Combinatorial multiple Eisenstein series

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Multiple Eisenstein series

Building blocks of the algebra of modular forms for $\mathrm{Sl}_2(\mathbb{Z})$ are the Eisenstein series, for $k \geq 4$ even we have

$$\mathbb{G}_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n, \qquad q = e^{2\pi i \tau}$$

$$= \sum_{\lambda \succ 0} \frac{1}{\lambda^k}.$$

Here $\succ$ is an order on $\mathbb{Z}\tau + \mathbb{Z}$ defined by $m_1\tau + n_1 \succ m_2\tau + n_2$, iff $m_1 > m_2$ or $m_1 = m_2$ and $n_1 > n_2$. This allows also to give a multiple version of these objects.

Definition

For $k_1 \geq 3,\ k_2, \ldots, k_r \geq 2$, the **multiple Eisenstein series (MES)** is defined by

$$\mathbb{G}_{k_1,\ldots,k_r}(\tau) = \sum_{\lambda_1 \succ \cdots \succ \lambda_r \succ 0} \frac{1}{\lambda_1^{k_1} \ldots \lambda_r^{k_r}}.$$

Multiple Eisenstein series

It is $\mathbb{G}_{k_1,\dots,k_r}(\tau) = \mathbb{G}_{k_1,\dots,k_r}(\tau+1)$, so MES possess a Fourier expansion

Theorem (Gangl-Kaneko-Zagier, Bachmann)

For all $k_1 \geq 3,\ k_2,\dots,k_r \geq 2$, it is

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \zeta(k_1,\dots,k_r) + \sum_{j=1}^{r-1} \hat{g}^*(k_1,\dots,k_j)\zeta(k_{j+1},\dots,k_r) + \hat{g}^*(k_1,\dots,k_r),$$

for some explicitly defined $\hat{g}^*(k_1,\dots,k_j) \in \mathbb{Q}[\pi i][[q]]$.

The constant term is a multiple version of the Riemann zeta values.

Question

Is there an extension of the $\mathbb{G}_{k_1,\dots,k_r}$ to all $k_1,\dots,k_r \geq 1$ satisfying relations, which can be viewed as a lift of the relations between the $\zeta(k_1,\dots,k_r)$?

To study the question we will introduce combinatorial bi-multiple Eisenstein series, which consider only the 'rational part' and capture the appearing derivatives.

Stuffle product

- Let $Z = \{z_d^k \mid k \geq 1,\ d \geq 0\}$ be an alphabet,
- $\mathbb{Q}\langle Z \rangle$ denotes the free non- commutative algebra over Z,
- Words are monic monomials in $\mathbb{Q}\langle Z \rangle$,
- The empty word is denoted by $\mathbf{1}$.

Definition

We define the **stuffle product** $*$ on $\mathbb{Q}\langle Z \rangle$ recursively by $\mathbf{1} * w = w * \mathbf{1} = w$ and

$$z_{d_1}^{k_1} v * z_{d_2}^{k_2} w = z_{d_1}^{k_1}(v * z_{d_2}^{k_2} w) + z_{d_2}^{k_2}(z_{d_1}^{k_1} v * w) + z_{d_1+d_2}^{k_1+k_2}(v * w).$$

E.g., it is

$$\begin{aligned} z_1^2 z_3^4 * z_5^6 &= z_1^2(z_3^4 * z_5^6) + z_5^6 z_1^2 z_3^4 + z_6^8 z_3^4 \\ &= z_1^2 z_3^4 z_5^6 + z_1^2 z_5^6 z_3^4 + z_1^2 z_8^{10} + z_5^6 z_1^2 z_3^4 + z_6^8 z_3^4. \end{aligned}$$

Theorem (Hoffman)

The pair $(\mathbb{Q}\langle Z \rangle, *)$ is a commutative, associative $\mathbb{Q}$-Hopf algebra with unit.

Combinatorial (bi)-multiple Eisenstein series

Theorem (Bachmann-B., 2022)

There is a family of q-series $G\binom{k_1,\dots,k_r}{d_1,\dots,d_r} \in \mathbb{Q}[[q]]$, $r \geq 1$, satisfying the following properties

- There is an algebra homomorphism
$$(\mathbb{Q}\langle Z\rangle, *) \to (\mathbb{Q}[[q]], \cdot), \qquad z_{d_1}^{k_1} \dots z_{d_r}^{k_r} \mapsto G\binom{k_1,\dots,k_r}{d_1,\dots,d_r}.$$
- For $k \geq 2$ even, the element $G\binom{k}{0}$ is exactly the classical Eisenstein series with rational coefficients.
- For each $r \geq 1$, the generating series of the $G\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$ is swap invariant.

We call these q-series **combinatorial bi-multiple Eisenstein series (CbMES)**. If the lower row equals zero, we denote
$$G(k_1,\dots,k_r) = G\binom{k_1,\dots,k_r}{0,\dots,0}$$
and call these elements **combinatorial multiple Eisenstein series (CMES)**.

Combinatorial (bi-)multiple Eisenstein series: Examples

For $r = 1$, we have the explicit expression

$$G\binom{k}{d} = -\delta_{d,0}\frac{B_k}{2k!} - \delta_{k,1}\frac{B_{d+1}}{2(d+1)} + \frac{1}{(k-1)!}\sum_{m,n\geq 1} m^d n^{k-1} q^{mn}.$$

- For $k > d \geq 0$ and $k - d$ even, this is essentially the d-th derivative of the classical Eisenstein series $G(k-d)$,

$$G\binom{k}{d} = \frac{(k-d-1)!}{(k-1)!}\left(q\frac{d}{dq}\right)^d G(k-d)\,.$$

- The combinatorial Eisenstein series of odd weight $k \geq 3$ are of the form

$$G(k) = \frac{1}{(k-1)!}\sum_{m,n\geq 1} n^{k-1} q^{mn} = \frac{1}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n) q^n.$$

Swap invariance in general

Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra.

Definition

A formal power series $\mathfrak{P} \in \mathcal{A}[[X_1, \ldots, X_r, Y_1, \ldots, Y_r]]$ is called swap invariant if

$$\mathfrak{P}\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} = \mathfrak{P}\begin{pmatrix} Y_1 + \cdots + Y_r, Y_1 + \cdots + Y_{r-1}, \ldots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \ldots, X_2 - X_3, X_1 - X_2 \end{pmatrix}.$$

This substitution of variables is closely related to conjugation of partitions.

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Swap invariance of a power series

$$\mathfrak{P}\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} = \sum_{\substack{k_1, \ldots, k_r \geq 1 \\ d_1, \ldots, d_r \geq 0}} P\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix} X_1^{k_1 - 1} \ldots X_r^{k_r - 1} \frac{Y_1^{d_1}}{d_1!} \ldots \frac{Y_r^{d_r}}{d_r!}$$

gives linear relations among its coefficients. E.g., we obtain for $r = 1, 2$

$$P\begin{pmatrix} k_1 \\ d_1 \end{pmatrix} = \frac{d_1!}{(k_1 - 1)!} P\begin{pmatrix} d_1 + 1 \\ k_1 - 1 \end{pmatrix},$$

$$P\begin{pmatrix} k_1, k_2 \\ d_1, d_2 \end{pmatrix} = \sum_{a=0}^{d_1} \sum_{b=0}^{k_2 - 1} \frac{(-1)^b}{a!\,b!} \frac{d_1!}{(k_1 - 1)!} \frac{(d_2 + a)!}{(k_2 - 1 - b)!} P\begin{pmatrix} d_2 + 1 + a, d_1 + 1 - a \\ k_2 - 1 - b, k_1 - 1 + b \end{pmatrix}.$$

Combinatorial (bi-)multiple Eisenstein series: Algebraic structure

Definition

Define the $\mathbb{Q}$-vector spaces spanned by all combinatorial (bi-)multiple Eisenstein series

$$\mathcal{G}^{bi} = \mathbb{Q} + \left\langle G\begin{pmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{pmatrix} \mid r \geq 1, k_1, \dots, k_r \geq 1, d_1, \dots, d_r \geq 0 \right\rangle_{\mathbb{Q}},$$

$$\mathcal{G} = \mathbb{Q} + \left\langle G(k_1, \dots, k_r) \mid r \geq 1, k_1, \dots, k_r \geq 1 \right\rangle_{\mathbb{Q}}.$$

Theorem

The pairs $(\mathcal{G}^{bi}, q\frac{d}{dq})$ and $(\mathcal{G}, q\frac{d}{dq})$ are differential algebras both containing the algebra of quasi modular forms.

For all $k_1, \dots, k_r \geq 1,\ d_1, \dots, d_r \geq 0$, it is

$$q\frac{d}{dq} G\begin{pmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{pmatrix} = \sum_{i=1}^{r} k_i G\begin{pmatrix} k_1, \dots, k_i + 1, \dots k_r \\ d_1, \dots, d_i + 1, \dots, d_r \end{pmatrix}.$$

Combinatorial bi-multiple Eisenstein series: Weight grading

For a word in $\mathbb{Q}\langle Z\rangle$, define the **weight** by

$$\mathrm{wt}(z_{k_1}^{d_1}\dots z_{k_r}^{d_r}) = k_1+\dots+k_r+d_1+\dots+d_r.$$

Then $(\mathbb{Q}\langle Z\rangle, *)$ is a weight-graded algebra.

Definition

For a combinatorial bi-multiple Eisenstein series $G\begin{pmatrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{pmatrix}$, we define its **weight** to be the number $k_1+\dots+k_r+d_1+\dots+d_r$.

Conjecture

The map

$$(\mathbb{Q}\langle Z\rangle, *)\to(\mathcal{G}^{bi},\cdot), \qquad z_{d_1}^{k_1}\dots z_{d_r}^{k_r}\mapsto G\begin{pmatrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{pmatrix}$$

is a morphism of graded algebras.

- The conjecture holds for the subalgebra of quasi-modular forms.
- The swap invariance gives linear relations among CbMES of the same weight.

Interlude: Multiple zeta values

Definition

- For integers $k_1 \geq 2, k_2, \ldots, k_r \geq 1$, we define the **multiple zeta value (MZV)**

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

  We call the number $k_1 + \cdots + k_r$ its **weight**.

- Let

$$\mathcal{Z} = \mathbb{Q} + \langle \zeta(k_1, \ldots, k_r) | r \geq 1, k_1 \geq 2, k_2, \ldots, k_r \geq 1 \rangle_{\mathbb{Q}},$$

  be the $\mathbb{Q}$-vector space spanned by the MZVs.

Multiple zeta values possess an expression via iterated integrals, e.g., it is

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_3} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}.$$

Interlude: Multiple zeta values

Proposition

The space $\mathcal{Z}$ is an algebra.

The two different expressions of MZVs give two different product formulas, e.g. we have for $r = 2$

$$\zeta(k_1)\zeta(k_2) = \begin{cases} \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2) & \textbf{(stuffle product)} \\ \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1 + k_2 - j) & \textbf{(shuffle product)}. \end{cases}$$

By comparing both formulas, we get the **double shuffle relations**. E.g., it is

$$\left.\begin{aligned} \zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5) \\ &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \end{aligned}\right\} \quad \zeta(5) = 2\zeta(3,2) + 6\zeta(4,1).$$

Interlude: Multiple zeta values

There are more relations between MZVs, which can be not obtained from the double shuffle relations. E.g. it was already known to Euler that

$$\zeta(3) = \zeta(2,1).$$

To capture these relations, one introduces stuffle- and shuffle-regularized multiple zeta values

$$\zeta^*(k_1,\dots,k_r),\ \zeta^{ш}(k_1,\dots,k_r) \in \mathcal{Z}$$

for all $k_1,\dots,k_r \geq 1$. By comparing both regularizations (Ihara-Kaneko-Zagier), we obtain the **extended double shuffle relations (EDS)** between MZVs. E.g., it is

$$\left.\begin{aligned} \zeta^*(2)\zeta^*(1) &= \zeta^*(2,1) + \zeta^*(1,2) + \zeta^*(3) \\ \zeta^{ш}(2)\zeta^{ш}(1) &= 2\zeta^{ш}(2,1) + \zeta^{ш}(1,2) \end{aligned}\right\} \quad \zeta(3) = \zeta(2,1).$$

Conjecture

All algebraic relations among MZVs are a consequence of the extended double shuffle relations. In particular, the algebra $\mathcal{Z}$ is graded by weight.

Combinatorial (bi-)multiple Eisenstein series: limits

Theorem

For any $k_1, \dots, k_r \geq 1$ we have

$$\overset{*}{\lim_{q \to 1}} (1-q)^{k_1+\dots+k_r} G(k_1, \dots, k_r) = \zeta^*(k_1, \dots, k_r).$$

In particular, the combinatorial multiple Eisenstein series are **q-analogs of multiple zeta values**.

There are two product expressions for the CbMES, which reduce under the (regularized) limit $q \to 1$ exactly to the extended double shuffle relations of MZVs. E.g. it is

$$G(3) = G(2,1) + q\frac{q}{dq}G(1) \qquad \overset{q \to 1}{\rightsquigarrow} \qquad \zeta(3) = \zeta(2,1)$$

By construction of the CMES, we also have the following

Proposition

The elements $\lim_{q \to 0} G(k_1, \dots, k_r) \in \mathbb{Q}$ are rational solutions to the extended double shuffle equations.

Bonus slide: Dimorphy of combinatorial bi-multiple Eisenstein series

We have two product expressions, the second one is obtained by using the swap invariance. E.g., we get for $r = 2$

$$G\binom{k_1}{d_1}\,G\binom{k_2}{d_2} = G\binom{k_1, k_2}{d_1, d_2} + G\binom{k_2, k_1}{d_2, d_1} + G\binom{k_1+k_2}{d_1+d_2}$$

$$= \sum_{k=1}^{k_1}\sum_{d=0}^{d_2}\binom{k_1+k_2-k-1}{k_1-k}\binom{d_1+d_2-d}{d_1}(-1)^{d_2-d}\,G\binom{k_1+k_2-k, k}{d, d_1+d_2-d}$$

$$+ \sum_{k=1}^{k_2}\sum_{d=0}^{d_1}\binom{k_1+k_2-k-1}{k_2-k}\binom{d_1+d_2-d}{d_2}(-1)^{d_1-d}\,G\binom{k_1+k_2-k, k}{d, d_1+d_2-d}$$

$$+ \binom{k_1+k_2-2}{k_1-1}\,G\binom{k_1+k_2-1}{d_1+d_2+1}.$$

These equations can be seen as a generalization/lift of the **extended double shuffle relations** of **multiple zeta values**.