Combinatorial Multiple Eisenstein Series

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partly joint work with H. Bachmann

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Outline

Multiple zeta values

Definition

- For integers $k_1 \geq 2, k_2, \ldots, k_d \geq 1$, we call

$$\zeta(k_1, \ldots, k_d) = \sum_{m_1 > \cdots > m_d > 0} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}$$

  a **multiple zeta value (MZV)** of weight $k_1 + \cdots + k_d$ and depth d.
- Let

$$\mathcal{Z} = \langle \zeta(k_1, \ldots, k_d) | d \geq 0, k_1 \geq 2, k_2, \ldots, k_d \geq 1 \rangle_{\mathbb{Q}},$$

  where $\zeta(\emptyset) = 1$, be the $\mathbb{Q}$-vector space spanned by the MZVs.

Multiple zeta values: EDS relations

Products of MZVs are $\mathbb{Q}$-linear combinations of MZVs, so the space $\mathcal{Z}$ is an algebra. There are two kinds of such formulas, e.g., in depth 2, we have

$$\zeta(k_1)\zeta(k_2) = \begin{cases} \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2) & \textbf{(stuffle product)} \\ \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j,k_1+k_2-j) & \textbf{(shuffle product)}. \end{cases}$$

We get the **double shuffle relations**.

Actually, the MZVs satisfy the **extended double shuffle relations (EDS)** obtained by regularization. Conjecturally, these are all relations between MZVs.

Formal multiple zeta values

Definition

Let $\mathcal{Z}^f$ be the algebra of **formal multiple zeta values**, i.e., the formal symbols $\zeta^f(k_1, \dots, k_d)$ satisfy the extended double shuffle relations and no other relations.

- The algebra $\mathcal{Z}^f$ is graded for the weight.
- We have a surjective algebra homomorphism

$$\mathcal{Z}^f \to \mathcal{Z},$$
$$\zeta^f(k_1, \dots, k_d) \mapsto \zeta(k_1, \dots, k_d).$$

Conjecturally, this morphism is an isomorphism.

Formal multiple zeta values: results and conjectures

Racinet introduced a pro-unipotent affine group scheme DM_0 with values in a graded Hopf algebra, such that we have for all $\mathbb{Q}$-algebras R

$$\mathrm{Hom}_{\mathbb{Q}\text{-alg}}\left(\mathcal{Z}^f \big/ (\zeta^f(2)), R \right) \simeq \mathrm{DM}_0(R).$$

By a deep theorem of Racinet, there is a bijection between DM_0 and its Lie algebra $\mathfrak{dm}_0$, from which we obtain

Theorem (Ecalle, Racinet)

The algebra $\mathcal{Z}^f$ of formal MZVs is a free polynomial algebra, more precisely

$$\mathcal{Z}^f \simeq \mathbb{Q}[\zeta^f(2)] \otimes \mathrm{Sym}\left(\mathrm{gr}\, \mathfrak{dm}_0^\vee \right).$$

It is conjectured by Ihara-Deligne that $\mathfrak{dm}_0$ is a free Lie algebra with exactly one generator in each odd degree ≥ 3. From this, we can deduce Zagier's dimension conjecture for formal MZVs:

$$\sum_{k \geq 0} \dim_{\mathbb{Q}}(\mathcal{Z}_k^f) t^k = \frac{1}{1 - t^2 - t^3} = \frac{1}{1 - t^2} \frac{1}{1 - t^3 - t^5 - t^7 - \dots}.$$

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Multiple q-zeta values

q-analog

A q-analog of an expression is a generalization involving the variable q, which returns the original expression by taking the limit $q \to 1$.

Abstract definition

- For integers $k_1 \geq 1, k_2, \ldots, k_d \geq 0$ and polynomials $R_1(t), R_2(t), \ldots, R_d(t) \in \mathbb{Q}[t]$, $R_1(0) = 0$, define the **multiple q-zeta value (qMZV)**

$$\zeta_q(k_1, \ldots, k_d; R_1, \ldots, R_d) = \sum_{m_1 > \cdots > m_d > 0} \frac{R_1(q^{m_1})}{(1 - q^{m_1})^{k_1}} \cdots \frac{R_d(q^{m_d})}{(1 - q^{m_d})^{k_d}}.$$

- Define the $\mathbb{Q}$-vector space generated by the qMZVs

$$\mathcal{Z}_q = \langle \zeta_q(k_1, \ldots, k_d; R_1, \ldots, R_d) | d \geq 0, k_1, \ldots, k_d \geq 1, \deg(R_j) \leq k_j \rangle_{\mathbb{Q}},$$

where $\zeta_q(\emptyset; \emptyset) = 1$.

For $k_1 > 1$, we obtain

$$\lim_{q \to 1} (1 - q)^{k_1 + \cdots + k_d} \zeta_q(k_1, \ldots, k_d; R_1, \ldots, R_d) = R_1(1) \cdots R_d(1) \zeta(k_1, \ldots, k_d).$$

Multiple q-zeta values

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Proposition

The space $\mathcal{Z}_q$ is an algebra for the power series multiplication.

For example, we have

$$\begin{aligned} \zeta_q(k_1; R_1)\zeta_q(k_2; R_2) &= \zeta_q(k_1, k_2; R_1, R_2) + \zeta_q(k_2, k_1; R_2, R_1) \\ &\quad + \zeta_q(k_1 + k_2; R_1 R_2) \end{aligned}$$

Since $\deg(R_1 R_2) \leq k_1 + k_2$, the product is indeed an element in $\mathcal{Z}_q$.

Multiple q-zeta values: SZ-model

Definition

For $k_1 \geq 1, k_2, \ldots, k_d \geq 0$, the Schlesinger-Zudilin qMZV is given by

$$\zeta_q^{SZ}(k_1, \ldots, k_d) = \sum_{m_1 > \cdots > m_d > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_d k_d}}{(1-q^{m_d})^{k_d}},$$

If we assume $k_1, \ldots, k_d \geq 1$, then the Schlesinger-Zudilin qMZVs satisfy the usual stuffle product.

Duality relation (Zhao)

There is an involution τ defined on the multi indices $\mathbf{k} \in \mathbb{N}^d$, such that

$$\zeta_q^{SZ}(\tau(\mathbf{k})) = \zeta_q^{SZ}(\mathbf{k}).$$

This involution will be explained in details in the next talk by Brindle.

Multiple q-zeta values: SZ-model

Proposition

The Schlesinger-Zudilin qMZV span the space $\mathcal{Z}_q$.

Definition

For a Schlesinger-Zudilin qMZV $\zeta_q^{SZ}(k_1, \dots, k_d)$, we call $k_1 + \dots + k_d + |\{i | k_i = 0\}|$ its **weight** and $d - |\{i | k_i = 0\}|$ its **depth**.

The notion of depth and weight endows the space $\mathcal{Z}_q$ with two filtrations.

- The q-stuffle product is filtered by weight and by depth.
- The involution τ preserves the weight and the depth.

Formal multiple q-zeta values (work in progress)

Definition

Denote by $\mathcal{Z}_q^f$ the graded $\mathbb{Q}$-algebra of formal multiple q-zeta values, i.e., $\mathcal{Z}_q^f$ is generated by formal symbols satisfying the weight graded q-stuffle product and the duality relation.

Similar to Racinet's work, we can construct a space $\mathfrak{dm}_q$ consisting of non-commutative formal power series, for which we expect the following

Conjecture

- The space $\mathfrak{dm}_q$ is a Lie algebra. Modulo higher depth, the Lie bracket is given by the ARI-bracket.
- There is an isomorphism

$$\mathcal{Z}_q^f \simeq \widetilde{\mathcal{M}}(\mathrm{Sl}_2(\mathbb{Z})) \otimes \mathrm{Sym}(\mathrm{gr}\,\mathfrak{dm}_q^\vee).$$

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Up to weight 12, the dimensions of the space $\mathfrak{dm}_q$ coincide with the conjectured dimensions of Bachmann-Kühn

$$\sum_{k\geq 0} \dim_{\mathbb{Q}}(\mathcal{Z}_{q,k}^f)t^k = \frac{1}{1-t-t^2-t^3+t^6+t^7+t^8+t^9}$$

$$= \frac{1}{(1-t^2)(1-t^4)(1-t^6)}\frac{1}{1-\mathrm{Gen}(t)+\mathrm{Rel}(t)},$$

$$\mathrm{Gen}(t) = \sum_{k\geq 1} \#\{\mathrm{generators}\}t^k = \frac{1}{1-t^2}\frac{t}{1-t^2} = t+2t^3+3t^5+\dots,$$

$$\mathrm{Rel}(t) = \sum_{k\geq 4} \#\{\mathrm{relations}\}t^k = \frac{1}{1-t^2}\sum_{k\geq 4} \dim(\mathcal{M}_k \oplus \mathcal{S}_k)t^k.$$

Conjecture

There is an isomorphism

$$\mathcal{Z}_q^f \to \mathcal{Z}_q.$$

Injectivity is out of reach for the moment, but surjectivity implies

Conjecture

There is a spanning set of the space $\mathcal{Z}_q$, which satisfies a weight-graded q-stuffle product and a duality relation.

In the following, we will attack this problem by using the bi-brackets, which form another spanning set of $\mathcal{Z}_q$. There is an explicit bijection

$$\{\text{SZ-qMZV } \& \ \tau\} \leftrightsquigarrow \{\text{bi-brackets } \& \text{ swap}\}.$$

Bi-brackets (Bachmann)

Definition

For integers $k_1, ..., k_d \geq 1$, $n_1, ..., n_d \geq 0$, define

$$g\binom{k_1, \ldots, k_d}{n_1, \ldots, n_d} = \sum_{\substack{u_1 > \cdots > u_d > 0 \\ v_1, \ldots, v_d > 0}} \frac{u_1^{n_1}}{n_1!} \cdots \frac{u_d^{n_d}}{n_d!} \cdot \frac{v_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{v_d^{k_d - 1}}{(k_d - 1)!} \cdot q^{u_1 v_1 + \cdots + u_d v_d}.$$

For a bi-bracket $g\binom{k_1, \ldots, k_d}{n_1, \ldots, n_d}$, we call $k_1 + \cdots + k_d + n_1 + \cdots + n_d$ its **weight** and d its **depth**.

Theorem (Bachmann-Kühn)

The space $\mathcal{Z}_q$ is spanned by the bi-brackets.

Bi-brackets: q-stuffle product

E.g., the q-stuffle product of the bi-brackets in depth 2 is given by

$$g\binom{k_1}{n_1} \cdot g\binom{k_2}{n_2} = g\binom{k_1, k_2}{n_2, n_2} + g\binom{k_2, k_1}{n_2, n_1} + \binom{n_1+n_2}{n_1} g\binom{k_1+k_2}{n_1+n_2}$$

$$+ \binom{n_1+n_2}{n_1} \sum_{j=1}^{k_1} \frac{(-1)^{k_2-1} B_{k_1+k_2-j}}{(k_1+k_2-j)!} \binom{k_1+k_2-j-1}{k_1-j} g\binom{j}{n_1+n_2}$$

$$+ \binom{n_1+n_2}{n_1} \sum_{j=1}^{k_2} \frac{(-1)^{k_1-1} B_{k_1+k_2-j}}{(k_1+k_2-j)!} \binom{k_1+k_2-j-1}{k_2-j} g\binom{j}{n_1+n_2}.$$

It is well-known that using generating series is a good way to study such kind of formulae. Thus, we define for $d \geq 1$

$$\mathfrak{g}\binom{X_1, \dots, X_d}{Y_1, \dots, Y_d} = \sum_{\substack{k_1,\dots,k_d \geq 1 \\ n_1,\dots,n_d \geq 0}} g\binom{k_1, \dots, k_d}{n_1, \dots, n_d} X_1^{k_1-1} \cdots X_d^{k_d-1} Y_1^{n_1} \cdots Y_d^{n_d}.$$

Bi-brackets: (graded) q-stuffle product

The q-stuffle product of the bi-brackets in depth 2 can be expressed as

$$\mathfrak{g}\binom{X_1}{Y_1}\mathfrak{g}\binom{X_2}{Y_2} = \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1} + \frac{1}{X_1 - X_2}\left(\mathfrak{g}\binom{X_1}{Y_1 + Y_2} - \mathfrak{g}\binom{X_2}{Y_1 + Y_2}\right)$$
$$+ 2\beta(X_2 - X_1)\left(\mathfrak{g}\binom{X_1}{Y_1 + Y_2} - \mathfrak{g}\binom{X_2}{Y_1 + Y_2}\right) - \frac{1}{2}\left(\mathfrak{g}\binom{X_1}{Y_1 + Y_2} + \mathfrak{g}\binom{X_2}{Y_1 + Y_2}\right)$$

where

$$\beta(X) = -\sum_{k\geq 2}\frac{B_k}{2k!}X^{k-1}.$$

If we omit the terms of lower weight, we obtain

$$\mathfrak{g}\binom{X_1}{Y_1}\mathfrak{g}\binom{X_2}{Y_2} = \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1} + \frac{1}{X_1 - X_2}\left(\mathfrak{g}\binom{X_1}{Y_1 + Y_2} - \mathfrak{g}\binom{X_2}{Y_1 + Y_2}\right).$$

A sequence of power series $\mathfrak{g} = \left(\mathfrak{g}\binom{X_1}{Y_1}, \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2}, \dots\right)$ satisfying the blue coloured formula is called **symmetril** (in depth 2), their coefficients fulfill the **(weight-)graded q-stuffle product**.

Bi-brackets: (graded) q-stuffle product

In depth 3 the q-stuffle product of the bi-brackets can be described as

$$\begin{aligned}
&\mathfrak{g}\binom{X_1}{Y_1}\mathfrak{g}\binom{X_2,X_3}{Y_2,Y_3} = \mathfrak{g}\binom{X_1,X_2,X_3}{Y_1,Y_2,Y_3} + \mathfrak{g}\binom{X_2,X_1,X_3}{Y_2,Y_1,Y_3} + \mathfrak{g}\binom{X_2,X_3,X_1}{Y_2,Y_3,Y_1}\\
&+ \frac{\mathfrak{g}\binom{X_1,X_3}{Y_1+Y_2,Y_3} - \mathfrak{g}\binom{X_2,X_3}{Y_1+Y_2,Y_3}}{X_1-X_2} + \frac{\mathfrak{g}\binom{X_2,X_1}{Y_2,Y_1+Y_3} - \mathfrak{g}\binom{X_2,X_3}{Y_2,Y_1+Y_3}}{X_1-X_3}\\
&+ 2\beta(X_2-X_1)\left(\mathfrak{g}\binom{X_1,X_3}{Y_1+Y_2,Y_3} - \mathfrak{g}\binom{X_2,X_3}{Y_1+Y_2,Y_3}\right)\\
&+ 2\beta(X_3-X_1)\left(\mathfrak{g}\binom{X_2,X_1}{Y_2,Y_1+Y_3} - \mathfrak{g}\binom{X_2,X_3}{Y_2,Y_1+Y_3}\right)\\
&- \frac{1}{2}\left(\mathfrak{g}\binom{X_1,X_3}{Y_1+Y_2,Y_3} + \mathfrak{g}\binom{X_2,X_3}{Y_1+Y_2,Y_3} + \mathfrak{g}\binom{X_2,X_1}{Y_2,Y_1+Y_3} + \mathfrak{g}\binom{X_2,X_3}{Y_2,Y_1+Y_3}\right)
\end{aligned}$$

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&+ \frac{\mathfrak{g}\binom{X_1, X_3}{Y_1 + Y_2, Y_3} - \mathfrak{g}\binom{X_2, X_3}{Y_1 + Y_2, Y_3}}{X_1 - X_2} + \frac{\mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1 + Y_3} - \mathfrak{g}\binom{X_2, X_3}{Y_2, Y_1 + Y_3}}{X_1 - X_3} \\
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&- \frac{1}{2}\left(\mathfrak{g}\binom{X_1, X_3}{Y_1 + Y_2, Y_3} + \mathfrak{g}\binom{X_2, X_3}{Y_1 + Y_2, Y_3} + \mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1 + Y_3} + \mathfrak{g}\binom{X_2, X_3}{Y_2, Y_1 + Y_3}\right)
\end{aligned}$$

A sequence of power series $\mathfrak{g}$ satisfying the blue coloured formula is called **symmetril** (in depth 3), their coefficients fulfill the **(weight-)graded q-stuffle product**.

Bi-brackets: Swap invariance

Proposition (Bachmann)

For all $d \geq 1$, the generating series of the bi-brackets is **swap invariant**, i.e., we have

$$\mathfrak{g}\begin{pmatrix} X_1, \ldots, X_d \\ Y_1, \ldots, Y_d \end{pmatrix} = \mathfrak{g}\begin{pmatrix} Y_1 + \cdots + Y_d, Y_1 + \cdots + Y_{d-1}, \ldots, Y_1 \\ X_d, X_{d-1} - X_d, \ldots, X_1 - X_2 \end{pmatrix}.$$

The relations between bi-brackets obtained from the swap invariance of $\mathfrak{g}$ are homogeneous w.r.t. the weight.

E.g., in depth ≤ 2, we have

$$\mathfrak{g}\begin{pmatrix} X \\ Y \end{pmatrix} = \mathfrak{g}\begin{pmatrix} Y \\ X \end{pmatrix}, \quad \mathfrak{g}\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} = \mathfrak{g}\begin{pmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{pmatrix},$$

which leads to the following relations between bi-brackets:

$$g\binom{k}{n} = g\binom{n+1}{k-1},$$

$$g\begin{pmatrix} k_1, k_2 \\ n_1, n_2 \end{pmatrix} = \sum_{n=0}^{n_1} \sum_{k=0}^{k_2-1} (-1)^k \binom{k_1 - 1 + k}{k} \binom{n_2 + n}{n} g\begin{pmatrix} n_2 + n + 1, n_1 - n + 1 \\ k_2 - 1 - k, k_1 - 1 + k \end{pmatrix}.$$

Combinatorial multiple Eisenstein series (CMES)

Now we reformulate the previous conjecture:

Conjecture

There is a spanning set of the space $\mathcal{Z}_q$, whose generating series are symmetril and swap invariant.

Towards this conjecture, we have the following:

Theorem (Bachmann-Kühn-Matthes, Bachmann-B.)

There is a sequence $\mathfrak{G} = \left(\mathfrak{G}\binom{X_1}{Y_1}, \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2}, \mathfrak{G}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3}\right)$ of power series with coefficients in $\mathcal{Z}_q$, which is symmetril and swap invariant.

The combinatorial multiple Eisenstein series $G\binom{X_1, \dots, X_d}{Y_1, \dots, Y_d}$ of depth $d \leq 3$ are defined to be the coefficients of the series $\mathfrak{G}$:

$$\mathfrak{G}\binom{X_1, \dots, X_d}{Y_1, \dots, Y_d} = \sum_{\substack{k_1, \dots, k_d \geq 1 \\ n_1, \dots, n_d \geq 0}} G\binom{k_1, \dots, k_d}{n_1, \dots, n_d} X_1^{k_1 - 1} \cdots X_d^{k_d - 1} \frac{Y_1^{n_1}}{n_1!} \cdots \frac{Y_d^{n_d}}{n_d!}$$

For $d = 1, 2, 3$, the CMES of depth $\leq d$ span the space $\mathrm{Fil}_d^D(\mathcal{Z}_q)$.

Combinatorial multiple Eisenstein series: special cases

In depth 1, we have

$$G\binom{k}{n} = \delta_{n,0}\left(-\frac{B_k}{2k!}\right) + n!g\binom{k}{n}, \quad k > n+1 \geq 1.$$

- For $k \geq 2$ even, the elements $G\binom{k}{0}$ are the Eisenstein series of weight k (with rational coefficients),
- For $k+n \geq 2$ even, the elements $G\binom{k}{n}$ are the derivatives of Eisenstein series.

For $k_1 \geq 3, k_2 \geq 2$ even, the element $G\binom{k_1, k_2}{0, 0}$ equals the combinatorial double Eisenstein series of Gangl-Kaneko-Zagier. We get the graded q-stuffle product formula:

$$G\binom{k_1}{0}\, G\binom{k_2}{0} = G\binom{k_1, k_2}{0, 0} + G\binom{k_2, k_1}{0, 0} + G\binom{k_1 + k_2}{0}$$

Combinatorial multiple Eisenstein series: properties

- The $\mathbb{Q}$-vector space generated by the CMES of depth $d \leq 2$ contains the quasi-modular forms with rational coefficients.
- The space generated by the CMES is closed under taking the derivative $q\frac{d}{dq}$.
- The CMES can be seen as a bi-version of the multiple Eisenstein series, since they are constructed analogously to the Fourier expansion of the multiple Eisenstein series.
- The CMES are linear combinations of q-analogs of MZVs. In particular, we have:

$$\lim_{q\to 1}(1-q)^{k_1+\cdots+k_d} G\binom{k_1,\ldots,k_d}{0,\ldots,0} = \zeta(k_1,\ldots,k_d)$$

  for $k_1 \geq 2, k_2,\ldots,k_d \geq 1,\ d \leq 3$.
- The elements $\lim_{q\to 0} G\binom{k_1,\ldots,k_d}{0,\ldots,0}$, $d \leq 3$, are rational numbers satisfying the stuffle and shuffle product.

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$$\lim_{q \to 1} (1-q)^{k_1+\cdots+k_d} G\binom{k_1, \ldots, k_d}{0, \ldots, 0} = \zeta(k_1, \ldots, k_d)$$

for $k_1 \geq 2, k_2, \ldots, k_d \geq 1,\ d \leq 3$.

- The elements $\lim_{q \to 0} G\binom{k_1, \ldots, k_d}{0, \ldots, 0}$, $d \leq 3$, are rational numbers satisfying the stuffle and shuffle product.

Combinatorial multiple Eisenstein series: properties

- The $\mathbb{Q}$-vector space generated by the CMES of depth $d \leq 2$ contains the quasi-modular forms with rational coefficients.
- The space generated by the CMES is closed under taking the derivative $q\frac{d}{dq}$.
- The CMES can be seen as a bi-version of the multiple Eisenstein series, since they are constructed analogously to the Fourier expansion of the multiple Eisenstein series.
- The CMES are linear combinations of q-analogs of MZVs. In particular, we have:

$$\lim_{q\to 1}(1-q)^{k_1+\cdots+k_d}\, G\begin{pmatrix} k_1,\ldots,k_d \\ 0,\ldots,0 \end{pmatrix} = \zeta(k_1,\ldots,k_d)$$

for $k_1 \geq 2, k_2,\ldots,k_d \geq 1,\ d \leq 3$.

- The elements $\lim_{q\to 0} G\begin{pmatrix} k_1,\ldots,k_d \\ 0,\ldots,0 \end{pmatrix}$, $d \leq 3$, are rational numbers satisfying the stuffle and shuffle product.

The CMES satisfy a bi-version of the double shuffle relations. E.g., in depth 2 they are given by

$$
\begin{aligned}
G\binom{k_1}{n_1} G\binom{k_2}{n_2} &= G\binom{k_1, k_2}{n_1, n_2} + G\binom{k_2, k_1}{n_2, n_1} + G\binom{k_1 + k_2}{n_1 + n_2} \\
&= \sum_{k=1}^{k_1} \sum_{n=0}^{n_2} \binom{k_1 + k_2 - k - 1}{k_1 - k} \binom{n_1 + n_2 - n}{n_1} (-1)^{n_2 - n} G\binom{k_1 + k_2 - k, k}{n, n_1 + n_2 - n} \\
&\quad + \sum_{k=1}^{k_2} \sum_{n=0}^{n_1} \binom{k_1 + k_2 - k - 1}{k_2 - k} \binom{n_1 + n_2 - n}{n_2} (-1)^{n_1 - n} G\binom{k_1 + k_2 - k, k}{n, n_1 + n_2 - n} \\
&\quad + \binom{k_1 + k_2 - 2}{k_1 - 1} G\binom{k_1 + k_2 - 1}{n_1 + n_2 + 1}.
\end{aligned}
$$

If $n_1 = n_2 = 0$, taking the limit $q \to 1$ (and multiplying with $(1-q)^{k_1+k_2}$) yields the classical double shuffle relations for double zeta values.

Combinatorial multiple Eisenstein series: construction

Proposition (Drinfeld+Furusho, Racinet)

There is a (non-unique) sequence $\beta(X_1), \beta(X_1, X_2), \ldots$ with rational coefficients satisfying the stuffle and shuffle product.

Explicit formulas for β in low depths are given by Gangl-Kaneko-Zagier, Brown and Écalle.

Proposition (Bachmann-Matthes-Kühn)

We can lift every series $\beta(X_1), \beta(X_1, X_2), \ldots$ with coefficients satisfying the stuffle and shuffle product to a sequence $\beta\binom{X_1}{Y_1}, \beta\binom{X_1, X_2}{Y_1, Y_2}, \ldots$ of power series, which is symmetril and swap invariant.

In the following, we fix such a sequence $\beta\binom{X_1}{Y_1}, \beta\binom{X_1, X_2}{Y_1, Y_2}, \ldots$ of power series with rational coefficients. In particular, we have in depth 1:

$$\beta\binom{X_1}{Y_1} = \beta(X_1) + \beta(Y_1) = -\sum_{k\geq 2} \frac{B_k}{2k!}(X_1^{k-1} + Y_1^{k-1}).$$

Combinatorial multiple Eisenstein series: main result

Theorem (Bachmann-Kühn-Matthes $d=2$, Bachmann-B. $d=3$)

There is an explicit symmetril sequence

$$\mathfrak{g}\binom{X_1}{Y_1}, \mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2}, \mathfrak{g}^{il}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} \in \mathcal{Z}_q[[X_1, X_2, X_3, Y_1, Y_2, Y_3]],$$

such that the following sequence is symmetril and swap invariant:

$$\begin{aligned}
\mathfrak{G}\binom{X_1}{Y_1} &= \mathfrak{g}\binom{X_1}{Y_1} + \beta\binom{X_1}{Y_1}, \\
\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} &= \mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_1}{Y_1}\beta\binom{X_2}{Y_2} + \beta\binom{X_1, X_2}{Y_1, Y_2}, \\
\mathfrak{G}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} &= \mathfrak{g}^{il}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} + \mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2}\beta\binom{X_3}{Y_3} \\
&\quad + \mathfrak{g}\binom{X_1}{Y_1}\beta\binom{X_2, X_3}{Y_2, Y_3} + \beta\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3}.
\end{aligned}$$

Construction of the $\mathfrak{g}^{il}$

The generating series of bi-brackets of depth $d \geq 1$ can also be written as

$$\mathfrak{g}\binom{X_1, \dots, X_d}{Y_1, \dots, Y_d} = \sum_{m_1 > \cdots > m_d > 0} L_{m_1}\binom{X_1}{Y_1} \cdots L_{m_d}\binom{X_d}{Y_d}$$

where

$$L_m\binom{X}{Y} = \frac{e^{X+mY} q^m}{1 - e^X q^m}.$$

In order to build from the series $\mathfrak{g}$ a symmetril series $\mathfrak{g}^{il}$, we will define a symmetril multiple version of the L_m.

Construction of the $\mathfrak{g}^{il}$

Definition

For $d, m \geq 1$, define the power series

$$\beta^R\begin{pmatrix} X_1, \ldots, X_d \\ Y_1, \ldots, Y_d \end{pmatrix} = \sum_{i=0}^{d} \frac{(-1)^i}{2^i i!} \beta\begin{pmatrix} -X_{i+1}, \ldots, -X_d \\ Y_1, \ldots, Y_{d-i} \end{pmatrix},$$

$$L_m\begin{pmatrix} X_1, \ldots, X_d \\ Y_1, \ldots, Y_d \end{pmatrix} = \sum_{j=1}^{d} \beta\begin{pmatrix} X_1 - X_j, \ldots, X_{j-1} - X_j \\ Y_1, \ldots, Y_{j-1} \end{pmatrix} L_m\begin{pmatrix} X_j \\ Y_1 + \cdots + Y_d \end{pmatrix}$$
$$\cdot\, \beta^R\begin{pmatrix} X_j - X_d, \ldots, X_j - X_{j+1} \\ -Y_d, \ldots, -Y_{i+1} \end{pmatrix}.$$

Proposition (Bachmann-B.)

- The series β^R are swap invariant up to signs.
- Up to depth 3, the β^R are symmetril up to signs.
- Up to depth 3, the L_m, $m \geq 1$, are symmetril.

Construction of the $\mathfrak{g}^{il}$

The symmetrility of the L_m, $m \geq 1$, implies:

Proposition

The following sequence of power series is symmetril

$$\mathfrak{g}\binom{X_1}{Y_1} = \sum_{m>0} L_m \binom{X_1}{Y_1},$$

$$\mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2} = \sum_{m_1>m_2>0} L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2}{Y_2} + \sum_{m>0} L_m \binom{X_1, X_2}{Y_1, Y_2},$$

$$\begin{aligned}\mathfrak{g}^{il}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} = & \sum_{m_1>m_2>m_3>0} L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2}{Y_2} L_{m_3}\binom{X_3}{Y_3} \\ & + \sum_{m_1>m_2>0} \left(L_{m_1}\binom{X_1, X_2}{Y_1, Y_2} L_{m_2}\binom{X_3}{Y_3} + L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2, X_3}{Y_2, Y_3} \right) \\ & + \sum_{m>0} L_m \binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3}.\end{aligned}$$

This conclusion works for all depths. The crucial point in arbitrary depths is to show that the L_m, $m \geq 1$, are symmetril and that the $\mathfrak{G}$ are swap invariant.

Summary

- The known models for qMZVs give a filtered product on the space $\mathcal{Z}_q$.
- For the space $\mathcal{Z}_q^f$ of formal graded qMZVs, there exists an associated space $\mathfrak{dm}_q$, which has conjecturally a Lie algebra structure.
- We expect that there is a spanning set of $\mathcal{Z}_q$, which satisfies a graded product formula and a duality relation/swap invariance.
- Up to depth 3, we have constructed the CMES. They form a spanning set of $\operatorname{Fil}_3^D(\mathcal{Z}_q)$ and there generating series is symmetril and swap invariant.
- The CMES are itself interesting objects, they contain the quasi-modular forms and have a direct connection to multiple Eisenstein series.