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Winter 2017/2018
Exercise class: 30th January 2018

Computer Tomography

Exercise Sheet 7

Exercise 1. Tikhonov regularization and normal equation

Let $A \in K(X,Y)$ be a compact operator between real Hilbert spaces. Show that the Tikhonov-Regularization $R_\alpha g := F_\alpha(A^*A)A^*g$ with the Tikhonov filter

$$F_\alpha(\lambda) = \frac{1}{\lambda+\alpha}, \qquad \alpha > 0, 0 \le \lambda \le \|A\|^2$$

is the unique solution of the regularized normal equation

$$(A^*A + \alpha I) f = A^* g\,.$$

Exercise 2. Tikhonov regularization

Let $A \in K(X,Y)$ be a compact operator between real Hilbert spaces with singular system (σ_j, v_j, u_j) and $g \in \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp = D(A^+)$. Let $g^\delta \in Y$ with $g^\delta \notin D(A^+)$ and f_α^δ the minimizer of the Tikhonov functional $J_\alpha : X \to \mathbb{R}$:

$$f_\alpha^\delta = \mathrm{argmin} J_\alpha(f) \qquad \text{with } J_\alpha(f) = \left\|Af - g^\delta\right\|_Y^2 + \alpha \|f\|_X^2\,.$$

Show the following properties

1. $\Psi(\alpha) := \left\|f_\alpha^\delta\right\|_X^2$ is monotone decreasing in α with
$$\lim_{\alpha\to 0} \Psi(\alpha) = \infty \qquad \lim_{\alpha\to\infty} \Psi(\alpha) = 0\,.$$
2. The residual $r(\alpha) := \left\|Af_\alpha^\delta - g^\delta\right\|_Y^2$ is monotone increasing in α with
$$\lim_{\alpha\to 0} r(\alpha) = \left\|P_{\overline{\mathcal{R}(A)}} g^\delta - g^\delta\right\|^2 \qquad \lim_{\alpha\to\infty} r(\alpha) = \left\|g^\delta\right\|^2\,.$$
3. The value of the Tikhonov functional $J_\alpha\left(f_\alpha^\delta\right)$ at the minimizer f_α^δ is monotone increasing in α with
$$\lim_{\alpha\to 0} J_\alpha\left(f_\alpha^\delta\right) = \left\|P_{\overline{\mathcal{R}(A)}} g^\delta - g^\delta\right\|^2 \qquad \lim_{\alpha\to\infty} J_\alpha\left(f_\alpha^\delta\right) = \left\|g^\delta\right\|^2\,.$$
4. Compare $\Psi'(\alpha)$ and $r'(\alpha)$ in order to show that $\frac{\mathrm{d}\Psi}{\mathrm{d}r}(\alpha)$ is negative for all $\alpha > 0$.

Exercise 3. Tikhonov-regularization and parameter selection
Consider again the integral operator $A : L^2(0,1) \to L^2(0,1)$,

$$Af(x) = \int_0^x f(t)\,\mathrm{d}t\,.$$

and its discretization (see exercise 6.5). Moreover, let the functions

$$f_1(x) = \operatorname{sign}\left(x - \frac{1}{2}\right) \qquad \text{and } f_2(x) = \sin(\pi x)$$

be given.

1. Create for the function f_2 and $N = 300$ noisy data g^δ such that the difference to the real data $g = Af_2$ is 5 %, i.e. it should hold that

$$\frac{\|g - g^\delta\|}{\|g\|} = 0.05\,.$$

2. Implement the Tikhonov-regularization. Determine the regularization parameter α visually.

3. Determine now the regularization parameter with help of the discrepancy principle of Morozov under the assumption that the noise level is bounded by $\delta = 0.4$. Plot both solutions and compare them to the original function.

4. Repeat all steps for the function f_1.

5* Determine the regularization parameter with help of the L-curve. Therefore, take a sequence of candidates of regularization parameter $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_M$ and plot the points

$$\left(\log\left\|AR_\alpha\left(g^\delta\right) - g^\delta\right|, \log\left\|R_\alpha g^\delta\right|\right)\,.$$

The curve has typically the shape of the letter L with a smooth corner. The optimal regularization parameter α is thought to be found as near the corner as possible.

Exercise * Reconstruction from real Xray Data
Solve the following limited angle X-ray problem: The noisy sinogram g (30 projections) and the observation matrix $A \in \mathbb{R}^{4920\times 26896}$ can be downloaded from Stine (XrayData.mat). Reconstruct the unknown phantom on the grid of size 164×164 pixel. What do you see?
Remark: In the case that you have memory problems, use the second data set instead, where the image resolution is reduced (82×82 pixel) such that the matrix is smaller.