THE MIRROR OF A FANO VARIETY

Talk held at the Institut Fourier, Grenoble, 2010

The first part of the talk will review the Gross-Siebert approach to mirror symmetry. The second part will then provide an interpretation of this approach in terms of tropical curve counting. This in turn allows us to construct the mirror of a Fano manifold and attack the mirror symmetry conjectures.

# 1 Introduction to the Gross-Siebert approach to mirror symmetry

The common slogan you have probably heard of is that the mirror of a Fano manifold $X$ ia a so called Landau-Ginzburg model, i.e. a non-compact manifold $X$ together with a holomorphic function. Mirror symmetry is then supposed to exchange the complex and symplectic geometry of the Fano manifold and its mirror, for instance the derived category of coherent sheaves on should be equivalent to the derived category of vanishing cycles of $W$

$$\mathrm{D}^b(\mathrm{coh}(\check{X})) \sim \mathrm{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W))$$

In case of surfaces $\mathrm{Lag}_{\mathrm{vc}}(W)$ is indeed a honest category whose objects are vanishing cycles, viewed as Lagrangian submanifolds together with an ordering, morphisms are order preserving intersections of these vanishing cycles and compositions are given by counting holomorphic triangles between three cycles.

Of course, the question arises: What is the mechanism behind such mirror symmetry relations?

An explanation comes from the well-known SYZ conjecture: Namely, in complement of anticanonical Divisor $D$, $\check{X} \setminus D$ and $X$ should admit dual singular torus fibrations. So we have two maps

$$\check{X} \setminus D \searrow B \swarrow X$$

with same target $B$ and the fibers over a general point in $B$ are dual Lagrangian tori.

So it remains to clarify the origin of the LG-potential $W$. $W$ is supposed to count pseudoholomorphic Maslov index 2 disks in $\check{X}$ with boundary on a torus fiber. Such disks are precisely the obstruction to the definition of Floer cohomology, so the set of critical points of the potential should consists of unobstructed tori (well-defined Floer cohomology given by counting disks between two Lagrangians in a torus fiber).

Our aim is now to give an algebro-geometric construction of $W$ by counting tropical disks.

We will do that in three steps:

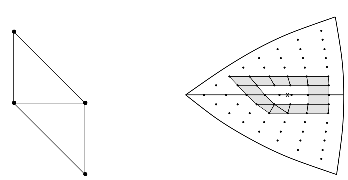
$$\begin{tikzcd} \check{X} \arrow[d, "I"] & (X,W) \\ \check{B} \arrow[r, "II"] & B \arrow[u, "III"'] \end{tikzcd}$$

**Step I: From complex to singular affine manifolds** The essential point here is that we not only consider a Fano manifold $X$ but a degeneration $\mathfrak{X} \to \mathbb{C}$ of it, such that the following holds:

1. First, the degenerate fiber $X_0$ is union of toric varieties which intersect in toric strata. It follows that there is an inclusion preserving one to one correspondence between toric strata of the degenerate fiber and integral convex polytopes. Namely, given a polytope $\sigma$, we can take its cone, then we take the monoid ring of its integral points which is graded by the height of the cone and apply the Proj construction:

   $$\begin{array}{ccc} \text{Convex integral polytope in } B & \longleftrightarrow & \text{Toric stratum in } \mathfrak{X}_0\text{:} \\ \sigma & \longleftrightarrow & \operatorname{Proj} \mathbb{C}[\operatorname{cone} \sigma \cap \mathbb{Z}^n \times \mathbb{N}] \end{array}$$

   So the degenerate fiber determines a polyhedral complex as its intersection complex and vice versa. For instance, a standard 2-simplex $\sigma \subset \mathbb{R}^2$ corresponds to a projective plane $\mathscr{P}^2 = \operatorname{Proj} \mathbb{C}[x,y,z]$ (where $x,z,y$ are vertices of $\sigma \times \{1\}$, which generate the integral points in the cone $\mathbb{R}_{\leq 0}(\sigma \times \{1\})$). Hence the intersection complex(es) below

   correspond to two projective planes intersecting in a projective line.

   In particular, the two polytopes here provide two charts of our intersection complex $B$, namely the two polytopes, but these do not cover $B$, as we don't have any chart around the interior (horizontal) line. As this is precisely the singular locus of the degenerate fiber, such charts come from additional information on the degenerating family. Namely:

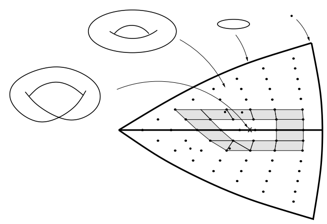
2. Locally apart from a codimension 2 set, $\mathfrak{X} \to \mathbb{C}$ is affine toric. That means, that near a vertex $v$, there is a piecewise linear function such that the local degeneration is given as follows: Take the cone *spanned* by the graph of $\varphi$, this defines a monoidring such that its last coordinate is identified with the deformation parameter. This is our local toric

model of the degeneration at a zero stratum corresponding to the vertex $v$.

| | | |
|---|---|---|
| PL germ at $v \in B^{[0]}$ | $\longleftrightarrow$ | Affine toric model of $\mathfrak{X} \to \mathbb{C}$ at $v$: |
| $\varphi$ | $\longleftrightarrow$ | $\mathbb{C}[t] \to \mathbb{C}[\mathrm{span}_{\mathbb{N}}\{(m, \varphi(m))_{m \in \mathbb{Z}^n}\}], t \mapsto z^{(0,1)}$ |

This provides indeed an integral chart at the vertices, hence apart from a codimension two locus $\Delta$, we have covered $B$ by charts whose transition maps lie in the integral affine group $GL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$. This is called a singular integral affine structure.

Now, as integral affine manifold there are precisely two intersection complexes which belong to a (simple) degeneration to two projective planes, namely the toric one we considered a minute go and the other where one corner has been smoothed in favor of a smooth boundary here. The price for this smoothing is the appearance of monodromy.



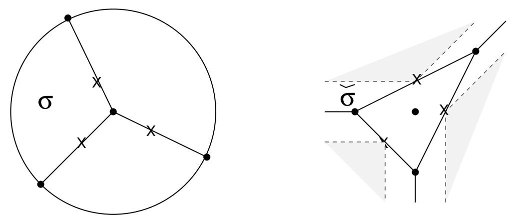
Remember that we have constructed the integral affine manifold as intersection complex of (the degenrate fiber of) a degeneration of $X$, but we can alternatively view it as the base of a singular torus fibration of the Fano manifold $X$: Namely, the fibers over regular points are regular tori, while approaching the boundary the tori collpase to lower dimensional tori (a circle and then a point). The fiber over the singular locus is precisely a selfintersecting sphere, for dimensional reasons drawn as pinched torus. Now if we degenerate the Fano, this piched torus will collapse further to a circle such that we recover $B$ as intersection complex.

[Symplectically, the integral points are the integral periods of the symplectic form w.r.t. the fibration.]

**Step II: Legendre transform** Now we come to the easy part of our diagram. Namely, the central idea of the Gross-Siebert program is that

Mirror dual manifolds admit degenerations with
*dual intersection complexes*:

**Example:** Intersection complex of a projective degenerating into a complete intersection of three projective planes and its dual:




As cell complex, this is just the Poincare dual, for instance maximal cell $\sigma$ maps to a vertex $\breve{\sigma}$. But in addition, this is a duality of integral affine tructures: Remember that we have a PL function $\varphi$ at each vertex which encoded the local model of the degeneration near the corresponding 0-stratum af the degenerate fiber. Now this PL function encodes precisely the polytope of the dual intersection complex by a discrete Legendre Transform:

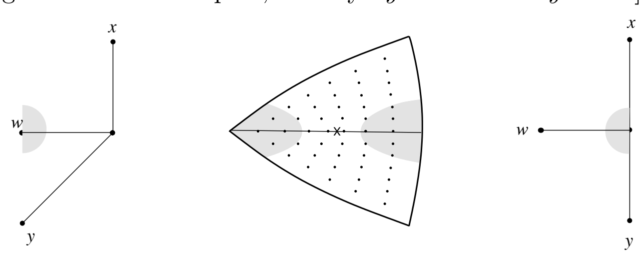
$\varphi(x) = \sup \langle \sigma, x \rangle$ for $\varphi \in PL_{B,\breve{\sigma}}$

In particular, if the boundary of our intersection complex was an affine geodesic, the dual unbounded rays will be all parallel, hence the integral affine structure is obtained by cutting out the grey triangles and identifying the cut faces along the invariant direction.

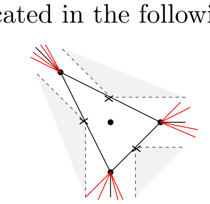
**Step III: From singular affine to complex manifolds** We are now row ready for our third step, namely: How to reconstruct $\mathfrak{X} \to \mathbb{C}$ from $B$?

The problem is that the local toric models $\operatorname{Spec} R_\varphi \to \operatorname{Spec} \mathbb{C}[t]$ of $\mathfrak{X} \to \mathbb{C}$ only glue canonically if the integral affine structure of $B$ has no singular locus $\Delta$, hence there is no monodromy.

[Example: Due to monodromy, have two local models at the right vertex of the following intersection complex, namely $xy = t$ versus $xy = wt$:]




The conjectural solution proposed by physicists is that pseudoholomorphic disks should define walls such that crossing a wall leads to automorphisms of the local models. What we will do now is replacing the disks by tropical disks that emanate from $\Delta$ as indicated in the following picture:



This will then indeed recover the Gross-Siebert algorithm.

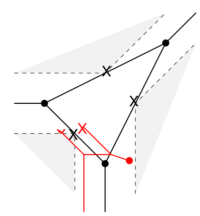
## 2 Construction of Fano mirror duals by counting tropical disks

**Virtual tropical disks** Let $B$ be an integral affine manifold with singular locus $\Delta$.

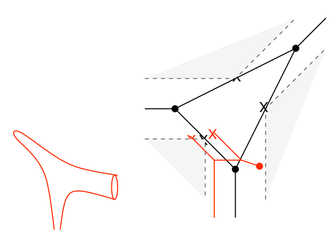
*Definition* 1. A *virtual tropical disk* is an immersed weighted tree $h : \Gamma \to B \setminus \Delta$, one example drawn in red below. From this example you can extract the following four properties:

1. First, each edge maps onto an integral affine line in $B$
2. Setting al weights to 1, the primitive tangent vectors at an internal vertex sum up to zero,
3. The bounded Leaves emanate from "general small perturbations" of $\Delta$ in the monodromy invariant direction. You should think of this distance as infinitesimally small, so technecically work in the tangent space, and "general" means that the perturbations of the singular points here lie in a certain complement on a nowhere dense subset of codim 1.
4. Finally, unbounded leaves map onto asymptotic rays, and their weighted nmber is called half the Maslov index.

*Example:*



The intuition behind these definitions is again given by torus fibrations: If you consider a cycle in a torus fiber, then in order to contract it and hence to sweep out a disk, it has to desintegrate into vanishing cycles of the singularities, so either to a pinched torus (which corresponds to a bounded leaf) or smaller tori (unbounded leaf), while balancing means that the cycle classes over an interior vertex match to form a pair of pants. The attribute "virtual" refers to pulling apart the bounded leaves in order to distinguish different intersection types with the singular fiber. The number of ounbounded leaves then corresponds to the intersection number with the anticanonical divisor, hence half the Maslov index.



- The moduli space of such virtual disks has a natural stratification by type, which is what remains of the disk if you forget about the length of the immersed edges, so keep only the graph and the local embedding at each internal vertex, i.e. the tangent vectors for each vertex parallel transported to the root.

  For instance, if an edge contracts, then you change to a lower dimensional stratum.

- A virtual disk $h$ is *general* if "a small deformation of the leaves lifts to a deformation of $h$ preserving its type".

  For instance, the disks below are not stable, as a 4-valent vertex can be deformed into two 3-valent ones, or such a 3-valent disk does no longer exist if we move one of the leaves away from the drawing plane..


The central property of general strata is now that they are of the expected dimension, namely the dimension of $B$ plus half the Maslov index minus 1.

In particular this implies that if we fix the root we can count (types of) general virtual disks of Maslov index $\leq 2$. We will count with *virtual multiplicity*, defined e.g. for $\dim B = 2$ by the rational number

$$\mathrm{mult}(h) := \frac{1}{\mathrm{Aut}(h)} \prod_{V \in \Gamma^{[0]}} \begin{cases} \frac{(-1)^{|m_V|}}{|m_V|^2} & V \text{ a leaf vertex of weight } |m_V| \\ |m_V \wedge m'_V| & V \text{ is trivalent} \end{cases}$$

**Construction of $X$** Now come to our central definition: Namely, we can define the counting function in the formal completion of our local toric models by interpretating the coefficient of a monomial with exponents $(m, k,)$ as the number of general virtual Maslov index zero disks with root tangent vector $m$ and area $k$, where the area is defined as the total change in the $\varphi$ slope along the disk

$$\log f_x = \sum_{(m,k)} \#\{h \in \mathcal{M}_0^{general} |\ m \text{ tangent vector of } h \text{ at } x, k = \mathrm{area}(h)\} |m| z^{(m,k)}$$

This function of course depends on the position of the root, so we get a map $B \to R_\varphi/(t^k), x \mapsto \log f_x$, As the moduli space of index zero disks with arbirary root had dimension equal to the dimension of $B$ -1, the counting function is trivial except on polyhedral subsets of codimension one, wich are our walls.

*Theorem* 1 (Conjecture for $\dim B \geq 3$). The theorem is now that these walls refine the cell decomposition of $B$. Crossing a wall $\rho$ leads to an automorphism of (a certian localization of) the local toric model $R_\varphi/(t^k)$ of our degeneration which is given simply by multiplying a monomial with the exponential of the counting function, but to the power of the exponenent projected to the primitive conormal vector of the wall:

$$z^{(m,k)} \mapsto z^{(m,k)} f_{\rho,x}^{\langle\rho^\perp,m\rangle}$$

The claim now is that these wall crossing automorphisms give rise to consistent gluings of the local models such that we get a formal scheme over Spec $\mathbb{C}[t]$, and this recovers in fact the Gross-Siebert degeneration $\mathfrak{X} \to \operatorname{Spec} \mathbb{C}[\![t]\!]$.

**Construction of the LG-potential** $W$ Similarily, we can now define the local LG-potential $W \in R_\varphi$ by counting virtual general Maslov index 2 disks:

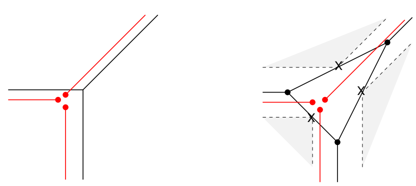
$$W := \sum_{(m,k)} \#\{h \in \mathcal{M}_2^{general} |\ m \text{ root tangent vector of } h, k = \operatorname{area}(h)\} z^{(m,k)}$$

As our degenerations were constructed by counting zero disks and zero disks are just the legs of Maslov index two disks, it is no astonishing that this function behaves well under wall crossing, i.e. defines a function on our degeneration $\mathfrak{X}$:

*Theorem* 2. $W \in \mathcal{O}(\mathfrak{X})$ (that is, $W$ behaves consistently under wall crossing).

For example, if we take the toric mirror of the projectve plane, the LG-potential defined by counting Maslov index 2 disk is just the usual Hori-Vafa mirror: Namely, 2-disks correspond to the generators of the rays of the normal fan. $W = x + y + z \in \mathbb{C}[x,y,z]/(xyz - t)$

The same is still true for the non-toric mirror of the projective plane, but here the expression holds only on the interior cell corresponding to an open subset of the total space, and gluing in the other pieces corresponds to a compactification of the fibers of the potential, i.e. a properification.



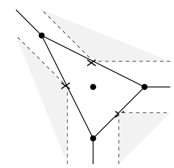
**Properties of the potential** $W$

1. *(LG-CY-correspondence)* $W$ is proper
   $\Leftarrow$: $\partial\check{B}$ is an affine submanifold
   $\Leftarrow$: $\partial\check{B}$ is the intersection complex of a degeneration of a smooth Calabi-Yau divisor $\mathfrak{D} \subset \check{\mathfrak{X}}$.

   Note that this is not the case for the Hori-Vafa mirror, where the boundary divisor is an intersection of toric varieties and in particular not a smooth Calabi-Yau.

2. Moreover, if in addition our Fano manifold is a toric surface and the boundary divisor ample, then the intersection complex is unique, for instance the proper version of the mirror of the projective plane is precisely the picture we've seen so many times:

$$\mathbb{P}^2\text{-degeneration} \longleftrightarrow B =$$



3. *(Mirror symmetry)* Last but not least, if $\check{X}$ is a Fano surface $D^b(\mathrm{coh}(X)) \sim D^b(\mathrm{Lag}_{\mathrm{vc}}(W))$.

   [In fact, parts of this equivalence can be seen directly from our integral affine manifolds: Namely, we can perform a hyperKähler rotation such that the torus fibration over $B$ becomes the Lefshetz fibration of the LG-potential, and simple Maslov index zero disks in the dual base correspond to Lagrangian vanishing cycles of $W$, hence we can directly read of intersection numbers from the dual intersection complex. Then we get a quiver which corresponds to the tilting object $\mathcal{O}e_1 \oplus \mathcal{T}(-1)e_2 \oplus \mathcal{O}(1)e_3 \oplus_{i>3} \mathcal{O}(p_i)e_i$ in $D^b(\widetilde{\mathbb{P}^2})$ as in Auroux-Katzarkov-Orlov...]