mathcal{H}^{N-1}$-measurable,

$$\mathcal{H}^{N-1}(h_t(A)) = \int_A Jh_t(x)\, d\mathcal{H}^{N-1}(x) \le C(\delta, ||D\Lambda^*||_\infty)\mathcal{H}^{N-1}(A), \qquad (3.1.6)$$

which implies (3.1.5) for $A = M = \partial I$.
Since $\pi^* \circ h_t = \mathrm{Id}$, the chain rules implies that Jh_t is nonsingular everywhere on M, therefore, it is a diffeomorphism and hence ∂I_t is an $(N-1)$-manifold of class C^1, which is also compact, being image of a compact set through a continuous function.
We also observe that this implies that I_t is a set of finite perimeter $\forall t \in (0, \delta)$, since, $\forall \phi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$, $||\phi||_\infty \le 1$,

$$\left| \int_{I_t} \mathrm{div}\phi\, dx \right| = \left| \int_{\partial I_t} \phi \cdot \nu_t\, d\mathcal{H}^{N-1} \right| \le \mathcal{H}^{N-1}(\partial I_t) < \infty$$

by (3.1.5).

3. *For $\mathcal{L}^1$-a.e. $t \in (0, \delta)$, conditions 1 and 2 in Lemma 3.1.1 hold for $E = I_t$*
First, we know that $F_\epsilon(x) \to F(x)$ for $\mathcal{L}^N$-a.e x, so there exists a set $A \subset \mathbb{R}^N$ with $|A| = 0$ such that we have pointwise convergence $\forall x \notin A$. By Lemma 1.4.2, since ψ^* is a differentiable function on U_δ^*, we have

$$\mathcal{H}^{N-1}((\psi^*)^{-1}(t) \cap (A \cap U_\delta^*)) = 0 \ \text{ for } \ \mathcal{L}^1\text{-a.e.} t \in (0, \delta).$$

Since $\partial I_t \subset (\psi^*)^{-1}(t)$, this implies condition 1 for $t \notin S$, for some S with $\mathcal{L}^1(S) = 0$.
Then, we observe that, by Remark 1.1.2, there exists a set $Z \subset (0, \delta)$ such that $\mathcal{L}^1(Z) = 0$ and $||\mathrm{div}F||(\partial I_t) = 0$ $\forall t \notin Z$, since clearly the sets I_t satisfy the required hypotheses.
Therefore, conditions 1 and 2 hold $\forall t \in (0, \delta) \setminus (Z \cup S)$. From now on we will always consider t in this set.

4. *The signed measures defined by*

$$\sigma_t(B) := \int_{B \cap \partial I_t} F(x) \cdot \nu(x)\, d\mathcal{H}^{N-1}(x) \ \ \textit{for each Borel set } \ B \subset \mathbb{R}^N$$

along with their positive and negative parts σ_t^+ and σ_t^-, where $\sigma_t = \sigma_t^+ - \sigma_t^-$, and their total variation measures $||\sigma_t||$ are all weak-star converging for a suitable subsequence $t_k \to 0$:

$$(\sigma_{t_k}^+, \sigma_{t_k}^-, \sigma_{t_k}, ||\sigma_{t_k}||) \overset{*}{\rightharpoonup} (\sigma^+, \sigma^-, \sigma, ||\sigma||) \;\; in \;\; \mathcal{M}(\mathbb{R}^N). \tag{3.1.7}$$

By Remark 3.1.1, we have, for any Borel set B,

$$|\sigma_t|(B) \leq ||F||_\infty \mathcal{H}^{N-1}(\partial I_t \cap B).$$

Therefore, by Definition 1.1.2 and (3.1.5), the total variation norm of σ_t satisfies

$$\begin{aligned}
|||\sigma_t||| &= \sup \left\{ \sum_{k=0}^{+\infty} |\sigma_t(E_k)| : \{E_k\} \text{ Borel sets partitioning } \mathbb{R}^N \right\} \\
&\leq ||F||_\infty \sup \left\{ \sum_{k=0}^{+\infty} \mathcal{H}^{N-1}(\partial I_t \cap E_k) : \{E_k\} \text{ Borel sets partitioning } \mathbb{R}^N \right\} \\
&= ||F||_\infty \mathcal{H}^{N-1}(\partial I_t) \leq C||F||_\infty \mathcal{H}^{N-1}(\partial I) \;\; \forall t.
\end{aligned}$$

So $\{\sigma_t\}_{t>0}$ and $\{||\sigma_t||\}_{t>0}$ are bounded sets in $\mathcal{M}(\mathbb{R}^N)$, and also $\{\sigma_t^\pm\}_{t>0}$, since $||\sigma_t|| = \sigma_t^+ + \sigma_t^-$. Hence, by Banach-Alaoglu theorem, there exists a sequence $t_k \to 0$ and Radon measures $\sigma, \sigma^+, \sigma^-, ||\sigma||$ with $\sigma = \sigma^+ - \sigma^-$ and $||\sigma|| = \sigma^+ + \sigma^-$ such that (3.1.7) holds.

5. *The supports of the measures $\sigma, \sigma^+, \sigma^-$ are all contained in ∂I.*
It is enough to prove this for σ^+, since the other two cases are analogous.
By contradiction, let $x \in \text{supp}(\sigma^+) \setminus \partial I$ and choose $r > 0$ such that $B(x,r) \cap \partial I = \emptyset$. By the definition of the support of a Radon measure, there exists $\phi \in C_c(B(x,r))$ such that $\int_{B(x,r)} \phi \, d\sigma^+ \neq 0$.
By the weak-star convergence, $\int_{B(x,r)} \phi \, d\sigma_{t_k}^+ \to \int_{B(x,r)} \phi \, d\sigma^+ \neq 0$, and this implies that there exists a k_0 such that, $\forall k \geq k_0$, $\int_{B(x,r)} \phi \, d\sigma_{t_k}^+ \neq 0$.
This leads to a contradiction, since $I_{t_k} \subset I$, $\text{supp}(\sigma_{t_k}) \subset \partial I_{t_k}$, and $\partial I_{t_k} \cap B(x,r) = \emptyset$, for t_k small enough.

6. *We have*

$$\lim_{t_k \to 0} (\sigma_{t_k}^+, \sigma_{t_k}^-, \sigma_{t_k})(\partial I_{t_k}) = (\sigma^+, \sigma^-, \sigma)(\partial I). \tag{3.1.8}$$

Since $\sigma_{t_k}^\pm$ are positive Radon measures and their supports are in ∂I_{t_k}, Lemma 1.1.2 yields

$$\liminf_{t_k \to 0} \sigma_{t_k}^\pm(\partial I_k) = \liminf_{t_k \to 0} \sigma_{t_k}^\pm(\mathbb{R}^N) \geq \sigma^\pm(\mathbb{R}^N) = \sigma^\pm(\partial I).$$

Now we choose a compact set $K \supset \partial I \cup \partial I_{t_k}$ and then, by the previous step and Lemma 1.1.2, we have

$$\limsup_{t_k\to 0} \sigma^{\pm}_{t_k}(\partial I_{t_k}) = \limsup_{t_k\to 0} \sigma^{\pm}_{t_k}(K) \le \sigma^{\pm}(K) = \sigma^{\pm}(\partial I).$$

Combining these two inequalities, we obtain (3.1.8) for σ^+ and σ^-. Since $\sigma = \sigma^+ - \sigma^-$, we have the result for σ as well.

7. *The measure σ is well defined.*
Let $I_{t'_k}$ be another sequence of open sets for which Lemma 3.1.1 applies. Moreover, we can choose it in such a way that $t_k > t'_k$ for all k. Then we have

$$\begin{aligned}\mathrm{div}F(I_{t'_k} \setminus I_{t_k}) &= -\int_{\partial I_{t'_k}} F(x)\cdot\nu(x)\, d\mathcal{H}^{N-1}(x) + \int_{\partial I_{t_k}} F(x)\cdot\nu(x)\, d\mathcal{H}^{N-1}(x) \\ &= -\sigma_{t'_k}(\partial I_{t'_k}) + \sigma_{t_k}(\partial I_{t_k}).\end{aligned}$$

Since $I_{t'_k} \setminus I_{t_k} \subset I \setminus I_{t_k}$ is a monotone decreasing sequence of sets and

$$\bigcap_{k\ge 1} I \setminus I_{t_k} = \emptyset,$$

it follows that $||\mathrm{div}F||(I_{t'_k} \setminus I_{t_k}) \to 0$ and therefore that

$$\sigma_{t'_k}(\partial I_{t'_k}) - \sigma_{t_k}(\partial I_{t_k}) \to 0,$$

which shows that σ is well defined, since it does not depend on the particular subsequence chosen.

8. $||\sigma|| \ll \mathcal{H}^{N-1}\llcorner\partial I$.
Let $A \subset \partial I$ with $\mathcal{H}^{N-1}(A) = 0$. By Theorem 1.4.2, $||D\chi_I|| = \mathcal{H}^{N-1}\llcorner\partial I$ and hence $||D\chi_I||(A) = 0$.
Since $0 = ||D\chi_I||(A) = \inf\{||D\chi_I||(G) : A \subset G, G \text{ open}\}$, for each $\epsilon > 0$, there exists an open set $G \supset A$ such that $\mathcal{H}^{N-1}(G \cap \partial I) < \epsilon$.
Moreover, using (3.1.6), we obtain

$$\begin{aligned}||\sigma_{t_k}||(G) &\le \int_{G\cap\partial I_{t_k}} |F(x)\cdot\nu(x)|\, d\mathcal{H}^{N-1}(x) \le ||F||_\infty \mathcal{H}^{N-1}(G\cap\partial I_{t_k}) \\ &\le C(\partial I, N)||F||_\infty \mathcal{H}^{N-1}(h^{-1}_{t_k}(G\cap\partial I_{t_k})).\end{aligned}$$

By Lemma 1.1.2 and the continuity of $h^{-1}_{t_k} = \pi^*$, we have

$$\begin{aligned}||\sigma||(A) \le ||\sigma||(G) &\le \liminf_{t_k\to 0} ||\sigma_{t_k}||(G) \\ &\le C||F||_\infty \lim_{t_k\to 0} \mathcal{H}^{N-1}(h^{-1}_{t_k}(G\cap\partial I_{t_k})) \\ &= C||F||_\infty \mathcal{H}^{N-1}(G\cap\partial I) < \epsilon C||F||_\infty.\end{aligned}$$

Since ϵ is arbitrary, we can conclude that $||\sigma||(A) = 0$, as desired.

9. *We prove now* (3.1.3).
Since $F \in \mathcal{DM}^\infty_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$, then $F \in \mathcal{DM}^\infty(W;\mathbb{R}^N)$ for some bounded open set W such that $\overline{I} \subset\subset W$. By Theorem 2.4.2, for any $\phi \in C^1_c(\mathbb{R}^N)$ one has $\phi F \in \mathcal{DM}^\infty(W;\mathbb{R}^N)$, since clearly $\phi \in BV(W) \cap L^\infty(W)$ and it is locally Lipschitz, and
$$\text{div}(\phi F) = \phi \text{div} F + F \cdot \nabla\phi.$$
By Lemma 3.1.1, we have
$$\text{div}(\phi F)(I_{t_k}) = -\int_{\partial I_{t_k}} \phi(x)F(x)\cdot\nu(x)\,d\mathcal{H}^{N-1}(x) \quad \text{for} \quad t_k \in (0,\delta)\setminus(S\cup Z).$$
Since the sets I_{t_k} are nested and increasing as $t_k \to 0$, $I_{t_k} = \bigcup_{t_k \le t < \delta} I_t$ and $I = \bigcup_{0<t<\delta} I_t$, hence
$$\lim_{k\to+\infty} \text{div}(\phi F)(I_{t_k}) = \lim_{k\to+\infty} \text{div}(\phi F)(\bigcup_{t_k\le t<\delta} I_t) = \text{div}(\phi F)(I).$$
Thus, by step 4, if we let $k \to +\infty$, we obtain
$$\text{div}(\phi F)(I) = -\int_{\partial I} \phi\, d\sigma. \tag{3.1.9}$$

10. *The Radon-Nykodim derivative of σ with respect to $\mathcal{H}^{N-1} \llcorner \partial I$ is a function $\mathcal{F}_i\cdot\nu \in L^\infty(\partial I;\mathcal{H}^{N-1})$ such that* (3.1.4) *holds.*
Since $||\sigma|| \ll \mathcal{H}^{N-1}\llcorner\partial I$, Radon-Nykodim theorem implies that there exists $\mathcal{F}_i\cdot\nu \in L^1(\partial I;\mathcal{H}^{N-1})$ such that (3.1.9) can be written as (3.1.3).
By the Lebesgue-Besicovitch differentiation theorem (Theorem 1.1.2), we know that for $\mathcal{H}^{N-1}$-$x \in \partial I$ one has
$$|(\mathcal{F}_i\cdot\nu)(x)| = \lim_{r\to0}\left|\frac{\sigma(B(x,r))}{\mathcal{H}^{N-1}(\partial I\cap B(x,r))}\right|.$$
Finally, a sequence of balls $B(x,r_j)$ with $r_j \to 0$ can be chosen in such a way that $||\sigma||(\partial B(x,r_j)) = 0$, by Remark 1.1.2, in order to have, by Lemma 1.1.2,
$$\begin{aligned}
|(\mathcal{F}_i\cdot\nu)(x)| &= \lim_{r_j\to0}\lim_{t_k\to0}\left|\frac{\sigma(B(x,r_j))}{\mathcal{H}^{N-1}(\partial I\cap B(x,r_j))}\right| \\
&= \lim_{r_j\to0}\lim_{t_k\to0}\left|\frac{\int_{\partial I_{t_k}\cap B(x,r_j)} F(x)\cdot\nu(x)\,d\mathcal{H}^{N-1}(x)}{\mathcal{H}^{N-1}(\partial I\cap B(x,r_j))}\right| \\
&\le ||F||_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)} \lim_{r_j\to0}\lim_{t_k\to0}\frac{\mathcal{H}^{N-1}(\partial I_{t_k}\cap B(x,r_j))}{\mathcal{H}^{N-1}(\partial I\cap B(x,r_j))} \\
&\le C(\partial I,N)||F||_\infty \lim_{r_j\to0}\frac{\mathcal{H}^{N-1}(\partial I\cap B(x,r_j))}{\mathcal{H}^{N-1}(\partial I\cap B(x,r_j))} \\
&= C(\partial I,N)||F||_\infty
\end{aligned}$$

with the last inequality coming from (3.1.6). □

3.2 Gauss-Green formula on bounded sets of finite perimeter

We now establish a version of the Gauss-Green formula for $\mathcal{DM}^\infty(\Omega;\mathbb{R}^N)$ fields on bounded sets of finite perimeter. The method is analogous to the one Vol'pert used in order to prove Theorem 1.5.2 and it is based on the product rule established in the paper of Chen and Torres ([CT]). The results are similar to those presented in the paper of Chen, Torres and Ziemer ([CTZ]), but here we are not using their theory concerning the approximation of sets of finite perimeter by sets with smooth boundary.
We recall that for a bounded set of finite perimeter E we select the representative $E = E^1 \cup \partial^m E$.
We begin with the following result concerning fields with compact support, similar to Lemma 1.5.1.

Lemma 3.2.1. *If $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^N)$ has compact support in Ω, then*

$$\mathrm{div}F(\Omega) = 0.$$

Proof. Since F has compact support, we can extend it to

$$\hat{F}(x) = \begin{cases} F(x) & \text{if } \ x \in \Omega \\ 0 & \text{if } \ x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

and $\hat{F} \in \mathcal{DM}^\infty(\mathbb{R}^N;\mathbb{R}^N)$, by Remark 2.1.2.
With a little abuse of notation, we will denote this extension again by F.
So, $F = 0$ on $\mathbb{R}^N \setminus \Omega$. In particular, this implies that $||\mathrm{div}F||(A) = 0$ for each open set $A \subset \mathbb{R}^N \setminus \Omega$: indeed

$$0 = \int_{\mathbb{R}^N} F \cdot \nabla\phi \, dx = -\int_{\mathbb{R}^N} \phi \, d\mathrm{div}F \quad \forall \phi \in C_c^\infty(A)$$

and $||\mathrm{div}F||(A)$ is the supremum of these integrals over $\phi \in C_c^\infty(A)$ with $||\phi||_\infty \leq 1$, by Proposition 1.1.2. By the properties of positive Radon measures (Proposition 1.1.1), this implies $||\mathrm{div}F||(B) = 0$ for any Borel set $B \subset \mathbb{R}^N \setminus \Omega$.
We set $\Omega_k := \{x \in \mathbb{R}^N : k > \mathrm{dist}(x, \overline{\Omega}) \geq k-1\}$ for $k \geq 2$ and $\Omega_1 := \{x \in \mathbb{R}^N : 1 > \mathrm{dist}(x, \overline{\Omega}) \geq 0\} \setminus \Omega$. Then, $||\mathrm{div}F||(\mathbb{R}^N \setminus \Omega) = 0$ since $\mathbb{R}^N \setminus \Omega = \bigcup_{k=1}^{+\infty} \Omega_k$ and each one of these sets has $||\mathrm{div}F||$-measure zero.
Now let $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi = 1$ on $\Omega_1 \cup \Omega$.
Then it is clear that

$$\int_{\mathbb{R}^N} \phi \, d\mathrm{div}F = \int_{\mathbb{R}^N} d\mathrm{div}F$$

and, by the definition of the distributional derivative,

$$\int_{\mathbb{R}^N} \phi \, d\mathrm{div} F = -\int_{\mathbb{R}^N} F \cdot \nabla\phi \, dx = -\int_{\mathbb{R}^N\setminus(\Omega_1\cup\Omega)} F \cdot \nabla\phi \, dx = 0,$$

since F has support inside Ω. Thus $\mathrm{div}F(\mathbb{R}^N) = 0$, which implies $\mathrm{div}F(\Omega) = 0$. □

Theorem 3.2.1. (The Gauss-Green formula) *Let $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^N)$. If $E \subset\subset \Omega$ is a bounded set of finite perimeter, then there exist interior and exterior normal traces of F on $\partial^* E$; that is, $(\mathcal{F}_i \cdot \nu_E), (\mathcal{F}_e \cdot \nu_E) \in L^\infty(\partial^* E; \mathcal{H}^{N-1})$ such that*

$$\mathrm{div}F(E^1) = -2\overline{\chi_E F \cdot D\chi_E}(\partial^* E) = -\int_{\partial^* E} \mathcal{F}_i \cdot \nu_E \, d\mathcal{H}^{N-1}$$

and

$$\mathrm{div}F(E) = -2\overline{\chi_{E^0} F \cdot D\chi_E}(\partial^* E) = -\int_{\partial^* E} \mathcal{F}_e \cdot \nu_E \, d\mathcal{H}^{N-1},$$

where $\overline{\chi_E F \cdot D\chi_E}$ and $\overline{\chi_{E^0} F \cdot D\chi_E}$ are the weak star limits, respectively, of the sequences $\chi_E F \cdot \nabla(\chi_E * \rho_\delta)$ and $\chi_{E^0} F \cdot \nabla(\chi_E * \rho_\delta)$ as $\delta \to 0$, up to a subsequence. Moreover,

$$||\mathcal{F}_i \cdot \nu_E||_{L^\infty(\partial^* E;\mathcal{H}^{N-1})} \leq ||F||_{L^\infty(E^1;\mathbb{R}^N)}$$

and

$$||\mathcal{F}_e \cdot \nu_E||_{L^\infty(\partial^* E;\mathcal{H}^{N-1})} \leq ||F||_{L^\infty(\Omega\setminus E;\mathbb{R}^N)}.$$

Proof. By Theorem 2.4.2, it follows that

$$\begin{aligned}\mathrm{div}(\chi_E^2 F) &= \mathrm{div}(\chi_E(\chi_E F)) = \chi_E^* \mathrm{div}(\chi_E F) + \overline{\chi_E F \cdot D\chi_E} \\ &= \chi_E^*(\chi_E^* \mathrm{div}F + \overline{F \cdot D\chi_E}) + \overline{\chi_E F \cdot D\chi_E} \\ &= (\chi_E^*)^2 \mathrm{div}F + \chi_E^* \overline{F \cdot D\chi_E} + \overline{\chi_E F \cdot D\chi_E}, \end{aligned} \tag{3.2.1}$$

where χ_E^* is the precise representative of χ_E.
On the other hand,

$$\mathrm{div}(\chi_E^2 F) = \mathrm{div}(\chi_E F) = \chi_E^* \mathrm{div}F + \overline{F \cdot D\chi_E}. \tag{3.2.2}$$

Combining (3.2.1) with (3.2.2) yields

$$((\chi_E^*)^2 - \chi_E^*)\mathrm{div}F + \chi_E^* \overline{F \cdot D\chi_E} + \overline{\chi_E F \cdot D\chi_E} - \overline{F \cdot D\chi_E} = 0.$$

On the other hand, $\mathrm{div}F \ll \mathcal{H}^{N-1}$ by Corollary 2.3.1 and so Lemma 1.4.1 yields

$$((\chi_E^*)^2 - \chi_E^*)\mathrm{div}F = -\frac{1}{4}\chi_{\partial^* E}\mathrm{div}F. \tag{3.2.3}$$

By Theorem 2.4.2, $\overline{F\cdot D\chi_E} \ll ||D\chi_E||$ and $\overline{\chi_E F\cdot D\chi_E} \ll ||D\chi_E||$, therefore these two measures are supported on $\partial^* E$. In particular this implies that $\chi_E^*\overline{F\cdot D\chi_E} = \frac{1}{2}\overline{F\cdot D\chi_E}$.
From this fact and (3.2.3) we obtain

$$\frac{1}{2}\chi_{\partial^* E}\mathrm{div}F + \overline{F\cdot D\chi_E} - 2\overline{\chi_E F\cdot D\chi_E} = 0. \tag{3.2.4}$$

Therefore, if we subtract (3.2.4) from (3.2.2) we have

$$\begin{aligned}\mathrm{div}(\chi_E F) = \chi_{E^1}\mathrm{div}F + \frac{1}{2}\chi_{\partial^* E}\mathrm{div}F + \overline{F\cdot D\chi_E} - \frac{1}{2}\chi_{\partial^* E}\mathrm{div}F +\\ - \overline{F\cdot D\chi_E} + 2\overline{\chi_E F\cdot D\chi_E} = \chi_{E^1}\mathrm{div}F + 2\overline{\chi_E F\cdot D\chi_E}.\end{aligned}$$

On the other hand, if we add (3.2.4) to (3.2.2) we have

$$\begin{aligned}\mathrm{div}(\chi_E F) = \chi_{E^1}\mathrm{div}F + \frac{1}{2}\chi_{\partial^* E}\mathrm{div}F + \overline{F\cdot D\chi_E} + \frac{1}{2}\chi_{\partial^* E}\mathrm{div}F +\\ + \overline{F\cdot D\chi_E} - 2\overline{\chi_E F\cdot D\chi_E} = \chi_E\mathrm{div}F + 2\overline{F\cdot D\chi_E} - 2\overline{\chi_E F\cdot D\chi_E}.\end{aligned}$$

We also observe that $\overline{F\cdot D\chi_E} - \overline{\chi_E F\cdot D\chi_E}$ is the weak-star limit of the sequence

$$F\cdot\nabla(\chi_E * \rho_\delta) - \chi_E F\cdot\nabla(\chi_E*\rho_\delta) = (1-\chi_E)F\cdot\nabla(\chi_E*\rho_\delta) = \chi_{E^0}F\cdot\nabla(\chi_E*\rho_\delta)$$

and so $\overline{F\cdot D\chi_E} - \overline{\chi_E F\cdot D\chi_E} = \overline{\chi_{E^0}F\cdot D\chi_E}$[1].

Thus, we have found

$$\mathrm{div}(\chi_E F) = \chi_{E^1}\mathrm{div}F + 2\overline{\chi_E F\cdot D\chi_E}. \tag{3.2.5}$$

and

$$\mathrm{div}(\chi_E F) = \chi_E\mathrm{div}F + 2\overline{\chi_{E^0}F\cdot D\chi_E}. \tag{3.2.6}$$

Since $\chi_E F$ clearly has compact support in Ω, by Lemma 3.2.1 and (3.2.5) we have

$$0 = \mathrm{div}(\chi_E F)(\Omega) = \mathrm{div}F(E^1) + 2\overline{\chi_E F\cdot D\chi_E}(\Omega),$$

which implies, recalling that $\overline{\chi_E F\cdot D\chi_E}$ is supported on $\partial^* E$,

$$\mathrm{div}F(E^1) = -2\overline{\chi_E F\cdot D\chi_E}(\Omega) = -2\overline{\chi_E F\cdot D\chi_E}(\partial^* E). \tag{3.2.7}$$

In an analogous way, Lemma 3.2.1 and (3.2.6) yield

$$\mathrm{div}F(E) = -2\overline{\chi_{E^0}F\cdot D\chi_E}(\partial^* E). \tag{3.2.8}$$

Since $\overline{\chi_E F\cdot D\chi_E} \ll ||D\chi_E|| = \mathcal{H}^{N-1}\llcorner\partial^* E$, the Radon-Nikodym theorem implies that there exists a function $\mathcal{F}_i\cdot\nu_E \in L^1(\partial^* E;\mathcal{H}^{N-1})$ such that $2\overline{\chi_E F\cdot D\chi_E} =$

[1] χ_{E^0} has to be understood as $\chi_{\Omega\cap E^0}$.

$(\mathcal{F}_i \cdot \nu_E)\,\mathcal{H}^{N-1} \llcorner \partial^* E$.
Thus, we conclude that

$$\mathrm{div} F(E^1) = -\int_{\partial^* E} (\mathcal{F}_i \cdot \nu_E)\, d\mathcal{H}^{N-1}.$$

Analogously, $\overline{\chi_{E^0} F \cdot D\chi_E} \ll ||D\chi_E|| = \mathcal{H}^{N-1} \llcorner \partial^* E$ and so there exists a function $\mathcal{F}_e \cdot \nu_E \in L^1(\partial^* E; \mathcal{H}^{N-1})$ such that $2\overline{\chi_{E^0} F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu_E)\,\mathcal{H}^{N-1} \llcorner \partial^* E$; that is,

$$\mathrm{div} F(E) = -\int_{\partial^* E} (\mathcal{F}_e \cdot \nu_E)\, d\mathcal{H}^{N-1}.$$

We prove now the estimates of the L^∞-norm of the normal traces.
We set $\chi_\delta := \chi_E * \rho_\delta$ and we observe that $\overline{\chi_E D\chi_E} = \frac{1}{2} D\chi_E$: indeed for any $\phi \in C^1_c(\Omega; \mathbb{R}^N)$ we have

$$\begin{aligned}
\int_\Omega \chi_E \phi \cdot \nabla \chi_\delta \, dx &= \int_\Omega \chi_E \mathrm{div}(\chi_\delta \phi)\, dx - \int_\Omega \chi_E \chi_\delta \mathrm{div}\phi \, dx \\
&= -\int_\Omega \chi_\delta \phi \cdot \, dD\chi_E - \int_\Omega \chi_E \chi_\delta \mathrm{div}\phi \, dx.
\end{aligned}$$

If we let $\delta \to 0$ we obtain that the limit of the right hand side exists, therefore also the one of the left hand side must exist (at least in the sense of distributions); moreover, it holds

$$\begin{aligned}
\int_\Omega \phi \cdot \, d\overline{\chi_E D\chi_E} &= -\int_\Omega \chi_E^* \phi \cdot \, dD\chi_E - \int_\Omega \chi_E^2 \mathrm{div}\phi \, dx \\
&= -\int_\Omega \frac{1}{2}\phi \cdot \, dD\chi_E - \int_\Omega \chi_E \mathrm{div}\phi \, dx \\
&= -\int_\Omega \frac{1}{2}\phi \cdot \, dD\chi_E + \int_\Omega \phi \cdot dD\chi_E
\end{aligned}$$

since $\chi_E^* = \frac{1}{2}$ on $\partial^* E$ and $D\chi_E = \nu_E d\mathcal{H}^{N-1} \llcorner \partial^* E$. Therefore, by the density of $C^1_c(\Omega; \mathbb{R}^N)$ in $C_c(\Omega; \mathbb{R}^N)$ with respect to the supremum norm, we have

$$\int_\Omega \phi \cdot \, d\overline{\chi_E D\chi_E} = \int_\Omega \frac{1}{2}\phi \cdot \, dD\chi_E \quad \forall \phi \in C_c(\Omega; \mathbb{R}^N) \tag{3.2.9}$$

which implies $\overline{\chi_E D\chi_E} = \frac{1}{2} D\chi_E$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$.
We also have $\overline{\chi_{E^0} D\chi_E} = \frac{1}{2} D\chi_E$ since it is the weak-star limit of the sequence

$$\chi_{E^0} \nabla \chi_\delta = \nabla \chi_\delta - \chi_E \nabla \chi_\delta \overset{*}{\rightharpoonup} \left(1 - \frac{1}{2}\right) D\chi_E$$

as $\delta \to 0$.

By the Lebesgue-Besicovitch differentiation theorem (Theorem 1.1.2), we know that for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$ one has

$$|(\mathcal{F}_i \cdot \nu_E)(x)| = \lim_{r \to 0} \left| \frac{2\overline{\chi_E F \cdot D\chi_E}(B(x,r))}{\mathcal{H}^{N-1}(\partial^* E \cap B(x,r))} \right|.$$

Now we observe that the sequence $|\chi_E F \cdot \nabla \chi_\delta|$ is bounded in $\mathcal{M}(\Omega)$:

$$\begin{aligned} ||\chi_E F \cdot \nabla\chi_\delta||(\Omega) &\leq \sup\left\{ \int_\Omega \phi |\chi_E F \cdot \nabla \chi_\delta|\, dx : \phi \in C_c^\infty(\Omega), ||\phi||_\infty \leq 1 \right\} \\ &\leq \int_\Omega |\chi_E F \cdot \nabla\chi_\delta|\, dx \leq ||F||_{L^\infty(E^1;\mathbb{R}^N)} ||\nabla \chi_\delta||_{L^1(\mathbb{R}^N)} \\ &\leq ||F||_{L^\infty(E^1;\mathbb{R}^N)} ||D\chi_E||(\mathbb{R}^N) \end{aligned}$$

by Remark 1.4.7.
Thus, there exists a weak-star converging subsequence, which we label with δ_k, and let the positive measure $\lambda \in \mathcal{M}(\Omega)$ be its limit.
In an analogous way, we can prove that the sequence or Radon measures $|\chi_{E^0} F \cdot \nabla\chi_\delta|$ is bounded, we just need to put in the previous calculation the norm $||F||_{L^\infty(\Omega \setminus E;\mathbb{R}^N)}$. So there exists a weak-star converging subsequence, which we label again with δ_k, whose limit is the positive Radon measure λ_0.
Moreover, we observe that also the sequences $\chi_E|\nabla\chi_{\delta_k}|$ and $\chi_{E^0}|\nabla\chi_{\delta_k}|$ are bounded using the same argument as above. So there exist weak-star converging subsequences which we shall not relabel for simplicity of notation and which converge to positive measures $\mu, \mu_0 \in \mathcal{M}(\Omega)$.
By Remark 1.1.2, sequence of balls $B(x, r_j)$ with $r_j \to 0$ can be chosen in such a way that $||D\chi_E||(\partial B(x,r_j)) = \lambda(\partial B(x,r_j)) = \mu_0(\partial B(x,r_j)) = 0$ and hence, by Lemma 1.1.2,

$$\begin{aligned} \lim_{r_j \to 0} \left| \frac{2\overline{\chi_E F \cdot D\chi_E}(B(x,r_j))}{||D\chi_E||(B(x,r_j))} \right| &= \lim_{r_j\to 0} \left| \frac{\lim_{\delta_k \to 0} 2 \int_{B(x,r_j)} \chi_E F \cdot \nabla \chi_{\delta_k}\, dy}{\lim_{\delta_k\to 0} \int_{B(x,r_j)} |\nabla\chi_{\delta_k}|\, dy} \right| \\ \leq \lim_{r_j\to 0} \frac{2||F||_{L^\infty(E^1;\mathbb{R}^N)} \lim_{\delta_k\to 0} \int_{B(x,r_j)} \chi_E |\nabla\chi_{\delta_k}|\, dy}{\lim_{\delta_k\to 0}\int_{B(x,r_j)} |\nabla \chi_{\delta_k}|\, dy} \\ = 2||F||_{L^\infty(E^1;\mathbb{R}^N)} \lim_{r_j\to 0} \left(1 - \frac{\lim_{\delta_k\to 0} \int_{B(x,r_j)} \chi_{E^0}|\nabla\chi_{\delta_k}|\, dy}{\lim_{\delta_k\to 0}\int_{B(x,r_j)} |\nabla\chi_{\delta_k}|\, dy} \right) \end{aligned}$$

$$\leq 2||F||_{L^\infty(E^1;\mathbb{R}^N)} \lim_{r_j \to 0} \left(1 - \frac{\lim_{\delta_k \to 0} |\int_{B(x,r_j)} \chi_{E^0} \nabla \chi_{\delta_k} \, dy|}{\lim_{\delta_k \to 0} \int_{B(x,r_j)} |\nabla \chi_{\delta_k}| \, dy} \right)$$
$$= 2||F||_{L^\infty(E^1;\mathbb{R}^N)} \lim_{r_j \to 0} \left(1 - \frac{|\overline{\chi_{E^0} D\chi_E}(B(x,r_j))|}{||D\chi_E||(B(x,r_j))} \right)$$
$$= 2||F||_{L^\infty(E^1;\mathbb{R}^N)} \lim_{r_j \to 0} \left(1 - \frac{1}{2} \frac{|D\chi_E(B(x,r_j))|}{||D\chi_E||(B(x,r_j))} \right) = ||F||_{L^\infty(E^1;\mathbb{R}^N)}.$$

In the last passage we used the definition of reduced boundary: if $x \in \partial^* E$, then $|\nu_E|(x) = 1$, $||D\chi_E||(B(x,r)) > 0$ for $r > 0$ and $\nu_E(x) = \lim_{r \to 0} \frac{D\chi_E(B(x,r))}{||D\chi_E||(B(x,r))}$. This implies that

$$\lim_{r \to 0} \frac{|D\chi_E(B(x,r))|}{||D\chi_E||(B(x,r))} = |\nu_E(x)| = 1.$$

The estimate for the exterior normal trace $\mathcal{F}_e \cdot \nu_E$ can be obtained in a similar way, considering instead balls contained in Ω which satisfy $||D\chi_E||(\partial B(x,r_j)) = \lambda_0(\partial B(x,r_j)) = \mu(\partial B(x,r_j)) = 0$ and using the inequality

$$\left| \int_{B(x,r)} \chi_{E^0} F \cdot \nabla \chi_{\delta_k} \, dy \right| \leq ||F||_{L^\infty(\Omega \setminus E;\mathbb{R}^N)} \int_{B(x,r)} \chi_{E^0} |\nabla \chi_{\delta_k}| \, dy.$$

This completes the proof. □

Remark 3.2.1. Since the proof of Theorem 3.2.1 relies on the product rule for $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^N)$ and $g \in BV(\Omega) \cap L^\infty(\Omega)$ with compact support, then Remark 2.4.2 shows that this result is consistent with Theorem 1.5.2 in the case $F \in BV(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$.

An immediate corollary of this theorem is a way to represent the measure div$F$ on the reduced boundary of bounded sets of finite perimeter.

Corollary 3.2.1. *Let $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^N)$. If $E \subset\subset \Omega$ is a bounded set of finite perimeter, then*

$$\chi_{\partial^* E} \mathrm{div} F = 2\overline{\chi_E F \cdot D\chi_E} - 2\overline{\chi_{E^0} F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E) \, \mathcal{H}^{N-1} \llcorner \partial^* E, \quad (3.2.10)$$

which implies

$$\mathrm{div} F(B) = \int_B (\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E) \, d\mathcal{H}^{N-1} \quad (3.2.11)$$

for any Borel set $B \subset \partial^* E$, and

$$||\mathrm{div} F||(\partial^* E) = \int_{\partial^* E} |\mathcal{F}_i \cdot \nu_E - \mathcal{F}_e \cdot \nu_E| \, d\mathcal{H}^{N-1} \quad (3.2.12)$$

Proof. Equation (3.2.10) follows immediately from (3.2.4) in the proof of Theorem 3.2.1, from $\overline{F \cdot D\chi_E} - \overline{\chi_E F \cdot D\chi_E} = \overline{\chi_{E^0} F \cdot D\chi_E}$ and from the definition of the normal traces. Then we evaluate both measures in equation (3.2.10) over a Borel set B in $\partial^* E$ and we obtain (3.2.11). Finally, 3.2.12 immediately follows from (3.2.10) and properties of total variation. □

Theorem 3.2.2. *Let $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$ and $E \subset\subset \Omega$ be a bounded set of finite perimeter. Then, for any $\phi \in C_c^1(\mathbb{R}^N)$,*

$$\int_{E^1} \phi \, d\mathrm{div} F = - \int_{\partial^* E} \phi (\mathcal{F}_i \cdot \nu_E) \, d\mathcal{H}^{N-1} - \int_{E^1} F \cdot \nabla \phi \, dx \tag{3.2.13}$$

and, recalling that, up to a set of $\mathcal{H}^{N-1}$ measure zero, $E = E^1 \cup \partial^* E$,

$$\int_{E} \phi \, d\mathrm{div} F = - \int_{\partial^* E} \phi (\mathcal{F}_e \cdot \nu_E) \, d\mathcal{H}^{N-1} - \int_{E} F \cdot \nabla \phi \, dx. \tag{3.2.14}$$

Proof. By Theorem 2.4.2, we know that $\phi F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$ for any $\phi \in C_c^1(\mathbb{R}^N)$. Using Theorem 3.2.1, we obtain

$$\mathrm{div}(\phi F)(E^1) = -2 \int_{\partial^* E} d\overline{\phi \chi_E F \cdot D\chi_E}.$$

We have $\overline{\phi \chi_E F \cdot D\chi_E} = \phi \overline{\chi_E F \cdot D\chi_E}$, since, for any $\psi \in C_c(\Omega)$,

$$\begin{aligned} \int_\Omega \psi \, d\overline{\phi \chi_E F \cdot D\chi_E} &= \lim_{\delta \to 0} \int_\Omega \psi \, \phi \chi_E F \cdot \nabla \chi_\delta \, dx \\ &= \lim_{\delta \to 0} \int_\Omega (\psi \phi) \, \chi_E F \cdot \nabla \chi_\delta \, dx = \int_\Omega (\psi \phi) \, d\overline{\chi_E F \cdot D\chi_E}. \end{aligned}$$

Since $2\overline{\chi_E F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu_E) \, \mathcal{H}^{N-1} \llcorner \partial^* E$, we have

$$\int_{E^1} d\mathrm{div}(\phi F) = - \int_{\partial^* E} \phi (\mathcal{F}_i \cdot \nu_E) \, d\mathcal{H}^{N-1}.$$

On the other hand, Theorem 2.4.2 yields $\mathrm{div}(\phi F) = \phi \mathrm{div} F + F \cdot \nabla \phi$, which implies

$$\int_{E^1} \phi \, d\mathrm{div} F = - \int_{E^1} F \cdot \nabla \phi \, dx + \int_{E^1} d\mathrm{div}(\phi F)$$

and so the proof of equation (3.2.13) is complete. The proof of equation (3.2.14) requires the same steps applied to the second identiy of Theorem 3.2.1. □

Remark 3.2.2. A consequence of Theorem 3.2.2 is that, in the case $p = \infty$, the functional normal trace $(TF)_{\partial E}$ (Definition 2.3.1) can be represented by an essentially bounded function on $\partial^* E$:

$$(TF)_{\partial E}(\phi) = - \int_{\partial^* E} \phi (\mathcal{F}_e \cdot \nu_E) \, d\mathcal{H}^{N-1} \; \forall \phi \in C_c^\infty(\Omega).$$

From this it also follows that $\mathrm{supp}((TF)_{\partial E}) \subset \partial^* E$.

Theorem 3.2.3. (Consistency of the Normal Trace with the classical one) *Let $F \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^N) \cap C(\Omega;\mathbb{R}^N)$. If $E \subset\subset \Omega$ is a bounded set of finite perimeter, then the interior and exterior normal trace coincide and admit a representative which is in fact the classical dot product of F and the measure theoretic interior unit normal to E on $\partial^* E$.*

Proof. By Theorem 3.2.1, we have that $2\overline{\chi_E F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu_E)\, \mathcal{H}^{N-1} \llcorner \partial^* E$ in the sense of Radon measures and $\mathcal{F}_i \cdot \nu_E \in L^\infty(\partial^* E; \mathcal{H}^{N-1})$. This means that for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$ one has

$$(\mathcal{F}_i \cdot \nu_E)(x) = \lim_{r \to 0} 2 \frac{\overline{\chi_E F \cdot D\chi_E}(B(x,r))}{||D\chi_E||(B(x,r))}. \tag{3.2.15}$$

Moreover, if we let $\chi_\delta = \chi_E * \rho_\delta$ be a mollification of χ_E, we know that

$$\chi_E F \cdot \nabla \chi_\delta \overset{*}{\rightharpoonup} \overline{\chi_E F \cdot D\chi_E} \quad \text{in} \quad \mathcal{M}(\Omega),$$

which means that, $\forall \phi \in C_c(\Omega)$,

$$\int_\Omega \phi \chi_E F \cdot \nabla \chi_\delta \, dx \to \int_\Omega \phi \, d\overline{\chi_E F \cdot D\chi_E} \quad \text{as} \quad \delta \to 0.$$

We observe that $\phi F \in C_c(\Omega;\mathbb{R}^N)$ and, since $\chi_E \nabla \chi_\delta \overset{*}{\rightharpoonup} \overline{\chi_E D\chi_E}$, we have also

$$\int_\Omega (\phi F) \cdot \nabla \chi_\delta \chi_E \, dx \to \int_\Omega (\phi F) \cdot \, d\overline{\chi_E D\chi_E} \quad \text{as} \quad \delta \to 0.$$

Thus we can conclude that $\overline{\chi_E F \cdot D\chi_E} = F \cdot \overline{\chi_E D\chi_E} = \frac{1}{2} F \cdot D\chi_E$ (equation (3.2.9) in the proof of Theorem 3.2.1), which means that

$$2\overline{\chi_E F \cdot D\chi_E}(B(x,r)) = \int_{B(x,r)} F \cdot dD\chi_E.$$

Recalling the definition of λ from the proof of Theorem 3.2.1, Remark 1.1.2 implies that we can choose a sequence $r_j \to 0$ such that $\lambda(\partial B(x,r_j)) = 0$.
Moreover, by the continuity of F, the function $F \cdot \nu_E$ is well defined on $\partial^* E$ and is also in $L^1(\partial^* E; \mathcal{H}^{N-1})$.
Thus, from (3.2.15), for $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$, we obtain

$$\begin{aligned}
(\mathcal{F}_i \cdot \nu_E)(x) &= \lim_{j \to +\infty} \frac{\int_{B(x,r_j)} F(y) \cdot dD\chi_E(y)}{||D\chi_E||(B(x,r_j))} \\
&= \lim_{j \to +\infty} \frac{\int_{B(x,r_j)} F(y) \cdot \nu_E(y) d||D\chi_E||(y)}{||D\chi_E||(B(x,r_j))} \\
&= F(x) \cdot \nu_E(x),
\end{aligned}$$

by Lemma 1.1.2 and the Lebesgue-Besicovitch differentiation theorem (Theorem 1.1.2).

Applying the same steps to the measure $2\overline{\chi_{E^0} F \cdot D\chi_E}$ yields that it is equal to $F \cdot D\chi_E$ and hence, if we choose balls in Ω such that $\lambda_0(\partial B(x, r_j)) = 0$ (see the proof of Theorem 3.2.1), we find that also $\mathcal{F}_e \cdot \nu_E$ admits $F \cdot \nu_E$ as representative and hence it coincides with $\mathcal{F}_i \cdot \nu_E$ as class of L^∞ functions. □

From this theorem we see that continuous fields have no jump component in the divergence.

Corollary 3.2.2. *Let $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N) \cap C(\Omega; \mathbb{R}^N)$. Then, for any $E \subset\subset \Omega$ set of finite perimeter, we have*

$$||\text{div}F||(\partial^* E) = 0.$$

Proof. From equation (3.2.10) in Corollary 3.2.1 and from the proof of Theorem 3.2.3, we see that

$$\chi_{\partial^* E} \text{div}F = 2\overline{\chi_E F \cdot D\chi_E} - 2\overline{\chi_{E^0} F \cdot D\chi_E} = 0$$

which implies

$$||\chi_{\partial^* E} \text{div}F|| = 0.$$

Indeed, by definition of total variation measure, for any Borel set A it is equal to

$$\sup \left\{ \sum_{k=0}^{+\infty} |\text{div}F(B_k \cap \partial^* E)| : B_k \text{ Borel sets pairwise disjoint}, A = \bigcup_{k=0}^{+\infty} B_k \right\}$$

and this yields 0 since every term in the series is null. □

Remark 3.2.3. We observe that the L^∞ estimates in Theorem 3.2.1 are sharp since we can find continuous divergence measure fields F for which

$$||\mathcal{F}_i \cdot \nu_E||_{L^\infty(\partial^* E; \mathcal{H}^{N-1})} = ||\mathcal{F}_e \cdot \nu_E||_{L^\infty(\partial^* E; \mathcal{H}^{N-1})} = ||F||_{L^\infty(E^1)} = ||F||_{L^\infty(\Omega \setminus E)}.$$

Indeed suppose $E = [0, 1]^N \subset\subset \Omega$ and let $F(x) = e_1 = (1, 0, ..., 0)$. Then clearly $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N) \cap C(\Omega; \mathbb{R}^N)$ and $||F||_{L^\infty(E^1)} = ||F||_{L^\infty(\Omega \setminus E)} = 1$.
Moreover, on $\{0\} \times (0, 1)^{N-1}$, $\nu_E = e_1$ and so over this part of $\partial^* E$ we have $\mathcal{F}_i \cdot \nu_E = \mathcal{F}_e \cdot \nu_E = F \cdot \nu_E = 1$. This implies the identity of the norms.

Remark 3.2.4. If $F \in \mathcal{DM}^\infty(\mathbb{R}^N; \mathbb{R}^N)$ is such that $\text{div}F$ is a positive Radon measure, then we have a partial converse to Proposition 2.3.3. Indeed, if we take $E = B(x, r)$, Theorem 3.2.1 yields

$$\text{div}F(B(x, r)) = -\int_{\partial B(x,r)} (\mathcal{F}_i \cdot \nu_{B(x,r)})\, d\mathcal{H}^{N-1}.$$

Then $\text{div}F \geq 0$ implies

$$\text{div}F(B(x, r)) \leq ||F||_{L^\infty(B(x,r);\mathbb{R}^N)} N\omega_N r^{N-1} \leq C r^{N-1},$$

where $C := ||F||_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)} N\omega_N$.

Example 3.2.1. Let $N = 2$, then $F(x, y) = (\sin\left(\frac{1}{x-y}\right), \sin\left(\frac{1}{x-y}\right)) \in \mathcal{DM}^\infty(\mathbb{R}^2; \mathbb{R}^2)$ as in Example 2.2.2. We will show now that the interior normal trace on any segment of the line $\{(x, y) : y = x\}$ is indeed 0, as we suggested in Remark 2.2.2. Let $E := \{(x, y) \in \mathbb{R}^2 : x \le y \le x + 1; -2R \le x + y \le 2R\}$ for some $R > 0$ and $\phi \in C_c(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} \phi(x, y)\chi_E(x, y)F(x, y) \cdot \nabla\chi_\delta(x, y)\, dx\, dy \to \int_{\mathbb{R}^2} \phi(x, y)\, d\overline{\chi_E F \cdot D\chi_E}$$

and $2\overline{\chi_E F \cdot D\chi_E} = (\mathcal{F}_i \cdot \nu_E)\, \mathcal{H}^1 \llcorner \partial^* E$.

Let $I_{\frac{R}{2}} := \{(x, y) \in \mathbb{R}^2 : y = x, x \in [-\frac{R}{2}, \frac{R}{2}]\} \subset \partial E$, then we have $\nu_E|_{I_{\frac{R}{2}}} = \frac{1}{\sqrt{2}}(-1, 1)$ and

$$\begin{aligned} \nabla\chi_{\delta_k}(x, y) &= \int_{\mathbb{R}^2} \nabla_{x,y}\rho_{\delta_k}(x - u, y - v)\chi_E(u, v)\, du\, dv \\ &= -\int_{\mathbb{R}^2} \nabla_{u,v}\rho_{\delta_k}(x - u, y - v)\chi_E(u, v)\, du\, dv \\ &= \int_{\partial^* E} \rho_{\delta_k}(x - u, y - v)\nu_E(u, v)\, d\mathcal{H}^1(u, v). \end{aligned}$$

If $(x, y) \in I_{\frac{R}{2}} + \overline{B(0, \delta)}$ and $|(x - u, y - v)| < \delta_k$, then $(u, v) \in I_{\frac{R}{2}} + \overline{B(0, \delta + \delta_k)}$. Then, if $\phi \in C_c(I_{\frac{R}{2}} + \overline{B(0, \delta)})$ and if we take δ and δ_k small enough (for example, both $< \frac{R}{4}$) so that $(I_{\frac{R}{2}} + \overline{B(0, \delta + \delta_k)}) \cap \partial^* E = I_R$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} \phi(x, y)\chi_E(x, y)F(x, y) \cdot \nabla\chi_\delta(x, y)\, dx\, dy \\ &= \int_{(I_{\frac{R}{2}} + \overline{B(0,\delta)}) \cap E} \phi(x, y)F(x, y) \cdot \int_{\partial^* E} \rho_{\delta_k}(x - u, y - v)\nu_E(u, v)\, d\mathcal{H}^1(u, v)\, dx\, dy \\ &= \int_{(I_{\frac{R}{2}} + \overline{B(0,\delta)}) \cap E} \phi(x, y)F(x, y) \cdot \int_{I_R} \rho_{\delta_k}(x - u, y - v)\nu_E(u, v)\, d\mathcal{H}^1(u, v)\, dx\, dy = 0, \end{aligned}$$

since $F(x, y) \cdot \nu_E|_{I_R} = \frac{1}{2}\left(-\sin\left(\frac{1}{x-y}\right) + \sin\left(\frac{1}{x-y}\right)\right) = 0$ for $\mathcal{L}^2$-a.e $(x, y) \in (I_{\frac{R}{2}} + \overline{B(0, \delta)}) \cap E)$. Thus,

$$\int_{I_R} \phi(x, y)(\mathcal{F}_i \cdot \nu_E)(x, y)\, d\mathcal{H}^1(x, y) = 0$$

for any $\phi \in C_c(I_{\frac{R}{2}} + \overline{B(0, \delta)})$, which in particular implies $(\mathcal{F}_i \cdot \nu_E) = 0$ in $I_{\frac{R}{2}}$.

As a final remark, we notice that in special cases we can also recover an integration by parts formula.

Remark 3.2.5. Let $F \in \mathcal{DM}^\infty(\Omega; \mathbb{R}^N)$ such that $F = fv$ for some $f \in L^\infty(\Omega)$ and $v \in \mathbb{S}^{N-1}$. We have $\text{div}F = D_v f$ in the sense of distributions, since, for any $\phi \in C_c^\infty(\Omega)$,

$$-\int_\Omega \phi \, d\text{div}F = \int_\Omega F \cdot \nabla\phi \, dx = \int_\Omega fv \cdot \nabla\phi \, dx = \int_\Omega f\nabla_v\phi \, dx$$

and so $D_v f \in \mathcal{M}(\Omega)$ and $||D_v f||(\Omega)|| < \infty$.
Using the notation of [AFP] (Section 3.11), we denote by Π_v the hyperplane orthogonal to v passing through the origin and by Ω_v the orthogonal projection of Ω on Π_v. It is clear that, for any $y \in \Omega_v$, the section of Ω corresponding to y; that is, $\Omega_v^y := \{t \in \mathbb{R} : y + tv \in \Omega\}$, is not empty.
Also, for any function $f : B \subseteq \Omega \to \mathbb{R}$ and any $y \in \Omega_v$, the function $f_v^y : B_v^y \subseteq \Omega_v^y \to \mathbb{R}$ is defined by $f_v^y(t) := f(y + tv)$. $D_v f$ is called directional distributional derivative, and it is well defined if $f \in L^1_{\text{loc}}(\Omega)$, which is our case.
Theorem 3.103 in [AFP] states that if $f \in L^1_{\text{loc}}(\Omega)$ and $v \in \mathbb{S}^{N-1}$, then

$$||D_v f||(\Omega) = \int_{\Omega_v} ||D_t f_v^y||(\Omega_v^y) \, dy.$$

From this, it follows immediately that the functions f_v^y belong to $BV_{\text{loc}}(\Omega_v^y)$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega_v$. We also notice that, for any $\phi \in C_c^\infty(\Omega)$, Fubini's theorem implies

$$-\int_\Omega \phi \, D_v f = \int_\Omega f\nabla_v\phi \, dx = \int_{\Omega_v} \left(\int_{\Omega_v^y} f(y + tv) \frac{d}{dt}\phi(y + tv) \, dt \right) dy$$
$$= \int_{\Omega_v} \left(\int_{\Omega_v^y} f_v^y \frac{d}{dt}\phi_v^y \, dt \right) dy = -\int_{\Omega_v} \left(\int_{\Omega_v^y} \phi_v^y \, dD_t f_v^y \right) dy,$$

since $f_v^y \in L^1_{\text{loc}}(\Omega_v^y)$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Omega_v$, and so,

$$D_v f = D_t f_v^y \otimes \mathcal{H}^{N-1} \llcorner \Omega_v \tag{3.2.16}$$

in the sense of Radon measures in Ω.
Moreover, Remark 3.104 in [AFP] implies that if $E \subset\subset \Omega$ is a set of finite perimter, then E_v^y is of finite perimter in $\mathbb{R}$ for $\mathcal{H}^{N-1}$-a.e. $y \in E_v$. We can conclude that, for $\mathcal{H}^{N-1}$-a.e. $y \in E_v$, E_v^y is the union of a finite number of pairwise disjoint intervals $\{[a_{2j-1}, a_{2j}]\}_{j=1}^m$ in Ω_v^y (see [AFP], Proposition 3.52).
Therefore, for any $E \subset\subset \Omega$, we may apply (3.2.16) and Vol'pert's results (Theorem 1.5.2):

$$\int_{E^1} dD_v f = \int_{E_v} \left(\int_{E_v^y} dD_t f_v^y \right) dy = \int_{E_v} \sum_{j=1}^m (f_{-1}(y + a_{2j}v) - f_{+1}(y + a_{2j-1}v)) \, dy,$$

where $f_{-1}(y + a_{2j}v)$ is the approximate limit of $f(y + \cdot v)$ in a_{2j} from the left and $f_{+1}(y + a_{2j-1}v)$ is the approximate limit of $f(y + \cdot v)$ in a_{2j-1} from the right, since

the interior unit normal in one dimension is $+1$ or -1.
Thus we obtain an integration by part formula at least in the direction v.

Chapter 4

Final remarks and applications

In this chapter we will present some consequences and applications of the Gauss-Green formula.
We will in particular illustrate gluing and extension theorems for essentially bounded divergence-measure fields.
Then, we will show a particular form of the Gauss-Green formula in $\mathcal{DM}^1_{\text{loc}}$ which holds only on almost every ball and use it in order to find a condition for the risolubility of the equation $\text{div}F = \mu$ in $\mathbb{R}^N$, for μ positive Radon measure and $F \in \mathcal{DM}^p(\mathbb{R}^N;\mathbb{R}^N)$ with $1 \le p \le \frac{N}{N-1}$ (for this and related subjects, see also [PT]).
Finally, we will briefly recall the main features of hyperbolic systems of conservation laws, show how the theory of divergence-measure fields is strictly connected with Lax entropy inequality and exhibit an application of Theorems 3.2.1 and 3.2.2 to this context.

4.1 Gluing and extension theorems

Theorem 4.1.1. (Gluing theorem) *Let $W \subset\subset E \subset\subset \Omega$, where Ω and W are open sets and E is a set of finite perimeter. Let $F_1 \in \mathcal{DM}^\infty(\Omega;\mathbb{R}^N)$ and $F_2 \in \mathcal{DM}^\infty(\mathbb{R}^N \setminus \overline{W};\mathbb{R}^N)$. Then*

$$F(x) = \begin{cases} F_1(x) & \text{if } x \in E \\ F_2(x) & \text{if } x \in \mathbb{R}^N \setminus E \end{cases}$$

belongs to $\mathcal{DM}^\infty(\mathbb{R}^N;\mathbb{R}^N)$, and

$$\begin{aligned}||F||_{\mathcal{DM}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \le ||F_1||_{\mathcal{DM}^\infty(E^1;\mathbb{R}^N)} + ||F_2||_{\mathcal{DM}^\infty(\mathbb{R}^N\setminus E^1;\mathbb{R}^N)} \\ + ||\mathcal{F}_{i,1}\cdot\nu - \mathcal{F}_{i,2}\cdot\nu||_{L^1(\partial^* E;\mathcal{H}^{N-1})},\end{aligned}$$

where $\mathcal{F}_{i,1}\cdot\nu$ is the interior normal trace of F_1 on $\partial^* E$ and

$$(\mathcal{F}_{i,2}\cdot\nu)\,\mathcal{H}^{N-1}\llcorner\partial^* E = 2\overline{\chi_E F_2 \cdot D\chi_E},$$

which can be seen as an interior normal trace, even if E is not a subset of $\mathbb{R}^N \setminus \overline{W}$.

Proof. Obviously, $F \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and

$$||F||_{L^\infty(\mathbb{R}^N;\mathbb{R}^N)} \leq ||F_1||_{L^\infty(E;\mathbb{R}^N)} + ||F_2||_{L^\infty(\mathbb{R}^N \setminus E;\mathbb{R}^N)}.$$

Now, if $\phi \in C_c^1(\mathbb{R}^N \setminus \overline{W})$, we observe that we can take $\xi \in C_c^1(\mathbb{R}^N \setminus \overline{W})$ such that $\xi = 1$ on $\text{supp}(\phi)$: then we can extend ξF_2 to a divergence-measure field $\hat{F}_2$ on $\mathbb{R}^N$ by setting it equal to 0 in $\overline{W}$ (indeed, it has compact support, so we refer to Remark 2.1.2).
By Theorem 3.2.1, we know that $2\overline{\chi_E \hat{F}_2 \cdot D\chi_E} = (\hat{\mathcal{F}}_{i,2} \cdot \nu)\, \mathcal{H}^{N-1} \llcorner \partial^* E$ and, arguing as in the proof of Theorem 3.2.2, we can show that

$$\int_{E^1} d\text{div}(\phi \hat{F}_2) = -\int_{\partial^* E} \phi\, 2d\overline{\chi_E \hat{F}_2 \cdot D\chi_E}.$$

For any $\varphi \in C_c(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \varphi \chi_E \hat{F}_2 \cdot \nabla \chi_\delta \, dx = \int_{\mathbb{R}^N \setminus \overline{W}} (\varphi \xi) \chi_E F_2 \cdot \nabla \chi_\delta \, dx.$$

The set $\{\chi_E F_2 \cdot \nabla \chi_\delta\}$ is bounded in $\mathcal{M}(\mathbb{R}^N \setminus \overline{W})$:

$$||\chi_E F_2 \cdot \nabla \chi_\delta||(\mathbb{R}^N \setminus \overline{W}) \leq ||F||_{L^\infty(\mathbb{R}^N \setminus \overline{W};\mathbb{R}^N)} ||D\chi_E||(\mathbb{R}^N),$$

by Remark 1.4.7. Thus there exists a converging subsequence labeled with δ_k and, if $\text{supp}\varphi \subset \mathbb{R}^N \setminus \overline{W}$, we can conclude that $\overline{\chi_E \hat{F}_2 \cdot D\chi_E} = \overline{\chi_E F_2 \cdot \nabla \chi_E}$ in $\mathcal{M}(\mathbb{R}^N \setminus \overline{W})$. This implies $(\hat{\mathcal{F}}_{i,2} \cdot \nu) = (\mathcal{F}_{i,2} \cdot \nu)$ $\mathcal{H}^{N-1}$-a.e. $x \in \partial^* E$, by the definition.
Therefore we have

$$\int_{E^1} \nabla \phi \cdot \hat{F}_2 \, dx + \int_{E^1} \phi \, d\text{div} \hat{F}_2 = \int_{E^1} d\text{div}(\phi \hat{F}_2) = -\int_{\partial^* E} \phi (\mathcal{F}_{i,2} \cdot \nu) \, d\mathcal{H}^{N-1}$$

and, since $\xi = 1$ and $\nabla \xi = 0$ on $\text{supp}\phi$, $\text{div}(\hat{F}_2) = \xi \text{div} F_2 + \nabla \xi \cdot F_2$ in $\text{supp}\phi$, which implies

$$\int_{E^1 \cap \text{supp}\phi} \nabla \phi \cdot F_2 \, dx + \int_{E^1 \cap \text{supp}\phi} \phi \, d\text{div} F_2 = -\int_{\partial^* E} \phi (\mathcal{F}_{i,2} \cdot \nu) \, d\mathcal{H}^{N-1}.$$

Recalling that $|E \setminus E^1| = 0$,

$$\begin{aligned} -\int_{\mathbb{R}^N \setminus \overline{W}} \phi \, d\text{div} F_2 &= \int_{\mathbb{R}^N \setminus \overline{W}} F_2 \cdot \nabla \phi \, dx = \int_{\mathbb{R}^N \setminus E} F_2 \cdot \nabla \phi \, dx + \int_{E \cap \text{supp}\phi} F_2 \cdot \nabla \phi \, dx \\ &= -\int_{\partial^* E} \phi (\mathcal{F}_{i,2} \cdot \nu) \, d\mathcal{H}^{N-1} - \int_{E^1 \cap \text{supp}\phi} \phi \, d\text{div} F_2 + \int_{\mathbb{R}^N \setminus E} F_2 \cdot \nabla \phi \, dx \end{aligned}$$

which implies

$$-\int_{\mathbb{R}^N\setminus E^1}\phi\, d\mathrm{div}F_2 = -\int_{\partial^* E}\phi(\mathcal{F}_{i,2}\cdot\nu)\, d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N\setminus E} F_2\cdot\nabla\phi\, dx$$

for any $\phi \in C^1_c(\mathbb{R}^N \setminus \overline{W})$.
Therefore, choosing $\phi \in C^1_c(\mathbb{R}^N)$ with $||\phi||_\infty \leq 1$ and $\xi \in C^1_c(\mathbb{R}^N \setminus \overline{W})$ with $||\xi||_\infty \leq 1$ and $\xi = 1$ on $\mathrm{supp}(\phi) \cap (\mathbb{R}^N \setminus E)$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} F\cdot\nabla\phi\, dx &= \int_E F_1\cdot\nabla\phi\, dx + \int_{\mathbb{R}^N\setminus E} F_2\cdot\nabla\phi\, dx\\
&= \int_E F_1\cdot\nabla\phi\, dx + \int_{\mathbb{R}^N\setminus E} F_2\cdot\nabla(\phi\xi)\, dx\\
&= -\int_{E^1}\phi\, d\mathrm{div}F_1 - \int_{\mathbb{R}^N\setminus E^1}\phi\, d\mathrm{div}F_2 - \int_{\partial^* E}(\mathcal{F}_{i,1}\cdot\nu - \mathcal{F}_{i,2}\cdot\nu)\phi\, d\mathcal{H}^{N-1}\\
&\leq ||\mathrm{div}F_1||(E^1) + ||\mathrm{div}F_2||(\mathbb{R}^N\setminus E^1) + ||\mathcal{F}_{i,1}\cdot\nu - \mathcal{F}_{i,2}\cdot\nu||_{L^1(\partial^* E;\mathcal{H}^{N-1})}.
\end{aligned}$$

Thus, taking the supremum over ϕ on the left hand side, we have the desired result. □

Before we prove the extension theorem, we need the following result from measure theory.

Proposition 4.1.1. *Let $U \subset \mathbb{R}^N$ be an open bounded set with $\mathcal{H}^{N-1}(\partial U) < \infty$. Then there exists a sequence of bounded open sets $U_k \subset \overline{U}_k \subset U$ such that*

1. $|U \setminus U_k| \to 0$;
2. $\limsup_{k\to+\infty} \mathcal{H}^{N-1}(\partial U_k) \leq 4^{N-1}\frac{N\omega_N}{\omega_{N-1}}\mathcal{H}^{N-1}(\partial U)$.

Proof. By the definition of spherical measure, for each integer k, there exists a δ_k-covering of ∂U by balls $\partial U \subset \bigcup_{j=1}^\infty B(x_j, r_j)$, with $2r_j < \delta_k$ $\forall j$, such that

$$\sum_{j=1}^\infty \omega_{N-1} r_j^{N-1} \leq \mathcal{S}^{N-1}_{\delta_k}(\partial U) + \frac{1}{k} \leq \mathcal{S}^{N-1}(\partial U) + \frac{1}{k}. \tag{4.1.1}$$

Since ∂U is compact, there exists a finite covering $\{B(x_j, r_j)\}_{j=1}^{m_k}$ and so we set $V_k := \bigcup_{j=1}^{m_k} B(x_j, r_j)$. We observe that $\partial V_k \subset \bigcup_{j=1}^{m_k} \partial B(x_j, r_j)$. This and (1.1.1) imply

$$\mathcal{S}^{N-1}(\partial V_k) \leq \sum_{j=1}^{m_k} \mathcal{S}^{N-1}(\partial B(x_j, r_j)) \leq 2^{N-1}\frac{N\omega_N}{\omega_{N-1}}\sum_{j=1}^{m_k}\omega_{N-1} r_j^{N-1},$$

which, together with (4.1.1) and (1.1.1), yields

$$\mathcal{H}^{N-1}(\partial V_k) \leq 2^{N-1}\frac{N\omega_N}{\omega_{N-1}}\left(\mathcal{S}^{N-1}(\partial U) + \frac{1}{k}\right) \leq 2^{N-1}\frac{N\omega_N}{\omega_{N-1}}\left(2^{N-1}\mathcal{H}^{N-1}(\partial U) + \frac{1}{k}\right) \tag{4.1.2}$$

for any k. We set $U_k := U \setminus \overline{V_k}$ and so, by (4.1.1), we have

$$\begin{aligned}|U \setminus U_k| = |U \cap V_k| \leq \sum_{j=1}^{m_k} \omega_N r_j^N &< \frac{\delta_k}{2}\frac{\omega_N}{\omega_{N-1}}\sum_{j=1}^{m_k}\omega_{N-1} r_j^{N-1} \\ &\leq \frac{\delta_k}{2}\frac{\omega_N}{\omega_{N-1}}\left(2^{N-1}\mathcal{H}^{N-1}(\partial U) + \frac{1}{k}\right),\end{aligned}$$

which goes to zero as $\delta_k \to 0$.
Finally, $\partial U_k = \partial V_k \cap U$ and so (4.1.2) implies 2. □

Definition 4.1.1. An open set $U \subset \mathbb{R}^N$ is called an *extension domain for* $F \in \mathcal{DM}^\infty(U;\mathbb{R}^N)$ if there exists $\hat{F} \in \mathcal{DM}^\infty(\mathbb{R}^N;\mathbb{R}^N)$ such that $\hat{F} = F$ on U.

Theorem 4.1.2. (Extension theorem) *A bounded open set U satisfying $\mathcal{H}^{N-1}(\partial U) < \infty$ is an extension domain for any $F \in \mathcal{DM}^\infty(U;\mathbb{R}^N)$.*

Proof. We define an extension of F by $\hat{F}(x) := \chi_U(x)F(x)\ \forall x \in \mathbb{R}^N$.
We just need to show that $||\mathrm{div}\hat{F}||(\mathbb{R}^N) < \infty$.
Let U_k be the sequence of approximating sets given in Proposition 4.1.1: we observe that, by Remark 1.4.1, each U_k is a set of finite perimeter since $\mathcal{H}^{N-1}(\partial U_k) < \infty$, $U_k^1 \subset \overline{U}_k$, since, by Remark 1.4.2, $\mathbb{R}^N \setminus \overline{U}_k \subset U_k^0 = \mathbb{R}^N \setminus (U_k^1 \cup \partial^m U_k)$ and $|U_k \Delta U_k^1| = 0$.
Hence, for any $\phi \in C_c^\infty(\mathbb{R}^N)$ with $||\phi||_\infty \leq 1$, we may apply the Gauss-Green formula (Theorem 3.2.2):

$$\int_{U_k} F \cdot \nabla\phi\, dx = -\int_{\partial^* U_k} \phi(\mathcal{F}_i \cdot \nu_{U_k})\, d\mathcal{H}^{N-1} - \int_{U_k^1} \phi\, d\mathrm{div}F.$$

Thus, by Proposition 4.1.1,

$$\begin{aligned}\left|\int_{U_k} F \cdot \nabla\phi\, dx\right| &\leq ||\mathrm{div}F||(U_k^1) + ||F||_{L^\infty(U_k^1;\mathbb{R}^N)}\mathcal{H}^{N-1}(\partial^* U_k) \\ &\leq ||\mathrm{div}F||(\overline{U}_k) + ||F||_{L^\infty(\overline{U}_k;\mathbb{R}^N)}\mathcal{H}^{N-1}(\partial U_k) \\ &\leq ||\mathrm{div}F||(U) + ||F||_{L^\infty(U;\mathbb{R}^N)}\mathcal{H}^{N-1}(\partial U_k).\end{aligned}$$

Letting $k \to +\infty$, Lebesgue's dominated convergence theorem and Proposition 4.1.1 yield

$$\left|\int_U F \cdot \nabla\phi\, dx\right| \leq ||\mathrm{div}F||(U) + 4^{N-1}\frac{N\omega_N}{\omega_{N-1}}||F||_{L^\infty(U;\mathbb{R}^N)}\mathcal{H}^{N-1}(\partial U) < \infty.$$

Since we have

$$\int_U F \cdot \nabla\phi \, dx = \int_{\mathbb{R}^N} \hat{F} \cdot \nabla\phi \, dx,$$

it follows that

$$||\text{div}\hat{F}||(\mathbb{R}^N) = \sup\left\{\int_{\mathbb{R}^N} \hat{F} \cdot \nabla\phi \, dx : \phi \in C_c^\infty(\mathbb{R}^N), ||\phi||_\infty \leq 1\right\} < \infty,$$

which concludes the proof. □

Corollary 4.1.1. *Let $U \subset \mathbb{R}^N$ be a bounded open set with $\mathcal{H}^{N-1}(\partial U) < \infty$, $F_1 \in \mathcal{DM}^\infty(U; \mathbb{R}^N)$ and $F_2 \in \mathcal{DM}^\infty(\mathbb{R}^N \setminus \overline{U}; \mathbb{R}^N)$. Then if we set*

$$F(x) := \begin{cases} F_1(x) & \text{if } x \in U \\ F_2(x) & \text{if } x \in \mathbb{R}^N \setminus \overline{U} \end{cases}$$

we have $F \in \mathcal{DM}^\infty(\mathbb{R}^N; \mathbb{R}^N)$.

Proof. We can apply Theorem 4.1.2 to F_1 in order to obtain that $\hat{F}_1 := \chi_U F_1 \in \mathcal{DM}^\infty(\mathbb{R}^N; \mathbb{R}^N)$.
Now, we argue as in proof of Proposition 4.1.1, and we set $W_k := U \cup V_k$, thus $W_k \subset \overline{W}_k \subset\subset B(0, R)$, for some $R > 0$ large enough, $\partial W_k = \partial V_k \setminus \overline{U}$ and

$$|(B(0, R) \setminus \overline{U}) \setminus (B(0, R) \setminus \overline{W}_k)| = |V_k \setminus \overline{U}| \leq |V_k| \to 0.$$

We also have $(B(0, R) \setminus \overline{W}_k)^1 \subset \overline{B(0, R) \setminus \overline{W}_k}$, $\mathcal{H}^{N-1}(\partial W_k) \leq \mathcal{H}^{N-1}(\partial V_k) < \infty$, which implies also

$$\limsup_{k\to+\infty} \mathcal{H}^{N-1}(\partial W_k) \leq 4^{N-1}\frac{N\omega_N}{\omega_{N-1}}\mathcal{H}^{N-1}(\partial U).$$

Thus, for any $\phi \in C_c^\infty(\mathbb{R}^N)$ with $||\phi||_\infty \leq 1$, we can apply the Gauss-Green formula (Theorem 3.2.2) to the set $B(0, R) \setminus W_k$ and the field F_2:

$$\int_{B(0,R)\setminus\overline{W}_k} F_2 \cdot \nabla\phi \, dx = -\int_{\partial B(0,R)\cup\partial^* W_k} \phi(\mathcal{F}_{i,2} \cdot \nu_{B(0,R)\setminus W_k}) \, d\mathcal{H}^{N-1} - \int_{(B(0,R)\setminus\overline{W}_k)^1} \phi \, d\text{div}F_2.$$

Clearly, there exists R_0 such that, for any $R \geq R_0$, $\text{supp}(\phi) \subset B(0, R)$ and therefore $\int_{\partial B(0,R)} \phi(\mathcal{F}_{i,2} \cdot \nu_{B(0,R)\setminus W_k}) \, d\mathcal{H}^{N-1} = 0$. Hence we have

$$\begin{aligned} \left|\int_{B(0,R)\setminus\overline{W}_k} F_2 \cdot \nabla\phi \, dx\right| &\leq ||\text{div}F_2||((B(0, R) \setminus \overline{W}_k)^1) + ||F_2||_{L^\infty(\mathbb{R}^N\setminus\overline{U};\mathbb{R}^N)}\mathcal{H}^{N-1}(\partial^* W_k) \\ &\leq ||\text{div}F_2||(\overline{B(0, R) \setminus \overline{W}_k}) + ||F_2||_{L^\infty(\mathbb{R}^N\setminus\overline{U};\mathbb{R}^N)}\mathcal{H}^{N-1}(\partial W_k) \\ &\leq ||\text{div}F_2||(\mathbb{R}^N \setminus \overline{U}) + ||F_2||_{L^\infty(\mathbb{R}^N\setminus\overline{U};\mathbb{R}^N)}\mathcal{H}^{N-1}(\partial W_k). \end{aligned}$$

Letting $k \to +\infty$, we obtain, by the previous remarks on W_k and Lebesgue's dominated convergence theorem,

$$\left| \int_{B(0,R)\setminus\overline{U}} F_2 \cdot \nabla\phi \, dx \right| \le ||\mathrm{div} F_2||(\mathbb{R}^N \setminus \overline{U}; \mathbb{R}^N) + 4^{N-1} \frac{N\omega_N}{\omega_{N-1}} ||F_2||_{L^\infty(\mathbb{R}^N\setminus\overline{U})} \mathcal{H}^{N-1}(\partial U)$$

for any $R \ge R_0$ and thus

$$\int_{B(0,R)\setminus\overline{U}} F_2 \cdot \nabla\phi \, dx = \int_{\mathrm{supp}(\phi)\setminus\overline{U}} F_2 \cdot \nabla\phi \, dx = \int_{\mathbb{R}^N\setminus\overline{U}} F_2 \cdot \nabla\phi \, dx.$$

Hence, if we set $\hat{F}_2(x) = \chi_{\mathbb{R}^N\setminus\overline{U}}(x) F_2(x) \; \forall x \in \mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N\setminus\overline{U}} F_2 \cdot \nabla\phi \, dx = \int_{\mathbb{R}^N} \hat{F}_2 \cdot \nabla\phi \, dx,$$

which implies

$$||\mathrm{div}\hat{F}_2||(\mathbb{R}^N) \sup \left\{ \int_{\mathbb{R}^N} \hat{F}_2 \cdot \nabla\phi \, dx : \phi \in C_c^\infty(\mathbb{R}^N), ||\phi||_\infty \le 1 \right\} < \infty.$$

We have therefore proved that $\hat{F}_2 \in \mathcal{DM}^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and, since $F = \hat{F}_1 + \hat{F}_2$, we have also $F \in \mathcal{DM}^\infty(\mathbb{R}^N; \mathbb{R}^N)$. $\square$

Finally, we state a result concerning the Gauss-Green formula on certain unbounded sets of finite perimeter.

Proposition 4.1.2. *Let W be a bounded open set, $F \in \mathcal{DM}^\infty(\mathbb{R}^N \setminus \overline{W}; \mathbb{R}^N)$ and $E \supset\supset W$ be a bounded set of finite perimeter. Then*

$$\int_{E^0} \phi \, d\mathrm{div} F = - \int_{\partial^* E} \phi (\mathcal{F}_i \cdot \nu_{\mathbb{R}^N\setminus E}) \, d\mathcal{H}^{N-1} - \int_{E^0} F \cdot \nabla\phi \, dx \tag{4.1.3}$$

for any $\phi \in C_c^1(\mathbb{R}^N)$.

Proof. Since E is bounded, there exists $R > 0$ such that $E \subset\subset B(0,R)$. Clearly $B(0,R)$ is a set of finite perimeter and $\partial^*(B(0,R) \setminus E) = \partial B(0,R) \cup \partial^* E$. Moreover, recalling that $E = E^1 \cup \partial^m E$ implies $B(0,R) \setminus E = B(0,R) \cap E^0$, we have $(B(0,R) \setminus E)^1 = B(0,R) \setminus E$.
Indeed, if $x \in B(0,R) \cap E^0$, then there exists $r_0 > 0$ such that $\forall r \le r_0$ we have $B(x,r) \subset B(0,R)$ and $x \in E^0$, so, for $r \le r_0$,

$$\frac{|B(x,r) \cap B(0,R) \cap E^0|}{|B(x,r)|} = 1 - \frac{|B(x,r) \cap E|}{|B(x,r)|} \to 1 \text{ as } r \to 0,$$

which implies $x \in (B(0,R) \setminus E)^1$. Hence $(B(0,R) \setminus E)^1 \supset B(0,R) \setminus E$.
On the other hand, if $x \in (B(0,R) \setminus E)^1$, then $x \in E^0$ since

$$\frac{|B(x,r) \cap E^0|}{|B(x,r)|} \ge \frac{|B(x,r) \cap B(0,R) \cap E^0|}{|B(x,r)|} \to 1 \text{ as } r \to 0.$$

If by contradiction $x \notin B(0,R)$, then either $x \in \mathbb{R}^N \setminus \overline{B(0,R)}$ or $x \in \partial B(0,R)$. In the first case, there exists $r_0 > 0$ such that $\forall r \leq r_0$ one has $B(x,r) \subset \mathbb{R}^N \setminus \overline{B(0,R)}$, which implies

$$\frac{|B(x,r) \cap B(0,R) \cap E^0|}{|B(x,r)|} = 0, \quad \forall r \leq r_0,$$

and this is a contradiction.
In the second case, $\frac{|B(x,r) \cap B(0,R)|}{|B(x,r)|} \to \frac{1}{2}$ but $\frac{|B(x,r) \cap B(0,R)|}{|B(x,r)|} \geq \frac{|B(x,r) \cap B(0,R) \cap E^0|}{|B(x,r)|} \to 1$, which is absurd.
Now, for any $\phi \in C_c^1(\mathbb{R}^N)$, we apply the Gauss-Green formula (Theorem 3.2.2) to the domain $(B(0,R) \setminus E)^1$:

$$\int_{(B(0,R)\setminus E)^1} \phi \, d\mathrm{div}F = -\int_{\partial B(0,R) \cup \partial^* E} \phi(\mathcal{F}_i \cdot \nu_{B(0,R)\setminus E}) \, d\mathcal{H}^{N-1} - \int_{B(0,R)\setminus E} F \cdot \nabla \phi \, dx.$$

Since E and ϕ are fixed, there exists R_0 such that, $\forall R \geq R_0$, $E \subset\subset B(0,R)$ and $\mathrm{supp}(\phi) \subset B(0,R)$, and therefore we can integrate over the whole space minus E in the first and the last integral, while $\int_{\partial B(0,R)} \phi(\mathcal{F}_i \cdot \nu_{B(0,R)\setminus E}) \, d\mathcal{H}^{N-1} = 0$ and clearly $(\mathcal{F}_i \cdot \nu_{B(0,R)\setminus E}) = (\mathcal{F}_i \cdot \nu_{\mathbb{R}^N\setminus E})$ on $\partial^* E$. We conclude that

$$\int_{\mathbb{R}^N\setminus E} \phi \, d\mathrm{div}F = -\int_{\partial^* E} \phi(\mathcal{F}_i \cdot \nu_{\mathbb{R}^N\setminus E}) \, d\mathcal{H}^{N-1} - \int_{\mathbb{R}^N\setminus E} F \cdot \nabla \phi \, dx$$

for any $\phi \in C_c^1(\mathbb{R}^N)$. Hence, since $\mathbb{R}^N \setminus E = \mathbb{R}^N \setminus (E^1 \cup \partial^m E) = E^0$, we have (4.1.3). □

Remark 4.1.1. We observe that this argument could be used in the proof of Theorem 4.1.1 as an alternative way to achieve a Gauss-Green formula for F_2 over the unbounded set of finite perimeter $\mathbb{R}^N \setminus E$.
Moreover, if $F \in \mathcal{DM}^\infty(\mathbb{R}^N;\mathbb{R}^N)$ and E is a bounded set of finite perimeter, then

$$(\mathcal{F}_e \cdot \nu_E) = -(\mathcal{F}_i \cdot \nu_{\mathbb{R}^N\setminus E}) \ \mathcal{H}^{N-1}\text{-a.e. on } \ \partial^* E, \tag{4.1.4}$$

where these functions are respectively the exterior normal trace of F on $\partial^* E$ and the interior normal trace on $\partial^* E$ taken with opposite orientation. Indeed, for any $\phi \in C_c^1(\mathbb{R}^N)$, by Theorem 3.2.2 and (4.1.3) we have

$$\begin{aligned}
\int_{\mathbb{R}^N} F \cdot \nabla \phi \, dx &= \int_E F \cdot \nabla \phi \, dx + \int_{\mathbb{R}^N\setminus E} F \cdot \nabla \phi \, dx = -\int_{\partial^* E} \phi(\mathcal{F}_e \cdot \nu_E) \, d\mathcal{H}^{N-1} \\
&- \int_E \phi \, d\mathrm{div}F - \int_{\partial^* E} \phi(\mathcal{F}_i \cdot \nu_{\mathbb{R}^N\setminus E}) \, d\mathcal{H}^{N-1} - \int_{\mathbb{R}^N\setminus E} \phi \, d\mathrm{div}F \\
&= -\int_{\partial^* E} \phi(\mathcal{F}_e \cdot \nu_E) \, d\mathcal{H}^{N-1} - \int_{\partial^* E} \phi(\mathcal{F}_i \cdot \nu_{\mathbb{R}^N\setminus E}) \, d\mathcal{H}^{N-1} - \int_{\mathbb{R}^N} \phi \, d\mathrm{div}F \\
&= -\int_{\partial^* E} \phi(\mathcal{F}_e \cdot \nu_E) \, d\mathcal{H}^{N-1} - \int_{\partial^* E} \phi(\mathcal{F}_i \cdot \nu_{\mathbb{R}^N\setminus E}) \, d\mathcal{H}^{N-1} + \int_{\mathbb{R}^N} F \cdot \nabla \phi \, dx,
\end{aligned}$$

which implies

$$\int_{\partial^* E} \phi(\mathcal{F}_e \cdot \nu_E) \, d\mathcal{H}^{N-1} = -\int_{\partial^* E} \phi(\mathcal{F}_i \cdot \nu_{\mathbb{R}^N \setminus E}) \, d\mathcal{H}^{N-1}.$$

Since this last identity is in particular true for any $\phi \in C_c^\infty(\mathbb{R}^N)$, we have proved (4.1.4).

4.2 An existence result in the subcritical case

We are going to study now a special case of the equation $\mathrm{div}F = \mu$ on $\mathbb{R}^N$, for $N \geq 2$. First we state a version of Gauss-Green formula on balls.

Theorem 4.2.1. *Let $F \in \mathcal{DM}^1_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N)$. Then for each $x \in \mathbb{R}^N$ and for $\mathcal{L}^1$-a.e. $r > 0$,*

$$\mathrm{div}F(B(x,r)) = \int_{\partial B(x,r)} F(y) \cdot \frac{(y-x)}{|y-x|} \, d\mathcal{H}^{N-1}(y). \tag{4.2.1}$$

Proof. Let F_ϵ be a mollification of F, then we have, for any $R > 0$,

$$||F_\epsilon - F||_{L^1(B(x,R))} \to 0 \quad \text{as} \quad \epsilon \to 0.$$

So we have

$$\int_{B(x,R)} |F_\epsilon - F| \, dy = \int_0^R \int_{\partial B(x,r)} |F_\epsilon - F| \, d\mathcal{H}^{N-1} \, dr \to 0 \quad \text{as} \quad \epsilon \to 0,$$

which implies that there exists a set $\mathcal{I}_R \subset (0, R)$ with $\mathcal{L}^1(\mathcal{I}_R) = 0$ such that $\int_{\partial B(x,r)} |F_\epsilon - F| \, d\mathcal{H}^{N-1} \to 0$ as $\epsilon \to 0$ for each $r \in (0, R) \setminus \mathcal{I}_R$.
We can repeat this argument with $R = n$ for every $n \in \mathbb{N}$ and so, if we set $\mathcal{I} = \bigcup_{n=1}^\infty \mathcal{I}_n$, we have $\int_{\partial B(x,r)} |F_\epsilon - F| \, d\mathcal{H}^{N-1} \to 0$ as $\epsilon \to 0$ $\forall r \in \mathbb{R} \setminus \mathcal{I}$.
$\forall \epsilon > 0$, the classical Gauss-Green formula yields

$$\int_{B(x,r)} \mathrm{div}F_\epsilon(y) \, dy = \int_{\partial B(x,r)} F_\epsilon(y) \cdot \frac{(y-x)}{|y-x|} \, d\mathcal{H}^{N-1}(y).$$

By Remark 1.1.2, there exists a set $\mathcal{J} \subset \mathbb{R}$ with $\mathcal{L}^1(\mathcal{J}) = 0$ such that, for any $r \notin \mathcal{J}$, $||\mathrm{div}F||(\partial B(x,r)) = 0$. We can now take $r \in \mathbb{R} \setminus (\mathcal{I} \cup \mathcal{J})$ and thus apply Lemma 1.1.2 and Remark 2.1.2 in order to obtain

$$\lim_{\epsilon \to 0} \int_{B(x,r)} \mathrm{div}F_\epsilon(y) \, dy = \mathrm{div}F(B(x,r)).$$

Hence, by observing that

$$\left| \int_{\partial B(x,r)} (F_\epsilon(y) - F(y)) \cdot \frac{(y-x)}{|y-x|} \, d\mathcal{H}^{N-1}(y) \right| \leq \int_{\partial B(x,r)} |F_\epsilon(y) - F(y)| \, d\mathcal{H}^{N-1}(y),$$

we have (4.2.1), $\forall r \in \mathbb{R} \setminus (\mathcal{I} \cup \mathcal{J})$. $\Box$

As it is shown in [PT], the choice of $\mathbb{R}^N$ as our domain allows us to use techniques from harmonic analysis to study properties of divergence-measure fields. Therefore we introduce the following notions of Riesz potential of order 1 and $(1,p)$-energy associated to a positive Radon measure.

Definition 4.2.1. Let $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N)$ be a positive Radon measure. We define the *Riesz potential of order 1* of μ as

$$I_1(\mu)(x) := \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-1}} \, d\mu(y).$$

Definition 4.2.2. Let $1 < p < \infty$. We say that $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N)$ has *finite* $(1,p)$-*energy* if

$$\int_{\mathbb{R}^N} (I_1\mu(x))^p \, dx < \infty.$$

Remark 4.2.1. We observe that the Riesz potential of order 1 of a positive Radon measure is always well defined, being $+\infty$ if the integral does not converge. Morevoer, for any $R > 0$ we have

$$I_1\mu(x) \geq \int_{B(0,R)} \frac{1}{|x-y|^{N-1}} \, d\mu(y) \geq \frac{\mu(B(0,R))}{(|x|+R)^{N-1}}.$$

This implies that, if $I_1\mu$ has finite $(1,p)$ energy, then we must have either $p > \frac{N}{N-1}$ or $1 < p \leq \frac{N}{N-1}$ and $\mu = 0$. So $\mu = 0$ is the only positive measure on $\mathbb{R}^N$ which has finite $(1,p)$ in the subcritical case $p \in (1, \frac{N}{N-1}]$.

The main result which we are going to show here states that the equation $\text{div}F = \mu$, with $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N)$ and positive, has a solution $F \in L^p(\mathbb{R}^N;\mathbb{R}^N)$ with $1 \leq p \leq \frac{N}{N-1}$ only if $\mu = 0$. Actually, since $F = 0$ is a solution, we could say that this equation has at least one solution in $L^p(\mathbb{R}^N;\mathbb{R}^N)$ if and only if $\mu = 0$.

Theorem 4.2.2. *Let* $1 \leq p \leq \frac{N}{N-1}$. *If* $F \in L^p(\mathbb{R}^N;\mathbb{R}^N)$ *satisfies* $\text{div}F = \mu$, *for some* $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R}^N)$ *positive, then* $\mu = 0$.

Proof. By the layer cake representation formula, we have

$$\begin{aligned}
I_1\mu(x) &= \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-1}} \, d\mu(y) \\
&= \int_0^\infty \mu(\{y \in \mathbb{R}^N : |x-y|^{1-N} > t\}) \, dt = \int_0^\infty \mu(B(x, t^{-\frac{1}{N-1}})) \, dt \\
&= \int_0^\infty \mu(B(x,r)) \frac{N-1}{r^N} \, dr \\
&= (N-1) \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \frac{\mu(B(x,r))}{r^N} \, dr,
\end{aligned}$$

where we are allowed to perform the change of variable $t = r^{1-N}$ since the function $t \to \mu(B(x, t^{-\frac{1}{N-1}}))$ is upper semicontinuous.
Now, since $\mathrm{div}F = \mu$, we apply Theorem 4.2.1 and we have

$$\begin{aligned} I_1\mu(x) &= (N-1) \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \frac{1}{r^N} \int_{\partial B(x,r)} F(y) \cdot \frac{(y-x)}{|y-x|} \, d\mathcal{H}^{N-1}(y) \, dr \\ &= (1-N) \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \int_{\partial B(x,r)} F(y) \cdot \frac{(x-y)}{|x-y|^{N+1}} \, d\mathcal{H}^{N-1}(y) \, dr \\ &= (1-N) \lim_{\epsilon \to 0^+} \int_{\{|x-y|>\epsilon\}} F(y) \cdot \frac{(x-y)}{|x-y|^{N+1}} \, dy. \end{aligned}$$

This last limit is known to exist for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$ and it is equal to $c(N) \sum_{j=1}^N \mathcal{R}_j F_j(x)$, where F_j is the j-th component of F and $R_j F_j$ is the j-th Riesz transform of the function F_j (see [St], Chapter II, § 4.2, Theorem 3 and § 4.5 Theorem 4). Moreover, we have that

$$||\mathcal{R}_j f||_{L^p(\mathbb{R}^N)} \leq C ||f||_{L^p(\mathbb{R}^N)}$$

for $1 < p < \infty$, and

$$||\mathcal{R}_j f||_{L^{1,\infty}(\mathbb{R}^N)} \leq C ||f||_{L^1(\mathbb{R}^N)}.$$

Thus we can conclude that, for $p \in (1, \frac{N}{N-1}]$,

$$||I_1\mu||_{L^p(\mathbb{R}^N)} \leq C ||F||_{L^p(\mathbb{R}^N;\mathbb{R}^N)} < \infty,$$

and so, by Remark 4.2.1, we must have $\mu = 0$.
On the other hand, if $p = 1$, we have

$$||I_1\mu||_{L^{1,\infty}(\mathbb{R}^N)} \leq C ||F||_{L^1(\mathbb{R}^N;\mathbb{R}^N)} < \infty.$$

We recall that the (quasi-)norm in the space $L^{1,\infty}$ is defined as

$$||f||_{L^{1,\infty}(\mathbb{R}^N)} := \sup_{t>0} t |\{x \in \mathbb{R}^N : |f(x)| > t\}|.$$

Since Remark 4.2.1 shows that, for any $R > 0$,

$$I_1\mu(x) \geq \frac{\mu(B(0,R))}{(|x| + R)^{N-1}},$$

we see that

$$\{x \in \mathbb{R}^N : I_1\mu(x) > t\} \supset \{x \in \mathbb{R}^N : \frac{\mu(B(0,R))}{(|x| + R)^{N-1}} > t\}$$

and so

$$\begin{aligned}\sup_{t>0} t|\{x \in \mathbb{R}^N : I_1\mu(x) > t\}| &\geq \sup_{t>0} t|\{x \in \mathbb{R}^N : \frac{\mu(B(0,R))}{(|x|+R)^{N-1}} > t\}| \\ &= \sup_{t>0} t\,\omega_N \left(\left(\frac{\mu(B(0,R))}{t}\right)^{\frac{1}{N-1}} - R\right)^N \\ &= \sup_{t>0} t^{-\frac{1}{N-1}}\,\omega_N \left(\mu(B(0,R))^{\frac{1}{N-1}} - t^{\frac{1}{N-1}}R\right)^N = +\infty,\end{aligned}$$

unless $\mu = 0$. Thus the statement is proved. □

Although we restrict ourselves to the case of a positive Radon measure μ, this does not diminish the interest of the equation $\text{div}F = \mu$. As we shall see (Remark 4.3.1), this is indeed the situation that occurs in the context of nonlinear hyperbolic systems of conservation laws.

4.3 Nonlinear hyperbolic systems of conservation laws

In this section we will describe an application of the theory of divergence-measure fields to the context of conservation laws. For completeness we will shortly summarize the main features of such systems of partial differential equations.

4.3.1 Brief introduction

Definition 4.3.1. A *general system of conservation laws* is the following initial value problem:

$$\begin{aligned} u_t + \text{div}_x f(u) = 0 \;\text{ in }\; \mathbb{R}^{d+1}_+ := (0,+\infty)\times\mathbb{R}^d, \qquad (4.3.1)\\ u = g \;\text{ on }\; \{0\}\times\mathbb{R}^d,\end{aligned}$$

where $u : \mathbb{R}^{d+1}_+ \to U \subset \mathbb{R}^m$, $f \in C^1(U;\mathbb{R}^{m\times d})$, $\text{div}_x f(u)$ is (at least formally) the divergence with respect to x of the matrix f; that is, the vector in $\mathbb{R}^m$ whose elements are the divergences of the rows of f, and u_0 is the initial datum.
u is called the *conserved quantity*, while f is the *flux*.

The terminology used above finds its origins in physics. If we suppose that the components of the vector valued function $u = u(t,x)$ are (smooth) densities of some conserved quantities, then, given any bounded set V with smooth boundary, the integral

$$\int_V u(t,x)\,dx$$

represents the total amount of these quantities within V at time t. Conservation laws in physics usually assert that the rate of change of such quantities is governed by a flux function $f : \mathbb{R}^m \to \mathbb{R}^{m\times d}$; that is, we have

$$\frac{d}{dt}\int_V u\,dx = \int_{\partial V} f(u)\cdot\nu\,d\mathcal{H}^d,$$

where ν is the unit interior normal to V. Supposing also that u and f are smooth enough and that we can apply the classical Gauss-Green formula, we obtain

$$\int_V u_t\,dx = -\int_V \mathrm{div}_x f(u)\,dx,$$

which gives the above system since the domain of integration V is arbitrary and the densities are supposed to be smooth.

Since f is differentiable, we may rewrite the system (4.3.1) in *nondivergence form* as

$$u_t + \sum_{j=1}^{d} B_j(u)u_{x_j} = 0, \tag{4.3.2}$$

where $B_j : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is the matrix whose components are $\{\frac{\partial f_{i,j}}{\partial u_k}\}_{i,k=1,\dots,m}$; that is, the jacobian of the j-th column vector of f, $\nabla_u f_j$.
In order to achieve well posedness for the initial value problem at least when f is linear, we make a further assumption of algebraic nature.

Definition 4.3.2. The system of convervation laws (4.3.1) is *strictly hyperbolic* if $B_j(u)$ is real diagonalizable $\forall j = 1,...,d$ and $\forall u \in \mathbb{R}^m$; that is, there exist m real distinct eigenvalues $\lambda_1(u) < ... < \lambda_m(u)$ and bases of left and right linearly independent eigenvectors, denoted by $l_1(u),...,l_m(u)$ and $r_1(u),...,r_m(u)$, and regarded as row vectors and column vectors respectively.

Example 4.3.1. The first elementary examples of hyperbolic systems of conservation laws are the linear ones.
Let $d = 1$ and $f(u) = Au$, where A is a $m \times m$ hyperbolic matrix, with real eigenvalues $\lambda_1 < ... < \lambda_m$ and right and left eigenvectors r_i, l_i, chosen such that $|r_i| = |l_i| = 1$ and $l_i \cdot r_j = \delta_{ij}$. Then the solution to the Cauchy problem

$$\begin{aligned} u_t + Au_x = 0 &\text{ in } \mathbb{R}^2_+ \\ u = g &\text{ on } \{0\}\times\mathbb{R}. \end{aligned}$$

with $g \in C^1(\mathbb{R})$, is

$$u(t,x) = \sum_{j=1}^{m}(l_j\cdot g(x-\lambda_j t))r_j.$$

This means that in the scalar case ($m = 1$) the inital profile is shifted with constant speed $\lambda = f'$, otherwise if $m > 1$ the initial profile is decomposed as a sum of m waves, each one travelling with one of the characteristic soppeds λ_j.

In the general case, f is nonlinear and, even if we restrict ourselves to the case $d = 1$, new features will arise in the solutions. Indeed, the eigenvectors depend on u and nontrivial interactions between different waves will occur. Also, the eigenvalues $\lambda_j(u)$ depend on u, so the shape of the travelling waves will vary in time and this may lead to shock formation in finite time.
This is actually the major problem connected with nonlinearity: the loss of regularity. It may be shown that, even if we assume the initial datum to be smooth, the classical solution may develop singularities and form shock waves in finite time.
An example of this fact is given by Burgers' equation ($m = d = 1$)

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad \text{in} \quad \mathbb{R}^2_+$$
$$u = g \quad \text{on} \quad \{0\} \times \mathbb{R}$$

with the initial data

$$g(x) = \begin{cases} 1 & \text{if } \ x \le 0 \\ 1 - x & \text{if } \ 0 \le x \le 1 \\ 0 & \text{if } \ x \ge 1 \end{cases} .$$

Indeed, any smooth solution is constant along the characteristics
$y(s) = (g(x_0)s + x_0, s)$, $s \ge 0$, for each $x_0 \in \mathbb{R}$ fixed. Then for $t \ge 1$ these lines cross, leading to discontinuity of u (for details, we refer to [E], Section 3.4.1, Example 1).

Since initially smooth solutions may become discontinuous within finite time, in order to construct global solutions we have to work in a space of discontinuous functions and to interpret the conservation laws in a distributional sense.

Definition 4.3.3. A function $u \in L^\infty(\mathbb{R}^{d+1}_+; \mathbb{R}^m)$ is a *weak solution* of the system of conservation laws (4.3.1) if, for any $\phi \in C^\infty_c([0, +\infty) \times \mathbb{R}^d; \mathbb{R}^m)$,

$$\int_0^\infty \int_{\mathbb{R}^d} u \cdot \phi_t + \sum_{j=1}^{d} f_j(u) \cdot \nabla_{x_j}\phi \, dx \, dt + \int_{\mathbb{R}^d} g(x) \cdot \phi(0, x) \, dx = 0,$$

where f_j is the j-th column of f.

However, extending the notion of solution from classical to weak introduces a new difficulty: we may loose the uniqueness.
An easy example of this fact is again provided by the Burgers' equation, with initial datum

$$g(x) = \begin{cases} 0 & \text{if } \ x < 0 \\ 1 & \text{if } \ x > 0 \end{cases} .$$

In this case, since the method of characteristics does not provide any information in the region $\{0 < x < t\}$, we have indeed at least two different weak solutions:

$$u_1(t,x) = \begin{cases} 0 & \text{if } x < \frac{t}{2} \\ 1 & \text{if } x > \frac{t}{2} \end{cases}$$

and

$$u_2(t,x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 < x < t \\ 1 & \text{if } x > t \end{cases}$$

(for details, we refer to [E], Section 3.4.2, Example 2).
The strategy to overcome this problem is to state some new condition which the acceptable solutions must satisfy. Historically the idea was to exclude 'nonphysical' solutions, and this is why these further requests are called *entropy conditions*. They indeed arise from a rough analogy with the thermodynamic principle that physical entropy is a non-decreasing function of time.
We will not describe here all these criteria, nor we will discuss the Rankine-Hugoniot conditions on jump discontinuities (for which we refer to [D] and [E]), since these fall outside the purpose of this section. Rather we will concentrate ourselves on the Lax entropy inequality.
First we need to define the concept of mathematical entropy.

Definition 4.3.4. We say that $\eta \in C^1(\mathbb{R}^m)$ is an *entropy* for (4.3.1), with associated *entropy flux* $q \in C^1(\mathbb{R}^m;\mathbb{R}^d)$, if

$$\nabla_u q_j(u) = \nabla_u \eta(u) \nabla_u f_j(u), \quad \text{for} \quad j = 1, ..., d, \tag{4.3.3}$$

We call $F^\eta_u = (\eta(u), q(u))$ an *entropy pair.* If η is convex, we say F^η_u is a convex entropy pair.

We can easily check that any C^1 solution of (4.3.1) satisfies also

$$\eta(u)_t + \text{div}_x q(u) = 0 \tag{4.3.4}$$

since

$$\eta(u)_t = \nabla_u \eta(u) u_t = \nabla_u \eta(u)(-\sum_{j=1}^d \nabla_u f_j(u) u_{x_j}) = -\sum_{j=1}^d \nabla_u q_j(u) \cdot u_{x_j} = -\text{div}_x q(u).$$

This means that, for any entropy pair (η, q), the additional conservation law (4.3.4) holds.

Definition 4.3.5. A function $u \in L^\infty(\mathbb{R}^{d+1}_+;\mathbb{R}^m)$ is an *weak entropy solution* of (4.3.1) if

$$\eta(u)_t + \text{div}_x q(u) \leq 0 \tag{4.3.5}$$

holds in the sense of distributions for any convex entropy pair (η, q); that is,

$$\int_0^{+\infty} \int_{\mathbb{R}^d} \eta(u)\phi_t + q(u) \cdot \nabla_x \phi \, dxdt \geq 0$$

for any $\phi \in C_c^\infty(\mathbb{R}^{d+1}_+)$ with $\phi \geq 0$.

Remark 4.3.1. Condition (4.3.5) implies that, for any convex entropy pair, the distribution $\operatorname{div}_{(t,x)}(\eta(u), q(u)) = \operatorname{div}_{(t,x)} F_u^\eta$ is nonpositive. Therefore, a corollary of the Riesz representation theorem (Lemma 1.1.3) shows that there exists a positive Radon measure on $\mathbb{R}^{d+1}_+$ μ_η such that

$$\operatorname{div}_{(t,x)} F_u^\eta = -\mu_\eta.$$

This is one of the main reasons of the original interest in the theory of divergence-measure fields by Chen and Frid (see [CF1], [CF2], [CF3]). Indeed, it is easy to show that $F_u^\eta \in \mathcal{DM}^\infty_{\text{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^{d+1}_+)$: since $u \in L^\infty(\mathbb{R}^{d+1}_+; \mathbb{R}^m)$ and η and q are continuously differentiable, $F_u^\eta \in L^\infty(\mathbb{R}^{d+1}_+)$, and Remark 4.3.1 shows that its divergence is actually a (nonpositive) Radon measure, though it is not necessarily finite.

4.3.2 Traces on hyperplanes

We will now present a result concerning the possibility of recovering traces for solution of hyperbolic systems of conservation laws on hyperplanes, following the paper [CT2].

We fix some notation: given $\tau > 0$, we set

$$\begin{aligned}
\Pi^\tau &:= \{(t, x) \in \mathbb{R}^{d+1} : t > \tau\}, \\
B^+((\tau, y), r) &:= B((\tau, y), r) \cap \Pi^\tau, \\
B^\tau(y, r) &:= B((\tau, y), r) \cap \partial\Pi^\tau = \{(\tau, x) \in \mathbb{R}^{d+1} : |x - y| < r\}, \\
C^+((\tau, y), r) &:= \{(t, x) \in \mathbb{R}^{d+1} : 0 < t - \tau < r, |x - y| < r\}.
\end{aligned}$$

We denote by $\bar{u}_r(\tau, y)$ the average of u over the half ball $B^+((\tau, y), r)$.

Definition 4.3.6. We say that u satisfies the *vanishing mean oscillation property for half balls on* $\partial\Pi^\tau$ if, for any continuous function $q \in C(\mathbb{R}^m; \mathbb{R}^d)$,

$$\lim_{r \to 0} \frac{1}{r^{d+1}} \int_{B^+((\tau,y),r)} |q(u(t, x)) - \overline{q(u)}_r(\tau, y)| \, dt \, dx = 0 \tag{4.3.6}$$

for $\mathcal{H}^d$-a.e. $(\tau, y) \in \partial\Pi^\tau$, where $\overline{q(u)}_r(\tau, y)$ is the average of $q(u)$ over the half ball $B^+((\tau, y), r)$.

We need the following result from measure theory.

Lemma 4.3.1. *Let μ be a positive Radon measure in $\mathbb{R}^{d+1}_+$. Then, for any $\tau > 0$ and for $\mathcal{H}^d$-a.e. $y \in \mathbb{R}^d$,*

$$\lim_{r\to 0} \frac{\mu(C^+((\tau, y), r))}{r^d} = 0.$$

Proof. Let

$$A_k := \left\{ y \in \mathbb{R}^d : \limsup_{r\to 0} \frac{\mu(C^+((\tau, y), r))}{r^d} > \frac{1}{k} \right\}$$

and, for $R > 0$,

$$A_k^R := \left\{ y \in B(0, R) : \limsup_{r\to 0} \frac{\mu(C^+((\tau, y), r))}{r^d} > \frac{1}{k} \right\},$$

where we denote here by $B(0, R)$ the open ball of radius R in $\mathbb{R}^d$. It is sufficient to show that $\mathcal{H}^d(A_k^R) = 0$ for each $k \in \mathbb{N}$ and for each $R > 0$, since then

$$\mathcal{H}^d(A_k) = \lim_{R\to+\infty} \mathcal{H}^d(A_k^R) = 0 \quad \forall k$$

and so

$$\mathcal{H}^d\left(\bigcup_{k=1}^{+\infty} A_k\right) = 0.$$

Given $y \in A_k^R$ and $\epsilon > 0$, there exists a number $r_y < \epsilon$ such that

$$\mu(C^+((\tau, y), r_y)) > \frac{1}{2k} r_y^d.$$

We can choose a sequence $y_j \in A_k$ such that $B(y_j, r_{y_j}) \cap B(y_i, r_{y_i}) = \emptyset$ if $j \neq i$ and $A_k^R \subset \bigcup_{j=1}^{\infty} B(y_j, 3r_{y_j})$ (see [G], Lemma 2.2, where we multiply ρ by ϵ). Then

$$\mathcal{H}^d(A_k^R) \le \omega_d \sum_{j=1}^{\infty} (3r_j)^d \le 2k\omega_d 3^d \sum_{j=1}^{\infty} \mu(C^+((\tau, y_j), r_{y_j})).$$

Since $r_{y_j} < \epsilon$ and $y_j \in B(0, R)$, then

$$C^+((\tau, y_j), r_{y_j}) \subset L_\epsilon^R := \{(t, y) \in \mathbb{R}^{d+1} : 0 < t - \tau < \epsilon, |(t - \tau, y)| < R + \sqrt{2}\epsilon\}$$

and hence

$$\mathcal{H}^d(A_k^R) < 2k\omega_d 3^d \mu(L_\epsilon^R) \quad \forall \epsilon > 0.$$

Since μ is a Radon measure, it is finite on the compact set $\overline{L_1^R}$ and so $\mu(L_\epsilon^R) \to 0$ as $\epsilon \to 0$, which implies $\mathcal{H}^d(A_k^R) = 0$. $\square$

Theorem 4.3.1. *Let (η, q) be any convex entropy pair and let $\tau > 0$. If $u \in L^\infty(\mathbb{R}^{d+1}_+; \mathbb{R}^m)$ is a weak entropy solution of (4.3.1) and satisfies the vanishing mean oscillation property on $\partial\Pi^\tau$, then $\eta(u)$ has a trace on $\partial\Pi^\tau$; that is, there exists a function $\eta(u)_{tr} \in L^\infty(\partial\Pi^\tau; \mathcal{H}^{N-1})$ such that, for $\mathcal{H}^d$-a.e. $(\tau, y) \in \partial\Pi^\tau$,*

$$\lim_{r\to 0} \frac{1}{\omega_d r^{d+1}} \int_{C^+((\tau,y),r)} \eta(u(t,x))\, dt\, dx = \eta(u)_{tr}(\tau, y). \tag{4.3.7}$$

In particular, if we choose $\eta(u) = u_j$, $j = 1, ..., m$, we obtain the trace for each component of u.

Proof. By Remark 4.3.1, $F^\eta_u = (\eta(u), q(u)) \in \mathcal{DM}^\infty_{\text{loc}}(\mathbb{R}^{d+1}_+; \mathbb{R}^{d+1})$ and so $F^\eta_u \in \mathcal{DM}^\infty(C^+((\tau, 0), 2R); \mathbb{R}^{d+1})$ for any $R > 0$. Therefore, by Theorem 3.2.1 there exists a function $\mathcal{F}_R \cdot \nu \in L^\infty(\partial^* C^+((\tau, 0), R); \mathcal{H}^d)$ which is the interior normal trace of F^η_u on

$$\partial^* C^+((\tau, 0), R) = B^\tau(0, R) \cup \{(t, x) \in \mathbb{R}^{d+1} : 0 < t - \tau < R, |x| = R\} \cup B^{\tau+R}(0, R).$$

Let $y \in \mathbb{R}^d$ and $r > 0$ such that $C^+((\tau, y), r) \subset C^+((\tau, 0), R)$, then we have that the interior normal trace to $C^+((\tau, y), r)$ satisfies $(\mathcal{F}_{y,r} \cdot \nu)(\tau, x) = (\mathcal{F}_R \cdot \nu)(\tau, x)$ $\mathcal{H}^d$-a.e. $(\tau, x) \in B^\tau(y, r)$. Indeed, by the Gauss-Green formula (Theorem 3.2.2), we have that, for any $\phi \in C^1_c(\mathbb{R}^{d+1})$,

$$\int_{C^+((\tau,0),R)} \phi\, d\text{div} F^\eta_u + \int_{C^+((\tau,0),R)} F^\eta_u \cdot \nabla\phi\, dx = -\int_{\partial^* C^+((\tau,0),R)} \phi(\mathcal{F}_R \cdot \nu)\, d\mathcal{H}^d$$

and

$$\int_{C^+((\tau,y),r)} \phi\, d\text{div} F^\eta_u + \int_{C^+((\tau,y),r)} F^\eta_u \cdot \nabla\phi\, dx = -\int_{\partial^* C^+((\tau,y),r)} \phi(\mathcal{F}_{y,r} \cdot \nu)\, d\mathcal{H}^d,$$

since $(C^+((\tau, y), r))^1 = C^+((\tau, y), r)$ for any $(\tau, y) \in \mathbb{R}^{d+1}$ and $r > 0$. Now we can take ϕ with compact support in $B((\tau, y), r)$, then $C^+((\tau, 0), R) \cap B((\tau, y), r) = C^+((\tau, y), r) \cap B((\tau, y), r) = B^+((\tau, y), r)$ and $\partial^* C^+((\tau, 0), R) \cap B((\tau, y), r) = \partial^* C^+((\tau, y), r) \cap B((\tau, y), r) = B^\tau(y, r)$, and so

$$\begin{aligned}\int_{B^+((\tau,y),r)} \phi\, d\text{div} F^\eta_u + \int_{B^+((\tau,y),r)} F^\eta_u \cdot \nabla\phi\, dx &= -\int_{B^\tau(y,r)} \phi(\mathcal{F}_R \cdot \nu)\, d\mathcal{H}^d \\ &= -\int_{B^\tau(y,r)} \phi(\mathcal{F}_{y,r} \cdot \nu)\, d\mathcal{H}^d.\end{aligned}$$

Since $\phi \in C^\infty_c(B((\tau, y), r))$ is arbitrary, one obtains the desired result.
Moreover, since R itself is arbitrary, the above argument shows the existence of an interior normal trace $\mathcal{F} \cdot \nu_{\Pi^\tau}$ over the whole hyperplane $\partial\Pi^\tau$, such that $(\mathcal{F}_{y,r} \cdot \nu)(\tau, x) = (\mathcal{F} \cdot \nu_{\Pi^\tau})(\tau, x)$ $\mathcal{H}^d$-a.e. $(\tau, x) \in B^\tau(y, r)$ and for any $(\tau, y) \in \partial\Pi^\tau$. We also notice that this implies $||\mathcal{F} \cdot \nu_{\Pi^\tau}||_{L^\infty(B^\tau(0,R);\mathcal{H}^d)} \le ||F^\eta_u||_{L^\infty(C^+((\tau,0),R);\mathbb{R}^{d+1})} \le$

$||F_u^\eta||_{L^\infty(\mathbb{R}^{d+1}_+;\mathbb{R}^{d+1})}$ for any R (Theorem 3.2.1), we can conclude that $(\mathcal{F}\cdot\nu_{\Pi^\tau}) \in L^\infty(\partial\Pi^\tau;\mathcal{H}^d)$.

Let $\mathcal{G}\subset\partial\Pi^\tau$ be the set of Lebesgue points of $\mathcal{F}\cdot\nu_{\Pi^\tau}$ for which Lemma 4.3.1 and property (4.3.6) hold; then we have $\mathcal{H}^d(\partial\Pi^\tau\setminus\mathcal{G})=0$. Therefore, for any $(\tau,y)\in\mathcal{G}$, we can choose a representative of the interior normal trace to $C^+((\tau,y),r)$ for which (τ,y) is a Lebesgue point, since $(\mathcal{F}_{y,r}\cdot\nu)(\tau,x)=(\mathcal{F}\cdot\nu_{\Pi^\tau})(\tau,x)$ $\mathcal{H}^d$-a.e. $(\tau,x)\in B^\tau(y,r)$. Without loss of generality, we may assume $(\tau,0)\in\mathcal{G}$.

Having taken $r>0$ and the interior normal trace as above (whose selected representative we shall denote simply by $\mathcal{F}\cdot\nu_{\Pi^\tau}$ on $B^\tau(0,r)$), we apply the Gauss-Green formula with the test function

$$\phi(t,x)=\varphi\left(\frac{x}{r}\right)\rho(t-\tau)(r+\tau-t),$$

such that $\varphi\in C_c^\infty(B^\tau(0,1))$ and $\rho\in C_c^\infty([-1,2R])$, $||\rho||_\infty\le 1$, $\rho=1$ on $[0,R]$. Since

$$\begin{aligned}\phi(\tau,x)&=r\varphi\left(\frac{x}{r}\right),\\ \phi(\tau+r,x)&=0,\\ \phi(t,x)&=0\ \text{ if }\ |x|=r,\end{aligned}$$

and

$$\frac{\partial\phi(t,x)}{\partial t}=\varphi\left(\frac{x}{r}\right)(\rho'(t-\tau)(r+\tau-t)-\rho(t-\tau))=-\varphi\left(\frac{x}{r}\right)\quad\forall t\in(\tau,\tau+r),$$

recalling that $F_u^\eta=(\eta(u),q(u))$, we have

$$\int_{C^+((\tau,0),r)}\phi\,d\mathrm{div}F_u^\eta+\int_{C^+((\tau,0),r)}(-\eta(u(t,x))\varphi\left(\frac{x}{r}\right)+q(u(t,x))\cdot\nabla_x\phi(t,x))\,dt\,dx$$
$$=-\int_{B^\tau(0,r)}r\varphi\left(\frac{x}{r}\right)(\mathcal{F}\cdot\nu_{\Pi^\tau})(\tau,x)\,d\mathcal{H}^d(x).$$

Now we divide both sides by r^{d+1} and we show that

$$\frac{1}{r^{d+1}}\int_{C^+((\tau,0),r)}q(u)\cdot\nabla_x\phi\,dt\,dx\to 0, \tag{4.3.8}$$

$$\frac{1}{r^{d+1}}\int_{C^+((\tau,0),r)}\phi\,d\mathrm{div}F_u^\eta\to 0. \tag{4.3.9}$$

We observe that

$$\int_{C^+((\tau,0),r)}\nabla_x\phi(t,x)\,dt\,dx=\int_0^r\int_{B^\tau(0,r)}\rho(t)(r-t)\nabla\varphi\left(\frac{x}{r}\right)dx\,dt=0 \tag{4.3.10}$$

since $\phi(\frac{\cdot}{r}) \in C_c^\infty(B^\tau(0,r))$. Therefore, if $\overline{q(u)}_{2r}$ denotes the average of $q(u)$ on the half ball $B^+((\tau,0),2r)$, then (4.3.10) implies

$$\begin{aligned}\left|\frac{1}{r^{d+1}}\int_{C^+((\tau,0),r)} q(u)\cdot\nabla_x\phi\,dt\,dx\right| &= \left|\frac{1}{r^{d+1}}\int_{C^+((\tau,0),r)} (q(u)-\overline{q(u)}_{2r})\cdot\nabla_x\phi\,dt\,dx\right| \\ &\le \frac{1}{r^{d+1}}||\nabla_x\phi||_\infty\int_{C^+((\tau,0),r)} |q(u)-\overline{q(u)}_{2r}|\,dt\,dx \\ &\le \frac{1}{r^{d+1}}||\nabla_x\phi||_\infty\int_{B^+((\tau,0),2r)} |q(u)-\overline{q(u)}_{2r}|\,dt\,dx \to 0,\end{aligned}$$

by property (4.3.6).
On the other hand, (4.3.9) follows from Lemma 4.3.1:

$$\left|\frac{1}{r^{d+1}}\int_{C^+((\tau,0),r)}\phi\, d\mathrm{div}F_u^\eta\right| \le ||\varphi||_\infty \frac{r}{r^{d+1}}||\mathrm{div}F_u^\eta||(C^+((\tau,0),r)) \to 0 \quad \text{as} \quad r\to 0.$$

Thus, we have

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{C^+((\tau,0),r)}\varphi\left(\frac{x}{r}\right)\eta(u(t,x))\,dt\,dx = \lim_{r\to 0}\frac{1}{r^d}\int_{B^\tau(0,r)}\varphi\left(\frac{x}{r}\right)(\mathcal{F}\cdot\nu_{\Pi^\tau})(\tau,x)\,d\mathcal{H}^d(x),$$

for any $\varphi\in C_c^\infty(B^\tau(0,1))$.

Since $C_c^\infty(B^\tau(0,1))$ is dense in $L^1(B^\tau(0,1))$, $\forall\epsilon>0$, there exists φ such that $||\varphi-\frac{1}{\omega_d}||_{L^1(B^\tau(0,1))}<\epsilon$. Hence, performing the change of variable $x=r\xi$, we have

$$\begin{aligned}&\left|\frac{1}{r^{d+1}}\int_{C^+((\tau,0),r)}\eta(u(t,x))\left(\varphi\left(\frac{x}{r}\right)-\frac{1}{\omega_d}\right)dt\,dx\right| \\ &\le ||\eta(u)||_{L^\infty(\mathbb{R}_+^{d+1})}\frac{1}{r^{d+1}}\int_{B^\tau(0,1)} r\left|\varphi(\xi)-\frac{1}{\omega_d}\right| r^d\,d\xi \\ &= ||\eta(u)||_{L^\infty(\mathbb{R}_+^{d+1})}||\varphi-(\omega_d)^{-1}||_{L^1(B^\tau(0,1))} < ||\eta(u)||_{L^\infty(\mathbb{R}_+^{d+1})}\epsilon.\end{aligned}$$

We can repeat the same kind of argument with $(\mathcal{F}\cdot\nu_{\Pi^\tau})$, obtaining

$$\begin{aligned}&\left|\frac{1}{r^d}\int_{B^\tau(0,r)}(\mathcal{F}\cdot\nu_{\Pi^\tau})(\tau,x)\left(\varphi\left(\frac{x}{r}\right)-\frac{1}{\omega_d}\right)dt\,dx\right| \\ &\le ||\mathcal{F}\cdot\nu_{\Pi^\tau}||_{L^\infty(B^\tau(0,r))}\frac{1}{r^d}\int_{B^\tau(0,1)}\left|\varphi(\xi)-\frac{1}{\omega_d}\right| r^d\,d\xi \\ &\le ||F_u^\eta||_{L^\infty(\mathbb{R}_+^{d+1},\mathbb{R}^{d+1})}||\varphi-(\omega_d)^{-1}||_{L^1(B^\tau(0,1))} < ||F_u^\eta||_{L^\infty(\mathbb{R}_+^{d+1},\mathbb{R}^{d+1})}\epsilon.\end{aligned}$$

Since $(\tau,0)$ is a Lebesgue point for $\mathcal{F}\cdot\nu_{\Pi^\tau}$, we have

$$\lim_{r\to 0}\frac{1}{\omega_d r^d}\int_{B^\tau(0,r)}(\mathcal{F}\cdot\nu_{\Pi^\tau})(\tau,x)\,dx = (\mathcal{F}\cdot\nu_{\Pi^\tau})(\tau,0). \tag{4.3.11}$$

Since ϵ is arbitrary, and using (4.3.11), we find

$$\lim_{r\to 0} \frac{1}{\omega_d r^{d+1}} \int_{C^+((\tau,0),r)} \eta(u(t,x))\,dt\,dx = (\mathcal{F}\cdot\nu_{\Pi^\tau})(\tau,0)$$

and we conclude that the desired trace is $\eta(u)_{tr}(\tau,0) := (\mathcal{F}\cdot\nu_{\Pi^\tau})(\tau,0)$. $\square$

Bibliography

[A] G. ANZELLOTTI, *Pairings between measures and functions and compensated compactness*, Anal. Mat. Pure Appl. 135 (1983), 293-318.

[AFP] L. AMBROSIO, N. FUSCO, D. PALLARA, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[B] A. BRESSAN, *Hyperbolic conservation laws: an illustrated tutorial*, Notes for a summer course, Cetraro 2009.

[CF1] G.-Q. CHEN, H. FRID, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Ration. Mach. Anal. **147** no. 2, 89-118 (1999).

[CF2] G.-Q. CHEN, H. FRID, *Extended divergence-measure fields and the Euler equations for gas dynamics*, Comm. Math. Phys. **236** no. 2, 251-280 (2003).

[CF3] G.-Q. CHEN, H. FRID, *On the theory of divergence-measure fields and its applications*, Bull. Braz. Math. Soc. (N.S.) **236** 401-433 (2001).

[CT1] G.-Q. CHEN, M. TORRES, *Divergence-measure fields, sets of finite perimeter, and conservation laws*, Arch. Ration. Mach. Anal. **175** no. 2, 245-267 (2005).

[CT2] G.-Q. CHEN, M. TORRES, *On the structure of solutions of nonlinear hyperbolic systems of conservation laws*, Comm. on Pure and Applied Mathematics, Vol. X, no. 4, 1011-1036 (2011).

[CTZ1] G.-Q. CHEN, M. TORRES, W.P. ZIEMER, *Gauss-Green Theorem for Weakly Differentiable Vector Fields, Sets of Finite Perimeters, and Balance Laws*, Comm. on Pure and Applied Mathematics, Vol. LXII, 0242-0304 (2009).

[CTZ2] G.-Q. CHEN, M. TORRES, W.P. ZIEMER, *Measure-Theoretic Analysis and Nonlinear Conservation Laws*, Pure and Applied Mathematics Quaterly, Vol. 3, no. 3, 841-879 (2007).

[D] C. M. DAFERMOS, *Hyperbolic Conservation Laws in Continuum Physics*, Springer-Verlag: New York, 1999.

[E] L. C. EVANS, *Partial Differential Equations*, volume 19 of *Graduate*

Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[EG] L. C. EVANS, R.F. GARIEPY, *Measure Theory and Fine Properties of Functions*, CRC Press: Boca Raton, FL, 1992.

[Fa] K. FALCONER, *Fractal geometry*, Chichester, Willey (1990).

[Fe] H. FEDERER, *Geometric measure theory*, Springer-Verlag, New-York, Heidelberg, 1969.

[Fu1] B. FUGLEDE, *Extremal length and functional completion*, Acta Mathematica (1957), Volume 98, 171-219.

[Fu2] B. FUGLEDE, *Extremal length and closed extensions of partial differential operators*, Jul. Gjellerups Boghandel, Copenhagen, 1960.

[G] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser Verlag, Basel, 1984.

[H] L. L. HELMS, *Introduction to potential theory*, Wiley-interscience, New York, 1969.

[HKM] J. HEINONEN, T. KILPELÄINEN, O. MARTIO, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1993.

[Mag] F. MAGGI, *Sets of Finite Perimeter and Geometric Variational Problems, An Introduction to Geometric Measure Theory*, Cambridge University Press, New York, 2012.

[Maz] V.G. MAZ'JA, *Sobolev Spaces*, Springer-Verlag, Springer Series in Soviet Mathematics, 1985.

[PT] N. C. PHUC, M. TORRES, *Characterizations of the existence and removalbe singularities of divergence-measure vector fields* Indiana Univ. Math. J. Vol. 57, no. 4, 1573-1598, 2008.

[S] M. ŠILHAVÝ, *Divergence measure fields and Cauchy's stress theorem* Rend. Sem. Mat. Univ. Padova 113 (2005), 15-45.

[St] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, no. 30, Princeton University Press, Princeton N.J:, 1970 MR 0290095 (44 7280).

[VH] A.I. VOL'PERT, S.I. HUDJAEV, *Analysis in Classes of Discontinuous Functions and Equation of Mathematical Physics*, Martinus Nijhoff Publishers, Dordrecht, 1985.

[W] H. WHITNEY, *Geometric Integration Theory*, Princeton University, Princeton, N. J., 1957.

[Z] W.P. ZIEMER, *Weakly Differentiable Functions*, Springer-Verlag, New York, 1989.