Infinite matroid theory exercise sheet 3

1. (a) Let $M_k|k \in K)$ be a family of matroids on the ground set E. Prove that there is a matroid $\bigvee_{i \in I} M_i$ whose independent sets are exactly the sets of the form $\bigcup_{k \in K} I_k$, where each I_k is independent in M_k. What is the rank function of $\bigvee_{k \in K} M_k$?

   (b) Deduce the tree covering theorem for graphs.

2. (a) Deduce the case $k = 2$ of the tree packing theorem for graphs from the matroid union theorem.

   (b)* Deduce the tree packing theorem for graphs from the matroid union theorem.

Questions 4 and 5 from sheet 2 remain open.

Reminder: Some theorems from graph theory

Theorem 0.1 (Packing theorem: Nash-Williams, Tutte)**.** *A finite multigraph has k edge-disjoint spanning trees if and only if every partition P of its vertex set has at least $k(|P|-1)$ crossing edges.*

Here a *crossing edge* is one with endvertices in different partition classes.

Theorem 0.2 (Covering theorem: Nash-Williams)**.** *A finite multigraph $G = (V, E)$ can be partitioned into at most k forests if and only if $e(U) \leq k(|U| - 1)$ for every nonempty set $U \subseteq V$.*

Here $e(U)$ denotes the number of edges with both endvertices in U.