Quantization of Integrable Systems with Monodromy

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Geometry and Quantization 2007

Outline

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Integrable System

Proper map $X^{2n} \to B^n$ on a symplectic manifold (X, ω) with

1. $\{H_i, H_j\} = 0$
2. $\boldsymbol{H}$ has only non-degenerate singularities.

$$\Downarrow$$

Regular connected fibers = tori $\boldsymbol{H}^{-1}(E) \cong \mathbb{R}^n/\mathbb{Z}^n$ labelled locally by *action coordinates* I_i

$$dI_i(E) := \int_{\gamma_i(E)} \omega$$

unique up to $Gl(n, \mathbb{Z}) \rtimes \mathbb{R}^n$.

$$\Downarrow$$

$\mathbb{Z}$-affine structure on regular part of $B := \widehat{\boldsymbol{H}(X)}$.

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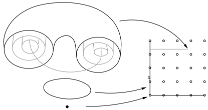
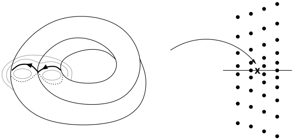
Non-degenerate singular fibers

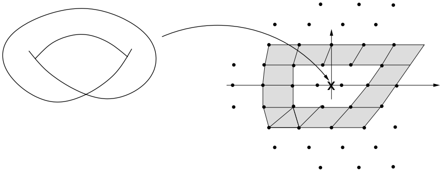
Elliptic Lower dimensional T^n orbits:



Hyperbolic (In)stable manifolds of saddle points:



Focus-focus Instable equilibrium with S^1-symmetry:



Reconstruction (Zung, Ngoc '01)

Any integrable system can be reconstructed from its base (B with $\mathbb{Z}$-affine structure and singular stratification) up to

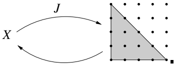
1. $H^1(B, Z^1_B/\Lambda_B)$ (w.r.t. some reference system)
2. Local fiber preserving symplectomorphism type near focus-focus and hyperbolic fibers.

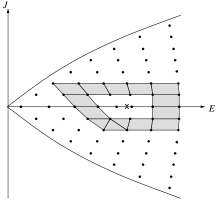
$$\Downarrow$$

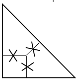
Any contractible B “determines" integrable system $X(B)$.

Examples

- $X(B)$ toric (compact with only elliptic singularities)

  $\iff$ B Delzant polytope. $\mathbb{CP}^2$: 

- Spherical pendulum: 

- K3-surface: Glue 8  to S^2.

Outline

Complex toric manifold

Integral Delzant polytope $B \longleftrightarrow \mathbb{C}$-projective manifold $\mathfrak{X} = \text{proj cone}_{\mathbb{Z}} B$ equal to torus compactification

$$(\mathbb{C}^*)^n \hookrightarrow \mathbb{CP}^k : x \mapsto [x^{\alpha_1}, \ldots, x^{\alpha_k}], \qquad \{\alpha_1, \ldots, \alpha_k\} = B \cap \mathbb{Z}^n$$

Relation to $X(B)$? $\rightsquigarrow T^n$-equivariant diffeomorphism

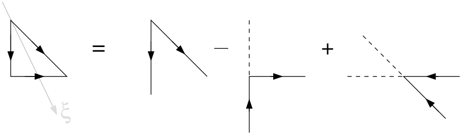
$$\mathfrak{X}(B) \cong X(B)$$
$$\Downarrow$$

- Action coordinates = Legendre transform of polar $(\mathbb{C}^*)^n$-coordinates $I_i = \frac{\partial f}{\partial \rho_i}, \quad \omega = \partial\bar{\partial} f$.
- Integral points $B \cap \mathbb{Z}^n \longleftrightarrow$ weight space basis of $H^0(\mathcal{O}_{\mathfrak{X}(B)}(1))$.

Induced Quantization

$\Rightarrow H^0(\mathcal{O}_{\mathfrak{X}}(k))$ canonical T^n-representations with classical limit:

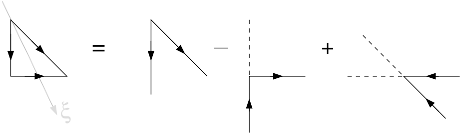
$$\begin{tikzcd}
& (2\pi\hbar)^n \sum_{\alpha\in\Delta\cap\hbar\mathbb{Z}^n} \delta_\alpha \arrow[dl, "\hbar\to 0"'] \arrow[dd] \\
\frac{1}{n!} J_* \omega^n \arrow[dd, "\mathcal{F}"'] & \\
& \exp \xi|_{H^0(X, \mathcal{O}_X(\frac{1}{\hbar}))} \arrow[dl] \\
\int_X \exp\left(iJ.\xi\right) \frac{\omega^n}{n!} &
\end{tikzcd}$$



Induced Quantization

$\Rightarrow H^0(\mathcal{O}_{\mathfrak{X}}(k))$ canonical T^n-representations with classical limit:

$$
\begin{tikzcd}
& (2\pi\hbar)^n \sum_{\alpha\in\Delta\cap\hbar\mathbb{Z}^n}\delta_\alpha \arrow[rr, equal] \arrow[dl, "\hbar\to 0"'] \arrow[dd] & & (2\pi\hbar)^n \sum_p \sum_{\alpha\in \mathrm{cone}_p(J)\cap\hbar\mathbb{Z}^n} \epsilon_p\delta_\alpha \arrow[dl] \arrow[dd, "\mathcal{F}"] \\
\frac{1}{n!}J_*\omega^n \arrow[rr, equal] \arrow[dd, "\mathcal{F}"'] & & (2\pi)^n \sum_p \epsilon_p d^n\alpha|_{\mathrm{cone}_p(J)} \arrow[dd] & \\
& \exp\xi|_{H^0(X,\mathcal{O}_X(\frac{1}{\hbar}))} \arrow[rr, equal] \arrow[dl] & & (2\pi\hbar)^n \sum_p \frac{\exp(iJ(p).\xi)}{\prod_j\left(1-\exp(i\hbar\alpha_j^{(p)}.\xi)\right)} \arrow[dl] \\
\int_X \exp(iJ.\xi)\frac{\omega^n}{n!} \arrow[rr, equal] & & \sum_p (2\pi i)^n \frac{\exp(iJ(p).\xi)}{\prod_j \alpha_j^{(p)}.\xi} &
\end{tikzcd}
$$



Outline

Rational equivalence

Problem: No $\mathbb{R}^n$-invariant $\mathbb{C}$ structure at focus-focus points.

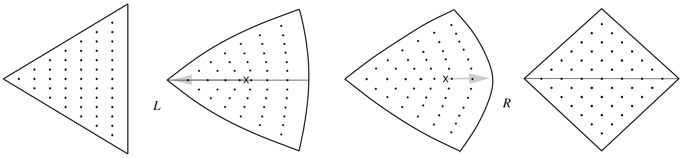
1. Idea: Rational equivalence preserves representation space
$\rightsquigarrow$ Symplectic analogue given by *two* types of blow up:

Problem: Both blow ups change symplectic volume.

Hamilton-Hopf bifurcations

2. Idea: Symplectic volume preserved by cuts and

Hamilton-Hopf bifurcations :

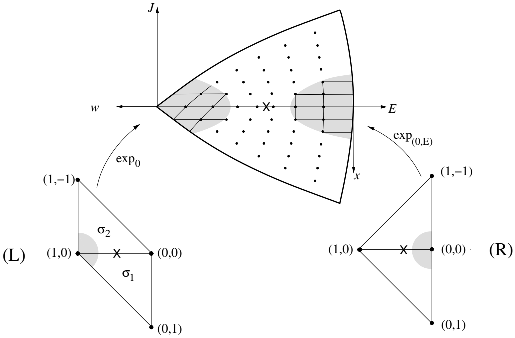


- relates two blow ups
- results are toric

$\rightsquigarrow$ need *both* bifurcations to keep monodromy information.

$\mathbb{C}$-analogue of bifurcations

View base as intersection complex $\{xy = 0\} \subset \mathbb{CP}^3$ and perturb it to toric models (L), (R) or *"both"*
$\mathfrak{X}_t := \{xy + t(zw + z^2) = 0\}$.



The two toric models coincide with the two bifurcations!

$\mathbb{C}$-analogue of integrable 4-manifolds

GROSS, SIEBERT: Any simple polyhedral decomposition of B determines such locally toric smoothings up to $H^1(\check{B}, i_*\Lambda_{\check{B}} \otimes_{\mathbb{Z}} \mathbb{C}^*)$.

$$\Downarrow$$

Lemma

Any compact integral 2-base B admits an integrable realization $X(B)$ and a polyhedral decomposition such that

$$\mathfrak{X}_t(B) \cong_{C^\infty} X(B)$$

Moreover, if B is contractible:

Lemma

$\mathfrak{X}_t$ is the result of Hamilton Hopf bifurcations determined by the polyhedral decomposition to a toric manifold.

Induced Quantization

Still bijection $\hbar\mathbb{Z}$-points in $B \longleftrightarrow$ base of $H^0\left(\mathcal{O}_{\mathfrak{X}_t}(\frac{1}{\hbar})\right)$ for all t.

$\rightsquigarrow$ Quantize cut normalizers via BT and retraction $\mathfrak{X}_t \to \mathfrak{X}_0$.

Outlook:

- How does monodromy translate in these BT-families?
- Meaning of A_∞ structure on the classical limit?