Combinatorial Multiple Eisenstein Series

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joint work with H. Bachmann

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Multiple zeta values

Definition

- For integers $k_1 \geq 2, k_2, \ldots, k_d \geq 1$, we call

$$\zeta(k_1, \ldots, k_d) = \sum_{m_1 > \cdots > m_d > 0} \frac{1}{m_1^{k_1} \cdots m_d^{k_d}}$$

  a **multiple zeta value (MZV)** of weight $k_1 + \cdots + k_d$ and depth d.
- Let

$$\mathcal{Z} = \langle \zeta(k_1, \ldots, k_d) | d \geq 0, k_1 \geq 2, k_2, \ldots, k_d \geq 1 \rangle_{\mathbb{Q}},$$

  where $\zeta(\emptyset) = 1$, be the $\mathbb{Q}$-vector space spanned by the MZVs.

Multiple zeta values - EDS relations

Products of MZVs are $\mathbb{Q}$-linear combinations of MZVs, so the space $\mathcal{Z}$ is an algebra. There are two kinds of such formulas, e.g., in depth 2, we have

$$\zeta(k_1)\zeta(k_2) = \begin{cases} \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2) & \textbf{(stuffle product)} \\ \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1 + k_2 - j) & \textbf{(shuffle product)}. \end{cases}$$

We get the **double shuffle relations**.

Actually, the MZVs satisfy the **extended double shuffle relations (EDS)** obtained by regularization. Conjecturally, these are all relations between MZVs.

Formal multiple zeta values

Definition

Let $\mathcal{Z}^f$ be the algebra of **formal multiple zeta values**, i.e., the formal symbols $\zeta^f(k_1, \ldots, k_d)$ satisfy the extended double shuffle relations and no other relations.

- The algebra $\mathcal{Z}^f$ is graded for the weight.
- We have a surjective algebra homomorphism

$$\mathcal{Z}^f \to \mathcal{Z},$$
$$\zeta^f(k_1, \ldots, k_d) \mapsto \zeta(k_1, \ldots, k_d).$$

Conjecturally, this morphism is an isomorphism.

Formal multiple zeta values

Racinet introduced a pro-unipotent affine group scheme DM_0 with values in a graded Hopf algebra, such that we have for all $\mathbb{Q}$-algebras R

$$\mathrm{Hom}_{\mathbb{Q}\text{-alg}}\left({\mathcal{Z}^f}/{(\zeta^f(2))}, R \right) \simeq \mathrm{DM}_0(R).$$

By a deep theorem of Racinet, there is a bijection between DM_0 and its Lie algebra $\mathfrak{dm}_0$, from which we obtain

Theorem (Ecalle, Racinet)

The algebra $\mathcal{Z}^f$ of formal MZVs is a free polynomial algebra.

It is conjectured by Ihara-Deligne that $\mathfrak{dm}_0$ is a free Lie algebra with exactly one generator in each odd degree ≥ 3. From this, we can deduce

Zagier's conjecture for formal MZVs

The dimensions of the homogeneous subspaces (w.r.t. the weight) of $\mathcal{Z}^f$ are given by

$$\sum_{k\geq 0} \dim_{\mathbb{Q}}(\mathcal{Z}^f_k)t^k = \frac{1}{1-t^2-t^3}.$$

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Multiple q-zeta values

q-analog

A q-analog of an expression is a generalization involving the variable q, which returns the original expression by taking the limit $q \to 1$.

For a natural number $n \geq 1$, the q-analog is given by

$$\{n\}_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

Schlesinger-Zudilin multiple q-zeta values

For $k_1 \geq 1, k_2, \ldots, k_d \geq 0$, a (modified) q-analog of the MZVs is given by

$$\zeta_q^{SZ}(k_1, \ldots, k_d) = \sum_{m_1 > \cdots > m_d > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_d k_d}}{(1-q^{m_d})^{k_d}},$$

We obtain for $k_1 \geq 2, k_2, \ldots k_d \geq 1$

$$\lim_{q \to 1} (1-q)^{k_1 + \cdots + k_d} \zeta_q^{SZ}(k_1, \ldots, k_d) = \lim_{q \to 1} \sum_{m_1 > \cdots > m_d > 0} \frac{q^{m_1 k_1}}{\{m_1\}_q^{k_1}} \cdots \frac{q^{m_d k_d}}{\{m_d\}_q^{k_d}}$$
$$= \zeta(k_1, \ldots, k_d).$$

Multiple q-zeta values

Definition

- For integers $k_1, \ldots, k_d \geq 1$ and polynomials $R_1(t), R_2(t), \ldots, R_d(t) \in \mathbb{Q}[t]$, $R_1(0) = 0$, define the **multiple q-zeta value (qMZV)**

$$\zeta_q(k_1, \ldots, k_d; R_1, \ldots, R_d) = \sum_{m_1 > \cdots > m_d > 0} \frac{R_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{R_d(q^{m_d})}{(1-q^{m_d})^{k_d}}.$$

- Define the $\mathbb{Q}$-vector space generated by the qMZVs

$$\mathcal{Z}_q = \langle \zeta_q(k_1, \ldots, k_d; R_1, \ldots, R_d) | d \geq 0, k_1, \ldots, k_d \geq 1, \deg(R_j) \leq k_j \rangle_{\mathbb{Q}},$$

where $\zeta_q(\emptyset; \emptyset) = 1$.

For $k_1 > 1$, we obtain

$$\lim_{q \to 1} (1-q)^{k_1 + \cdots + k_d} \zeta_q(k_1, \ldots, k_d; R_1, \ldots, R_d) = R_1(1) \cdots R_d(1) \zeta(k_1, \ldots, k_d).$$

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$$\mathcal{Z}_q = \langle \zeta_q(k_1, \ldots, k_d; R_1, \ldots, R_d) | d \geq 0, k_1, \ldots, k_d \geq 1, \deg(R_j) \leq k_j \rangle_{\mathbb{Q}},$$

  where $\zeta_q(\emptyset; \emptyset) = 1$.

The power series multiplication (**q-stuffle product**) endows $\mathcal{Z}_q$ with the structure of an algebra. For example, we have

$$\begin{aligned} \zeta_q(k_1; R_1)\zeta_q(k_2; R_2) = \zeta_q(k_1, k_2; R_1, R_2) + \zeta_q(k_2, k_1; R_2, R_1) \\ + \zeta_q(k_1 + k_2; R_1 R_2) \end{aligned}$$

Since $\deg(R_1 R_2) \leq k_1 + k_2$, the product is indeed an element in $\mathcal{Z}_q$.

Bi-brackets (Bachmann)

Definition

For integers $k_1, \ldots, k_d \geq 1, n_1, \ldots, n_d \geq 0$, define

$$\begin{bmatrix} k_1, \ldots, k_d \\ n_1, \ldots, n_d \end{bmatrix} = \sum_{m_1 > \cdots > m_d > 0} \prod_{j=1}^{d} \frac{m_j^{n_j}}{n_j!} \frac{P_{k_j}(q^{m_j})}{(1-q^{m_j})^{k_j}}$$

where the Eulerian polynomials $P_k(t)$ are defined by

$$\frac{P_k(t)}{(1-t)^k} = \sum_{r \geq 1} \frac{r^{k-1}}{(k-1)!} t^r.$$

For a bi-bracket $\begin{bmatrix} k_1, \ldots, k_d \\ n_1, \ldots, n_d \end{bmatrix}$, we call $k_1 + \cdots + k_d + n_1 + \cdots + n_d$ its **weight** and d its **depth**.

Filtrations on $\mathcal{Z}_q$

Theorem (Bachmann-Kühn)

The space $\mathcal{Z}_q$ is spanned by the bi-brackets.

We define a weight filtration

$$\mathrm{Fil}_k^W(\mathcal{Z}_q) = \left\langle \begin{bmatrix} k_1, \dots, k_d \\ n_1, \dots, n_d \end{bmatrix} \,\middle|\, 0 \leq d \leq k,\, k_1 + \cdots + k_d + n_1 + \cdots + n_d \leq k \right\rangle_{\mathbb{Q}}$$

and a depth filtration

$$\mathrm{Fil}_d^D(\mathcal{Z}_q) = \left\langle \begin{bmatrix} k_1, \dots, k_l \\ n_1, \dots, n_l \end{bmatrix} \,\middle|\, l \leq d \right\rangle_{\mathbb{Q}}$$

on the algebra $\mathcal{Z}_q$.

q-stuffle product of bi-brackets

Define the generating series of bi-brackets of depth $d \geq 1$ by

$$\mathfrak{g}\binom{X_1, \dots, X_d}{Y_1, \dots, Y_d} = \sum_{\substack{k_1, \dots, k_d \geq 1 \\ n_1, \dots, n_d \geq 0}} \begin{bmatrix} k_1, \dots, k_d \\ n_1, \dots, n_d \end{bmatrix} X_1^{k_1 - 1} \cdots X_d^{k_d - 1} Y_1^{n_1} \cdots Y_d^{n_d}.$$

The q-stuffle product of the bi-brackets in depth 2 can be expressed as

$$\mathfrak{g}\binom{X_1}{Y_1}\mathfrak{g}\binom{X_2}{Y_2} = \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1} + \frac{1}{X_1 - X_2}\left(\mathfrak{g}\binom{X_1}{Y_1 + Y_2} - \mathfrak{g}\binom{X_2}{Y_1 + Y_2}\right)$$
$$+ 2\beta(X_2 - X_1)\left(\mathfrak{g}\binom{X_1}{Y_1 + Y_2} - \mathfrak{g}\binom{X_2}{Y_1 + Y_2}\right) - \frac{1}{2}\left(\mathfrak{g}\binom{X_1}{Y_1 + Y_2} + \mathfrak{g}\binom{X_2}{Y_1 + Y_2}\right)$$

where

$$\beta(X) = -\sum_{k \geq 2} \frac{B_k}{2k!} X^{k-1}.$$

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where

$$\beta(X) = -\sum_{k \geq 2} \frac{B_k}{2k!} X^{k-1}.$$

We call a sequence of power series satisfying the blue coloured formula **symmetril**, their coefficients fulfill the **(weight-)graded q-stuffle product**.

q-stuffle product of bi-brackets

In depth 3 the q-stuffle product of the bi-brackets can be described as

$$
\begin{aligned}
&\mathfrak{g}\binom{X_1}{Y_1}\mathfrak{g}\binom{X_2, X_3}{Y_2, Y_3} = \mathfrak{g}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} + \mathfrak{g}\binom{X_2, X_1, X_3}{Y_2, Y_1, Y_3} + \mathfrak{g}\binom{X_2, X_3, X_1}{Y_2, Y_3, Y_1} \\
&+ \frac{\mathfrak{g}\binom{X_1, X_3}{Y_1+Y_2, Y_3} - \mathfrak{g}\binom{X_2, X_3}{Y_1+Y_2, Y_3}}{X_1 - X_2} + \frac{\mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1+Y_3} - \mathfrak{g}\binom{X_2, X_3}{Y_2, Y_1+Y_3}}{X_1 - X_3} \\
&+ 2\beta(X_2 - X_1)\left(\mathfrak{g}\binom{X_1, X_3}{Y_1+Y_2, Y_3} - \mathfrak{g}\binom{X_2, X_3}{Y_1+Y_2, Y_3}\right) \\
&+ 2\beta(X_3 - X_1)\left(\mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1+Y_3} - \mathfrak{g}\binom{X_2, X_3}{Y_2, Y_1+Y_3}\right) \\
&- \frac{1}{2}\left(\mathfrak{g}\binom{X_1, X_3}{Y_1+Y_2, Y_3} + \mathfrak{g}\binom{X_2, X_3}{Y_1+Y_2, Y_3} + \mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1+Y_3} + \mathfrak{g}\binom{X_2, X_3}{Y_2, Y_1+Y_3}\right)
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$$+ \frac{\mathfrak{g}\binom{X_1,X_3}{Y_1+Y_2,Y_3} - \mathfrak{g}\binom{X_2,X_3}{Y_1+Y_2,Y_3}}{X_1-X_2} + \frac{\mathfrak{g}\binom{X_2,X_1}{Y_2,Y_1+Y_3} - \mathfrak{g}\binom{X_2,X_3}{Y_2,Y_1+Y_3}}{X_1-X_3}$$

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$$+ 2\beta(X_3-X_1)\left(\mathfrak{g}\binom{X_2,X_1}{Y_2,Y_1+Y_3} - \mathfrak{g}\binom{X_2,X_3}{Y_2,Y_1+Y_3}\right)$$

$$- \frac{1}{2}\left(\mathfrak{g}\binom{X_1,X_3}{Y_1+Y_2,Y_3} + \mathfrak{g}\binom{X_2,X_3}{Y_1+Y_2,Y_3} + \mathfrak{g}\binom{X_2,X_1}{Y_2,Y_1+Y_3} + \mathfrak{g}\binom{X_2,X_3}{Y_2,Y_1+Y_3}\right)$$

We call a sequence of power series satisfying the blue coloured formula **symmetril**, their coefficients fulfill the **(weight-)graded q-stuffle product**.

Swap invariance of bi-brackets

Proposition (Bachmann)

For all $d \geq 1$, the generating series of the bi-brackets is **swap invariant**, i.e., we have

$$\mathfrak{g}\binom{X_1,\ldots,X_d}{Y_1,\ldots,Y_d} = \mathfrak{g}\binom{Y_1+\cdots+Y_d,\, Y_1+\cdots+Y_{d-1},\ldots,Y_1}{X_d,\, X_{d-1}-X_d,\ldots,X_1-X_2}.$$

The relations between bi-brackets obtained from the swap invariance of $\mathfrak{g}$ are homogeneous w.r.t. the weight.

E.g., in depth ≤ 2, we have

$$\mathfrak{g}\binom{X}{Y} = \mathfrak{g}\binom{Y}{X}, \quad \mathfrak{g}\binom{X_1,X_2}{Y_1,Y_2} = \mathfrak{g}\binom{Y_1+Y_2,\,Y_1}{X_2,\,X_1-X_2},$$

which leads to the following relations between bi-brackets:

$$\begin{bmatrix} k \\ n \end{bmatrix} = \begin{bmatrix} n+1 \\ k-1 \end{bmatrix},$$

$$\begin{bmatrix} k_1,k_2 \\ n_1,n_2 \end{bmatrix} = \sum_{n=0}^{n_1}\sum_{k=0}^{k_2-1}(-1)^k\binom{k_1-1+k}{k}\binom{n_2+n}{n}\begin{bmatrix} n_2+n+1,\,n_1-n+1 \\ k_2-1-k,\,k_1-1+k \end{bmatrix}.$$

Relations between qMZV

Conjecture (Bachmann)

All algebraic relations in the algebra $\mathcal{Z}_q$ of multiple q-zeta values can be obtained from combining the q-stuffle product and the swap invariance of bi-brackets.

Inspired by the classical MZVs, we expect the following:

Conjecture

There is a spanning set of the space $\mathcal{Z}_q$, which satisfies the graded q-stuffle product formula and whose generating series is swap invariant.

In the following, we will construct such a spanning set of $\mathcal{Z}_q$ up to depth 3, we will call these the combinatorial multiple Eisenstein series.

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Combinatorial multiple Eisenstein series

The multiplication of quasi-modular forms is graded, therefore we make the following ansatz:

$$G\binom{k}{n} = \delta_{n,0}\left(-\frac{B_k}{2k!}\right) + n!\begin{bmatrix} k \\ n \end{bmatrix}, \quad k > n+1 \geq 1.$$

- For $k \geq 2$ even, the elements $G\binom{k}{0}$ are the Eisenstein series of weight k (with rational coefficients),
- For $k+n \geq 2$ even, the elements $G\binom{k}{n}$ are the derivatives of Eisenstein series.

Let $G\binom{k_1, k_2}{0, 0}$, $k_1 \geq 3, k_2 \geq 2$ even, be the combinatorial double Eisenstein series of Gangl-Kaneko-Zagier, then we obtain the graded q-stuffle product formula:

$$G\binom{k_1}{0}G\binom{k_2}{0} = G\binom{k_1, k_2}{0, 0} + G\binom{k_2, k_1}{0, 0} + G\binom{k_1 + k_2}{0}$$

Combinatorial multiple Eisenstein series

Proposition (Drinfeld+Furusho, Racinet)

There is a (non-unique) sequence $\beta(X_1), \beta(X_1, X_2), \ldots$ with rational coefficients satisfying the stuffle and shuffle product.

Explicit formulas for β in low depths are given by Gangl-Kaneko-Zagier, Brown and Écalle.

Proposition (Bachmann-Matthes-Kühn)

We can lift every series $\beta(X_1), \beta(X_1, X_2), \ldots$ with coefficients satisfying the stuffle and shuffle product to a sequence $\beta\binom{X_1}{Y_1}, \beta\binom{X_1, X_2}{Y_1, Y_2}, \ldots$ of power series, which is symmetril and swap invariant.

In the following, we fix such a sequence $\beta\binom{X_1}{Y_1}, \beta\binom{X_1, X_2}{Y_1, Y_2}, \ldots$ of power series with rational coefficients. In particular, we have in depth 1:

$$\beta\binom{X_1}{Y_1} = \beta(X_1) + \beta(Y_1) = -\sum_{k\geq 2} \frac{B_k}{2k!}(X_1^{k-1} + Y_1^{k-1}).$$

Combinatorial multiple Eisenstein series

Theorem (Bachmann-Kühn-Matthes $d=2$, Bachmann-B. $d=3$)

There is an explicit symmetril sequence

$$\mathfrak{g}\binom{X_1}{Y_1}, \mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2}, \mathfrak{g}^{il}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} \in \mathcal{Z}_q[[X_1, X_2, X_3, Y_1, Y_2, Y_3]],$$

such that the following sequence is symmetril and swap invariant:

$$\begin{aligned}
\mathfrak{G}\binom{X_1}{Y_1} &= \mathfrak{g}\binom{X_1}{Y_1} + \beta\binom{X_1}{Y_1}, \\
\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} &= \mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_1}{Y_1}\beta\binom{X_2}{Y_2} + \beta\binom{X_1, X_2}{Y_1, Y_2}, \\
\mathfrak{G}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} &= \mathfrak{g}^{il}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} + \mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2}\beta\binom{X_3}{Y_3} \\
&\quad + \mathfrak{g}\binom{X_1}{Y_1}\beta\binom{X_2, X_3}{Y_2, Y_3} + \beta\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3}.
\end{aligned}$$

Combinatorial multiple Eisenstein series

Definition

The **combinatorial multiple Eisenstein series (CMES)** G in depth $d \leq 3$ are defined to be the coefficients of the series $\mathfrak{G}$:

$$\mathfrak{G}\begin{pmatrix} X_1, \dots, X_d \\ Y_1, \dots, Y_d \end{pmatrix} = \sum_{\substack{k_1,\dots,k_d \geq 1 \\ n_1,\dots,n_d \geq 0}} G\begin{pmatrix} k_1, \dots, k_d \\ n_1, \dots, n_d \end{pmatrix} X_1^{k_1-1} \cdots X_d^{k_d-1} \frac{Y_1^{n_1}}{n_1!} \cdots \frac{Y_d^{n_d}}{n_d!}$$

Towards the conjecture about a spanning set of $\mathcal{Z}_q$, we have:

Theorem

- The CMES satisfy the graded q-stuffle product and their generating series is swap invariant.
- For $d = 1, 2, 3$, the CMES of depth d span the space $\mathrm{Fil}_d^D(\mathcal{Z}_q)$.

Combinatorial multiple Eisenstein series

- The $\mathbb{Q}$-vector space generated by the CMES of depth $d \leq 2$ contains the quasi-modular forms with rational coefficients.
- The space generated by the CMES is closed under taking the derivative $q\frac{d}{dq}$.
- The CMES can be seen as a bi-version of the multiple Eisenstein series, since they are constructed analogously to the Fourier expansion of the multiple Eisenstein series.
- The CMES are linear combinations of q-analogs of MZVs. In particular, we have:

$$\lim_{q\to 1}(1-q)^{k_1+\cdots+k_d}\, G\binom{k_1,\ldots,k_d}{0,\ldots,0} = \zeta(k_1,\ldots,k_d)$$

for $k_1 \geq 2, k_2,\ldots,k_d \geq 1,\ d \leq 3$.

- The elements $\lim_{q\to 0} G\binom{X_1,\ldots,X_d}{0,\ldots,0}$, $d \leq 3$, are rational numbers satisfying the stuffle and shuffle product.

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  for $k_1 \geq 2, k_2,\ldots,k_d \geq 1,\ d \leq 3$.
- The elements $\lim_{q\to 0} G\begin{pmatrix} X_1,\ldots,X_d \\ 0,\ldots,0 \end{pmatrix}$, $d \leq 3$, are rational numbers satisfying the stuffle and shuffle product.

Combinatorial multiple Eisenstein series

- The $\mathbb{Q}$-vector space generated by the CMES of depth $d \leq 2$ contains the quasi-modular forms with rational coefficients.
- The space generated by the CMES is closed under taking the derivative $q\frac{d}{dq}$.
- The CMES can be seen as a bi-version of the multiple Eisenstein series, since they are constructed analogously to the Fourier expansion of the multiple Eisenstein series.
- The CMES are linear combinations of q-analogs of MZVs. In particular, we have:

$$\lim_{q \to 1} (1-q)^{k_1+\cdots+k_d} G\binom{k_1, \ldots, k_d}{0, \ldots, 0} = \zeta(k_1, \ldots, k_d)$$

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Combinatorial multiple Eisenstein series

The CMES satisfy the bi-version of the double shuffle relations. E.g., in depth 2 they are given by

$$
\begin{aligned}
&G\binom{k_1}{n_1}G\binom{k_2}{n_2} = G\binom{k_1,k_2}{n_1,n_2} + G\binom{k_2,k_1}{n_2,n_1} + G\binom{k_1+k_2}{n_1+n_2} \\
&= \sum_{k=1}^{k_1}\sum_{n=0}^{n_2}\binom{k_1+k_2-k-1}{k_1-k}\binom{n_1+n_2-n}{n_1}(-1)^{n_2-n}G\binom{k_1+k_2-k,k}{n,n_1+n_2-n} \\
&\quad + \sum_{k=1}^{k_2}\sum_{n=0}^{n_1}\binom{k_1+k_2-k-1}{k_2-k}\binom{n_1+n_2-n}{n_2}(-1)^{n_1-n}G\binom{k_1+k_2-k,k}{n,n_1+n_2-n} \\
&\quad + \binom{k_1+k_2-2}{k_1-1}G\binom{k_1+k_2-1}{n_1+n_2+1}.
\end{aligned}
$$

If $n_1 = n_2 = 0$, taking the limit $q \to 1$ (and multiplying with $(1-q)^{k_1+k_2}$) yields the classical double shuffle relations for double zeta values.

Proof: Construction of the $\mathfrak{g}^{il}$

Before we indicate the proof, recall:

Theorem (Bachmann-Kühn-Matthes $d = 2$, Bachmann-B. $d = 3$)

There is an explicit symmetril sequence

$$\mathfrak{g}\binom{X_1}{Y_1}, \mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2}, \mathfrak{g}^{il}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} \in \mathcal{Z}_q[[X_1, X_2, X_3, Y_1, Y_2, Y_3]],$$

such that the following sequence is symmetril and swap invariant:

$$\mathfrak{G}\binom{X_1}{Y_1} = \mathfrak{g}\binom{X_1}{Y_1} + \beta\binom{X_1}{Y_1},$$

$$\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} = \mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_1}{Y_1}\beta\binom{X_2}{Y_2} + \beta\binom{X_1, X_2}{Y_1, Y_2},$$

$$\begin{aligned}\mathfrak{G}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} &= \mathfrak{g}^{il}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} + \mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2}\beta\binom{X_3}{Y_3} \\ &\quad + \mathfrak{g}\binom{X_1}{Y_1}\beta\binom{X_2, X_3}{Y_2, Y_3} + \beta\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3}.\end{aligned}$$

Construction of the $\mathfrak{g}^{il}$

The generating series of bi-brackets of depth $d \geq 1$ can also be written as

$$\mathfrak{g}\binom{X_1,\ldots,X_d}{Y_1,\ldots,Y_d} = \sum_{m_1>\cdots>m_d>0} L_{m_1}\binom{X_1}{Y_1}\cdots L_{m_d}\binom{X_d}{Y_d}$$

where

$$L_m\binom{X}{Y} = \frac{e^{X+mY}q^m}{1-e^X q^m}.$$

In order to build from the series $\mathfrak{g}$ a symmetril series $\mathfrak{g}^{il}$, we will define a symmetril multiple version of the L_m.

Construction of the $\mathfrak{g}^{il}$

Definition

For $d, m \geq 1$, define the power series

$$\beta^R\begin{pmatrix} X_1, \dots, X_d \\ Y_1, \dots, Y_d \end{pmatrix} = \sum_{i=0}^{d} \frac{(-1)^i}{2^i i!} \beta\begin{pmatrix} -X_{i+1}, \dots, -X_d \\ Y_1, \dots, Y_{d-i} \end{pmatrix},$$

$$L_m\begin{pmatrix} X_1, \dots, X_d \\ Y_1, \dots, Y_d \end{pmatrix} = \sum_{j=1}^{d} \beta\begin{pmatrix} X_1 - X_j, \dots, X_{j-1} - X_j \\ Y_1, \dots, Y_{j-1} \end{pmatrix} L_m\begin{pmatrix} X_j \\ Y_1 + \dots + Y_d \end{pmatrix} \cdot \beta^R\begin{pmatrix} X_j - X_d, \dots, X_j - X_{j+1} \\ -Y_d, \dots, -Y_{i+1} \end{pmatrix}.$$

Proposition (Bachmann-B.)

- The series β^R are swap invariant up to signs.
- Up to depth 3, the β^R are symmetril up to signs.
- Up to depth 3, the L_m, $m \geq 1$, are symmetril.

Construction of the $\mathfrak{g}^{il}$

The symmetrility of the L_m, $m \geq 1$, implies:

Proposition

The following sequence of power series is symmetril

$$\mathfrak{g}\binom{X_1}{Y_1} = \sum_{m>0} L_m \binom{X_1}{Y_1},$$

$$\mathfrak{g}^{il}\binom{X_1, X_2}{Y_1, Y_2} = \sum_{m_1>m_2>0} L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2}{Y_2} + \sum_{m>0} L_m \binom{X_1, X_2}{Y_1, Y_2},$$

$$\begin{aligned}\mathfrak{g}^{il}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} &= \sum_{m_1>m_2>m_3>0} L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2}{Y_2} L_{m_3}\binom{X_3}{Y_3} \\ &\quad + \sum_{m_1>m_2>0} \left(L_{m_1}\binom{X_1, X_2}{Y_1, Y_2} L_{m_2}\binom{X_3}{Y_3} + L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2, X_3}{Y_2, Y_3} \right) \\ &\quad + \sum_{m>0} L_m \binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3}.\end{aligned}$$

This conclusion works for all depths. The crucial point in arbitrary depths is to show that the L_m, $m \geq 1$, are symmetril and that the $\mathfrak{G}$ are swap invariant.

Outlook: Formal graded multiple q-zeta values

Definition

Let $\mathcal{Z}_q^f$ be the graded $\mathbb{Q}$-algebra of **formal multiple q-zeta values**, i.e., the generating series of the formal symbols $\zeta_q^f\binom{k_1,\ldots,k_d}{n_1,\ldots,n_d}$ are swap invariant and symmetril and there are no other relations among these formal symbols.

Goal

- Construct a surjective algebra morphism $\mathcal{Z}_q^f \to \mathcal{Z}_q$.
- Show that this morphism is an isomorphism.

Conjecturally, the previously given construction of the generating series $\mathfrak{G}$ works for all depths. Thus, the association

$$\zeta_q^f\begin{pmatrix}k_1,\ldots,k_d\\n_1,\ldots,n_d\end{pmatrix} \mapsto G\begin{pmatrix}k_1,\ldots,k_d\\n_1,\ldots,n_d\end{pmatrix}, \quad d \leq 3,$$

should extend to a surjective algebra morphism $\mathcal{Z}_q^f \to \mathcal{Z}_q$.

Injectivity in general depths seems to be out of reach for the moment, though it is expected to be a lot easier as for MZVs.

Outlook: Formal graded multiple q-zeta values

Having a surjective algebra morphism $\mathcal{Z}_q^f \to \mathcal{Z}_q$, we can hopefully continue analogously to Racinet's work on MZVs:

- Construct dual graded Hopf algebras and reformulate the relations satisfied by the formal qMZV
  $\rightsquigarrow$ Obtain an affine group scheme and a corresponding Lie algebra.
- Prove that the algebra $\mathcal{Z}_q^f$ of formal q-MZVs is a free polynomial algebra.
- Show that the dimension conjectures on qMZVs (Bachmann-Kühn) hold for the formal qMZV.