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Exercise class: 20th December 2018

Inverse Problems

Exercise Sheet 9

Exercise 1. $L = A^*A + tB^+B$ **is continuously invertible.**
Let be $L \in \mathcal{L}(X)$ with $\langle Lx, x\rangle \geq \lambda \|x\|^2$ for all $x \in X$ and $\lambda > 0$. Then L is continuously invertible with $\|L^{-1}\| \leq 1/\lambda$.
Hint: In order to show that L is surjective, consider the function $\Psi(x) = x - \kappa(Lx - y)$, $y \in X$ and $\kappa > 0$ and show that

$$\|\Psi(x) - \Psi(z)\|^2 \leq \left(1 - 2\kappa\lambda + \kappa^2 \|L\|^2\right) \|x - z\|^2 \,,$$

i.e. for $\kappa \in (0, 2\lambda / \|L\|^2)$ the map Ψ is a contraction. Now you can apply the Banach fixed point theorem.

Exercise 2. n-times iterated Tikhonov regularization
Let $A \in \mathcal{K}(X, Y)$, $n \in \mathbb{N}$ and $R_{t,n} : Y \to X$, $R_{t,n}g = F_{t,n}(A^*A) A^*g$ the n-times iterated Tikhonov regularization with filter

$$F_{t,n}(\lambda) = \frac{(\lambda + t)^n - t^n}{\lambda(\lambda + t)^n} \,. \tag{1}$$

Show for $f_{t,n} = R_{t,n}g$

1. the characterization

$$f_{t,n} = \arg \min_{f \in \mathcal{N}(A)^\perp} \left(\|Af - g\|_Y^2 + t\|f - f_{t,n-1}\|^2\right) \,.$$

2. the recursive representation

$$\begin{aligned} f_{t,0} &= 0\,, \\ (A^*A + tI) f_{t,n} &= A^*g + tf_{t,n-1}\,. \end{aligned}$$

Exercise 3. Qualification of the n-times iterated Tikhonov regularization
Show that the filter (1) has the qualification $\mu_0 = 2n$.
Hint: Check the determination of the qualification of the classical Tikhonov regularization.

Exercise 4. Generalized Tikhonov-regularization with L-curve criteria
Consider again the integral operator $A : L^2(0,1) \to L^2(0,1)$,

$$Af(x) = \int_0^x f(t)\,\mathrm{d}t\,.$$

and its discretization (see exercise 7.4) as well as the functions

$$f_1(x) = \operatorname{sign}\left(x - \frac{1}{2}\right) \qquad \text{and } f_2(x) = \sin(\pi x)\,.$$

1. Create for the function f_2 and $N = 300$ noisy data g^δ such that the difference to the real data $g = Af_2$ is 5 %, i.e. it should hold that

$$\frac{\left\|g - g^\delta\right\|}{\|g\|} = 0.05\,.$$

2. Implement the generalized Tikhonov-regularization with differential operator B. Therefore, use a discretized differential operator by forward differences. Compare the result with the classical Tikhonov - regularization.

3. Determine the regularization parameter with help of the L-curve. Therefore, take a sequence of candidates of regularization parameter $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_M$ and plot the points

$$\left(\log\left\|AR_\alpha\left(g^\delta\right) - g^\delta\right\|, \log\left\|BR_\alpha g^\delta\right\|\right)\,.$$

The curve has typically the shape of the letter L with a smooth corner. The optimal regularization parameter α is thought to be found as near the corner as possible.

4. Repeat all steps for the function f_1.

Remark: Please prepare one Matlab file which executes your whole implementation (including plots etc) and sent it via mail with subject "Inverse problems"at the latest on Thursday 9am to me (christina.brandt@uni-hamburg.de). Please name your file in the form ex09_xy.m, where xy is your family name.