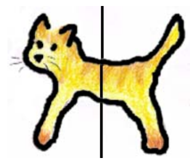
GAUGE THEORY OF THE FALLING CAT

A Review of R. Montgomery's treatement



Rapport du stage à l' Institut Fourier - Michael Carl

The configuration space of a rigid body. A rigid body is a Borel-mesure m on the euklidean $\mathbb{R}^3$ with compact connected support assigning to any Borel-set in $\mathbb{R}^3$ its mass. Its configuration space Q is the affine group of its rotations and translations

$$Q := \{q(g,b) : x \mapsto gx + b,\ x \in \mathbb{R}^3 \,|\, g \in SO(\mathbb{R}^3), b \in \mathbb{R}^3\} \cong SO(3) \rtimes \mathbb{R}^3$$

equipped with the invariant metric $\langle .,. \rangle$ induced by kinetic energy

$$L : TQ \to \mathbb{R} : L(q,\dot{q}) = \tfrac{1}{2} \int |\dot{q}(x)|^2 dm(x) =: \tfrac{1}{2} < \dot{q}, \dot{q} > .$$

So Q, viewed as $SO(3)$-bundle, carries a natural connection given by the orthogonal complement to the vertical distribution. It consists of states of vanishing **angular momentum** $\mu := I^{-1}\alpha$, where α denotes the corresponding connection form and $I :=_{Q^*} \langle .,. \rangle$ the inertia as in appendix .

More explicite we may identify the Liealgebra of $SO(3)$ with $(\mathbb{R}^3, \times)$ via the Hodge-isomorphism $\star$ which maps an $\omega \in \mathbb{R}^3$ to the generator of a rotation around ω of angle $|\omega|$. Then by (7) and (6) μ becomes the usual expression

$$\omega^t.\mu(q,\dot{q}) = \langle \omega_Q(q), \dot{q} \rangle = \int \omega \times q.\dot{q}\ dm = \omega^t. \int q \times \dot{q}\ dm. \tag{1}$$

Note that two rigid bodies m_1, m_1 produce the same metric and hence connection on Q if their masses $m_i(\mathbb{R}^3)$, their center of masses $c_i := \int x dm_i(x)$ and their basic inertias $I_i(id) \in (\mathbb{R}^3 \otimes \mathbb{R}^3)^*$ coincide. Indeed, in the identification $\Theta : TSO(3) \cong SO(3) \times \mathbb{R}^3$ via left translation $\dot{g} \mapsto \star \dot{g} g^{-1}$ L decomposes as

$$2L(\dot{g}, \dot{b}) = I(id)(\otimes^2 g^{-1}\Theta(\dot{g})) + |\dot{b}|^2 m(\mathbb{R}^3) + 2\dot{b}^t.\dot{g}c.$$

In this case we call the bodies equivalent $m_1 \sim m_2$.

The configuration space of the falling cat. We now model the falling cat by decomposing it into two rigid bodies m_1, m_2 (c. figure above), the front and the back half, which are mirror symetric up to equivalence

$$m_1 \sim m_2 \circ \sigma$$

with respect to a mirror $\sigma \in W$. Here W denotes the manifold of mirrors, i.e. orthogonal transformations $W \subset O(3)$ having eigenvalues $(-1,1,1)$, canonically diffeomorphic to the projetive space of mirror axes $P^2\mathbb{R}$ by

$$^\vee : W \to P^2\mathbb{R} : \sigma^\vee = \ker(\sigma + id_{\mathbb{R}^3}).$$

The cat's configuration space C is then defined as the cartesian product $Q \times Q$ of the body half's configurations spaces restricted by the following conditions:

(1) The total center of mass is fixed $q_1(c_1) + q_2(c_2) = 0$. This assumption is justified since any closed system in a homogenous exterior gravitation field splits into the dynamics of the center of mass and kinetic energy in the fixed center of mass system, and we are only interested in the latter.
(2) The bodies are joined, i.e. $q_1(0) = q_2(0)$ for all $q_i \in Q$.
(3) The body half's configurations are mirror symmetric with respect to *any* mirror $\sigma(q_1, q_2) \in W$, i.e. $q_1 = \sigma(q_1, q_2)q_2$. This condition avoids anatomically impossible twists of the cat.

The first two conditions determine the translational parts once the positions $g_i \in SO(3)$ are given, more explicite

$$i : SO(3)^2 \to Q \times Q : \; i(g_1, g_2) = (g_1, b; g_2, -b), b = \tfrac{1}{2}(g_2 c_2 - g_1 c_1)$$

definines an imbedding whose image $\operatorname{im} i$ is Q^2 restricted by conditions 1 and 2. The embedding i is equivariant with respect to the diagonal action of $SO(3)$ representing rigid rotations of the cat. Since this action is free and proper, $\operatorname{im} i$ becomes a principial $SO(3)$-bundle whose orbits $SO(3)(g_1, g_2) = SO(3)(e, g_1^{-1} g_2)$ can be represented by the "configuration difference" $g_1^{-1} g_2$ corresponding to the positions of the second half in the reference system of the first.

We now take care of the remaining no twist condition which we may write as

$$C = \{\, i(g, \sigma g \sigma) \in Q \mid \; g \in SO(3), \sigma \in W \subset O(3) \} \,.$$

The condition is well defined in the sense that it forbids twists but not rigid rotations: Indeed, $SO(3)$ continues to act freely and properly on C, but the base is reduced to the projective plane of mirror axes:

$$\pi(C) = \{g^{-1} \sigma g \sigma | g \in SO(3), \sigma \in W\} = W\sigma = W.$$

The configurations with fixed mirror $\sigma(q_1, q_2) = \sigma$ for all $(q_1, q_2) \in C$ form a subbundle

$$C_\sigma := \{i(g, \sigma g \sigma) | g \in SO(3)\}$$

of C whose structure group is the normalizer $N(T_\sigma)$ of rotations around the mirror axis $T_\sigma := \exp \ker(\sigma + id)$. So we may regard the base as the homogenous space $SO(3)/SO(3)_\sigma = SO(3)/N(T_\sigma) = \mathbb{R}^2 P$. This is just the Hopf fibration $SO(3) \mapsto SO(3)/T = S^2$ followed by an identification of the antipodals since the Weyl group $N(T_\sigma)/T_\sigma$ acts as reflection along the mirror axis. Thereby we have proved:

Lemma 1. *The mirror fixed configurations C_σ form a reduced $O(2)$-bundle of C isomorphic to the $\mathbb{Z}_2$ quotient of the Hopf bundle $S^3 \to S^2$, where $O(2)$ is the restriction of the normalizer $N(T_\sigma)$ of rigid rotations around the mirror axis $T := \exp \sigma^\vee$ to the mirror plane, and the Weyl group $N(T)/T$ acts as permutation of the alcoves.*

$$\begin{array}{ccc} S^3 & \xrightarrow{/S^1} & S^2 \\ {\scriptstyle /\mathbb{Z}_2}\downarrow & & {\scriptstyle /\mathbb{Z}_2}\downarrow \\ SO(3) & \xrightarrow{/N(T)} & P^2\mathbb{R} \end{array}$$

□

Reorientation of the cat. The induced metric on C (corresponding to total kinetic energy) remains invariant under rigid rotations. The corresponding natural connection $hTQ = \ker T\pi^{\perp}$ consists of states of vanishing total angular momentum

$$hTQ = \ker \mu, \quad \mu := \mu_1 + \mu_2.$$

Since by isotropy of the system J and hence μ is Noether conserved, the connection contains the possible states of the cat in free fall without initial angular momentum. We are intersted in the holonomy of the connection, i.e. loops in the base leading to a global rotation of the initial configuration.

Now any tangent vector T_pC for $p \in C_\sigma$ may decomposed into one respecting the initial mirror and a rigid rotation not respecting it, i.e.

$$T_pC = TpC_\sigma + (\ker \sigma - 1)_C(p).$$

The latter are clearly forbidden by the horizontal condition $m = 0$, wheras restricted to TC_σ the connection form α becomes

$$\alpha = \alpha_1 + \sigma^*\alpha_1 = (1 - \sigma)\alpha.$$

Indeed, we have $\sigma^*\mu_1 = -\sigma\mu_2$ by (1) since σ is an anti-automorphism of the Liealgebra $(\mathbb{R}^3, \times)$ and the body halfs are mirror symetric up to equivalence $m_1 \sim m_2 \circ \sigma$. But $\frac{1}{2}(1 - \sigma)$ is the orthogonal projector on the mirror axix $\sigma^\vee$, so the curvature only takes values in $\sigma^\vee$, i.e. the holonomy bundle lies in C_σ and the connection is reducible to C_σ.

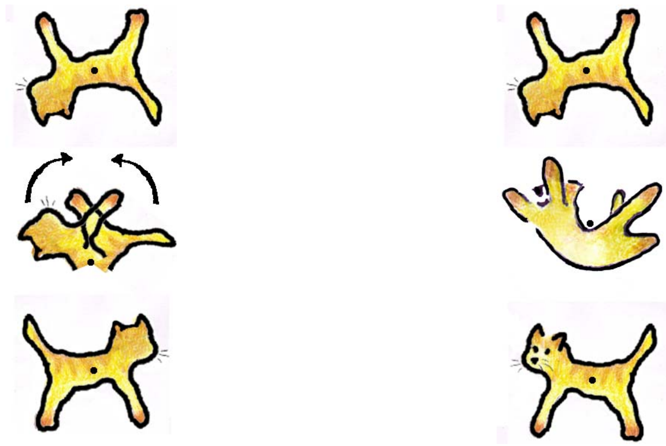
Theorem 1. *The mirror fixed bundle C_σ forms the no twist configurations accessible by the free falling cat without initial angular momentum.*

Proof: It remains to show that every holonomy can be realized. The induced connection on C_σ lifts to a connection on the universal covering Hopf bundle, and here we have:

Lemma 2. *The Hopf bundle $S^3 \mapsto S^2$ is geodesical complete in the subRiemann sense with respect to any connection, i.e. any two points in S^3 can be joint by a horizontal curve with minimal energy.*

By the lifting property it is enough to show that every holonomy can be realized by a sub-Riemannian geodesic. The Hopf bundle is simply connected but not trivial (the 2-dim homology of a section $id_{S^2} : S^2 \xrightarrow{s} S^3 \xrightarrow{\pi} S^2$ yields the contradiction $id_{\mathbb{Z}} : \mathbb{Z} \overset{H_2(s)}{\to} 0 \overset{H_2(\pi)}{\to} \mathbb{Z}$) so it does not admit a flat connection, i.e. $\Omega \neq 0$. This implies a non trivial holonomy group by the Gauß-Bonnet formula, so $hol = S^1$. Now by the Arzela Ascoli theorem any sequence of loops realizing a given holonomy with decreasing length (and so energy) converges. □

Reorientation schemes. A holonomy in the non trivial component of the structure group $O(2)$ corresponds the non-trivial class of the fundamental group $\pi_1(P^2\mathbb{R}) = \mathbb{Z}_2$ of the base. Its holonomy includes a reflection and therefore corresponds to a selfintersection of the cat, i.e. a penetration of the body halfs. So we have the following reorientation schemes:



Reorientation with (left) and without (right) selfintersection

Now for the contractible loops $c =\in [0] \in \pi^1(S)$ the structure group is abelian, hence we have the Gauß-Bonnet formula (3) for the holonomy = reorientation.

Optimal Reorientation in the spherical case. We now calculate the sub Riemannian geodesics representing optimal reorientation, i.e. horizontal lifts of loops minimizing the kinetic energy when the holonomy is fixed, in the following special case: First note that the connection on Q_σ is independent of the initial center of mass positions, so we set $c_i = 0$. We further suppose that the basic inertias $I_i(id) = I$ are isotropic. Together, kinetic energy of the C_σ becomes twice the pullback of I by the Cartan form $L = 2\Theta^* I$ coinciding up to scaling with the Killing metric on $SO(3)$, hence geodesics are 1-parameter subgroups and by invariance of the Killing form the natural connection will be invariant. So it coincides with the Levi Cevita connection and the bundle and its projection have constant sectional curvature. It follows that the sub Riemannia geodesics are horizontal lifts of *circles* by the following

Theorem 2. *Let P be the bundle of orthonormal frames over a Riemannian manifold of constant sectional curvature with respect to the Levi Cevita connection on P. Then the sub Riemannian geodesics are horizontal lifts of curves with constant curvatures.*

Proof after [Mon90]: The curve's curvatures are functions depending pointwise on $\langle x^{(j)}, x^{(i)} \rangle$, $x^{(j)}$ denoting the j'th (covariant) derivative along $x \in \Omega(S)$. By constant sectional curvature K, the Yang Mills equations (9) become $x^{(2)} = Ke \cdot \dot{x}$ and $\nabla_{\dot{x}} e = 0$. Since e is skew, $\langle x^{(j)}, x^{(i)} \rangle = 0$ if $j - i \in 2\mathbb{Z}$ and $\nabla_{\dot{x}} \langle x^{(j)}, x^{(i)} \rangle = 0$ otherwise. □

Finally, the horizontal geodesics in C_σ are exactly the self-intersection loops $t \mapsto (\exp(t\omega), \sigma \exp(t\omega))$ for a $\omega \in \ker(\sigma - 1)$ showing that the cat can't reorientate itself *passively*.

Appendix

Local geometry of G-bundles with connection. This section collects facts from [KN64] relevant for the falling cat.

Let G be a Lie group acting freely and properly from the left on a manifold Q such that the orbitspace $S := Q/G$ is itself a manifold, $Q \overset{\pi_S}{\to} S$ is then called a principal G-bundle over S. Local sections $s_i : U_i \to \pi^{-1}(U_i) : \pi s_i = id_{U_i}$ of π then define local trivialisations

$$h_i : \pi^{-1}(U_i) \to U_i \times G : \; h_i^{-1}(x,g) := s_i g.$$

Therefore the bundle is trivial iff it admits a global section. Two such local sections s_i, s_j differ by a gauge-transformation $s_i/s_j \in U_i \cap U_j \to G$ which we may consider as Check-1-cocycles in the sheaf G_S of local G-valued functions on S, in fact the isomorphism classes of G-bundles are easily shown to be given by $\check{H}^1(S, G_S)$.
Definition of a connection. A *connection* on Q is a smooth G-invariant distribution hTQ transversal to the vertical = fibertangential one given by $\ker T\pi_S$. It is equivalently characterised by a connection form defined as Ad-equivariant $\mathfrak{g}$-valued 1-form α rightinverse to the infinitesimal action $_Q : \mathfrak{g} \to \Gamma(TQ)$, i.e. $g^*\alpha = Ad_g.\alpha$ and $\alpha.\xi_Q = \xi$. The correspondence is the given by $hTQ = \ker\alpha$ and we may express the projection $h \in \Gamma\,\mathrm{End}(TQ)$ on the horizontal distribution as $h := id - v$, $v.X(q) = (\alpha.X)_Q(q)$.
Parallel transport. A fundamental property of the connection is that any curve $c : [0,1] \to S$ in the base has a unique horizontal lift once a lift of the start point is given, i.e. the path lifting map

$$\widehat{\ } : \{{}_q c \in Q \times \Omega(S) \mid c(0) = \pi(q)\} \to \{\widehat{c} \in \Omega(Q) \mid \alpha(\dot{\widehat{c}}) = 0\}$$

is bijektive. Explicitely ${}_q\widehat{c}$ is given by $c = g \cdot d$ where d is any lift $\pi d = c$ starting in q and g satisfies

$$\dot{g}g^{-1} = -\alpha.\dot{d} \tag{2}$$

with $g(0) = 1$. The lifting map defines a parallel transport of fibers

$$|| : \; (\Omega(S, x_0, x_1), \star) \to \mathrm{Iso}(G_{x_0}, G_{x_1}) : \quad ||_c q = {}_q\widehat{c}(1).$$

In particular it gives an action of the loop group $\Omega(S,x)$ with basepoint x on the fiber over x commuting with the G-action. One thus gets a homomorphism of $\Omega(S,x)$ in the holonomy groups Hol_q of the "'differences" $g_c : \; ||_c q = g_c q$ between start- end endpoint of the lift ${}_q\widehat{c}$. The connectivity of points by horizontal curves in fact partitions Q in (isomorphic) holnomy-bundles $[q]$ which are principal bundles over S with reduced structure group Hol_q.
Local Gauß-Bonnet-formula. Suppose now G is abelian. Then by (2) and Stokes theorem we get an explicite local holonomy formula as analogon of the local Gauss-Bonnet-formula:

$$g_c = \exp \int_{int(c)} s^*\Omega. \tag{3}$$

Here Ω denotes the curvature defined as covariant derivative $d^\alpha := h^*d$ of α, and s a local section whose domain contains a surface $int(c)$ with border c. Thus we may interpretate the curvature as infinitesimal holonomy

$$s^*\Omega(\partial_i f|_0, \partial_j f|_0) = \lim_{\epsilon \to 0} \frac{g_{\partial f([0,\epsilon]^2)}}{\epsilon^2}. \tag{4}$$

if $f : \mathbb{R}^2 \to S$ is an embedding. It immediately follows that $\mathfrak{hol}_\mathfrak{q}$ is the image of Ω restricted to $[q]$ (theorem of Ambrose und Singer).

Induced transport in associated bundles. If G operates from the right on any manifold F one defines the associated F-bundle $Q \times_G F$ as quotient of $Q \times F$ by the diagonal action $g \cdot (q, v) = (gq, vg)$. Then by

$$G \cdot (q, v) = G \cdot (q, w) \iff \exists g : \ (q, v) = g \cdot (q, w) \iff v = w \tag{5}$$

any $q \in Q$ defines a G-diffeomorphism of F in the fiber over $\pi(q)$ of the associated bundle, denoted by $v \mapsto qv$. Local sections of Q thus induce local sections of $Q \times_G F$, and we may identify the latter with equivariant maps $Q \to F$. Moreover, parallel transport on Q induces one in the associated bundle by $||_c v := \hat{c}_1 \hat{c}_0^{-1} v$. In particular, if F is a vector space and $X \in \Gamma(c^* E)$ a vector field, one defines its covariant derivative in direction of $\dot{c}(0)$ as

$$\nabla_{\dot{c}(0)} X := \lim_{t \to 0} \frac{\widehat{c}_0 \widehat{c}_t^{-1} X(c_t) - X(c_0)}{t} = \widehat{c}_0 \lim_{t \to 0} \frac{\widehat{c}_t^{-1} X(c_t) - \widehat{c}_0^{-1} X(c_0)}{t} = L_{\dot{\widehat{c}}_0} (\widehat{c}^{-1} X(c)).$$

In particular, we now may express the curvature Ω, considered as $Q \times_G \mathfrak{g}$-valued 2-form F on S, in the usual form

$$F(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Riemannian G-bundles.

The natural connection on Riemannian G-bundles. Let Q be a principal G-bundle over S and $J : T^*Q \to \mathfrak{g}^*$ be the associated momentum map defined as the pointwise dual map of the infinitesimal G-action $_Q : \mathfrak{g} \mapsto \Gamma(TQ)$, i.e.

$$p.\xi_Q = J(p).\xi \tag{6}$$

for all $p \in \Gamma(T^*Q)$.

Then any G-invariant metric $\langle ., \rangle$ on Q induces a **natural connection** on Q given by the orthogonal complement to the vertical distribution $\ker T\pi_S$. Its connection form $\alpha \in \mathfrak{g} \otimes \Gamma(TQ)$ is given by the composition

$$T_q Q \xrightarrow{\langle .,. \rangle} T_q^* Q \xrightarrow{J} \mathfrak{g}^* \xrightarrow{I^{-1}(q)} \mathfrak{g} \tag{7}$$

where the inertia $I \in \mathfrak{g} \otimes \mathfrak{g} \otimes C^\infty(Q)$ is defined by $I(\xi, \eta) = \langle \xi_Q, \eta_Q \rangle$. In particular we may identify the connection with the coisotropic submanifold $\ker J$ of T^*Q whose symplectic reduction $\ker J/G$ is T^*S.

Geodesics and Reduction. This section simply summarizes [Mon90] The geodesic flow on T^*Q is the Hamiltonian flow of the quadratic form $H(p_q) = \frac{1}{2}||p_g||^2$ (**kinetic energy**) associated to $<, >$. Geodesic orbits $t \mapsto \phi_t^{\mathcal{X}H}(q, p)$ in Q with same projected start conditions in TS, so coinciding projections to S, differ from each other by multiplication with a 1-parameter subgroup in G

$$\phi_t^{\mathcal{X}H}(qg, p_1) = \phi_t^{\mathcal{X}H}(q, p_2) g \exp(t(p_2 - p_1)^\flat)$$

according to the product rule. However, the projection is geodesic (in the induced metric so that π_Q Riemann submersion) if and only if its horizontal lifts are.

Now decompose kinetic energy in vertical and horizontal components with respect to the natural connection:

$$H_v(p_q) = \tfrac{1}{2}||v^* p_q||^2 = \tfrac{1}{2} I^{-1}(J(p_q), J(p_q))$$

$$H_h(p_q) = \tfrac{1}{2}||h^* p_q||^2 = \tfrac{1}{2}||p_q - J(p_q)\alpha(p)||^2 = H - H_v$$

where $h, v \in \Gamma \operatorname{End}(TQ)$ denote the orthogonal projectors on the horizontal resp. vertical distribution. Since H_v clearly drops out under symplectic reduction, the reduced geodesic flow $\pi \circ \phi^{\mathcal{X}H}$ is independent of the vertical energy resp. the vertical fibermetric $v^* <,>$. So the projected flows of H and H_h are identical and its projected orbits $c \in \Omega(S, x_0, x_1)$ are equivalently caracterized as follows:

(1) c is projection of a geodesic in Q.

(2) c is projection[2] of an orbit of H_h. H_h drops to $H(p_q, \nu) = ||p_q||^2_{g/G} + ||\nu||^2_{I/G}$ in a local trivialisation $G \cdot \mu \times T^*U$ of the reduced bundle $(G \cdot \mu)(\pi^* T^* S)$ with its magnetic symplectic form given above. The corresponding Hamiltonian equations are equivalent to

(3) c is the projection of a H_h-Orbit.

(4) $\hat{c}$ is sub Riemannian geodesic, i.e. has stationary lenth within the horizontal pathes joining given points in the holonomy bundle. This may be reformulated as variational problem on the path manifold $\Omega(Q, p, q)$ with horizontal path constraint $\gamma^* \alpha = 0$, i.e. stationarity of

$$S_{\mu \in \Omega(\mathfrak{g}^*)} : \Omega(Q, p, q) \to \mathbf{R} : (\gamma, \mu) \mapsto \int ||h\dot{\gamma}|| dt + \int \mu . \gamma^* \alpha$$

where μ is the Lagrange multiplier. First decompose a variational vector field (tangential vector of the path mfd) into vertical and horizontal component. The vertical generates the variation $\gamma_\epsilon(t) := \gamma(t) \exp(\epsilon \xi(t)), \quad \xi \in \Omega(\mathfrak{g}, e, e)$ so using $\gamma^* \alpha = 0$ we get by the product rule: $\frac{d}{d\epsilon}\big|_0 S(\gamma_\epsilon, \mu) = \int \mu . \dot{\xi} = -\int \dot{\mu} . \xi$. Thereforee stationarity yields the conservation law $\mu = const$ that we may interpretate as **conservation of charge**

$$\frac{d^\alpha e}{dt} = 0; \quad e := G \cdot (\gamma, \mu) \in \Omega(Q \times_G \mathfrak{g}^*) \tag{8}$$

Now we consider horizontal variational fields $H \subset T\Omega(Q, p, q)$. Then the variation of the first S-summand is equivalent to that of the usual geodesic lagrangian $X \mapsto -\int ||\dot{c}||^{-1} < \nabla_{\dot{c}} \dot{c}, X > dt$, and the variation of the Lagrange multiplier term is given by $X \mapsto \int L_X \gamma^* \alpha = \int \gamma^*(i(X) d\alpha) = \int \Omega(X, \dot{\gamma})$. So the S-stationarity with respect to H drops entirely to Euler-Lagrange-Equation on orbitspace: $0 = e.i(\dot{c})F - ||\dot{c}||^{-1} \nabla_{\dot{c}} \dot{c}^\flat$, where F is the $Q \times_G \mathfrak{g}$-valued 2-form induced by Ω. This implies $||\dot{c}|| = const$, so by rescaling $e \mapsto ||\dot{c}|| e$ we get the **Lorentz-Yang-Mills force law**:

$$\nabla_{\dot{c}} \dot{c} = e.(i(\dot{c})F)^\sharp . \tag{9}$$

(5) c has stationary length in the subset of pathes with given fixed parallel translation $\{\gamma \in \Omega(S, x_0, x_1) \mid ||\gamma = const \in \mathrm{Aut}(\pi^{-1}(x_0), \pi^{-1}(x_1))\}$.

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