Infinite matroid theory exercise sheet 2

1. Let M be a matroid, $X \subseteq E(M)$, and let $\mathcal{C}(X) = \{C \setminus X | C \in \mathcal{C}(M)\}$. Show that $\mathcal{C}(X)$ satisfies $(C3)$. By exercise 2 of the previous sheet, this means that the set of minimal nonempty elements of $\mathcal{C}(X)$ is the set of circuits of some matroid. Which matroid?

2. Let M be a matroid with rank function r. Define $r^* : \mathcal{P}(E) \to \mathbb{Z}_{\geq 0}$ via $r^*(X) = |X| - r(E) + r(E \setminus X)$. Show directly that r^* satisfies the rank axioms. Then prove that r^* is the rank function of the dual M^* of M.

3. Let $E = E_1 \dot\cup E_2$ be a partition of the ground set. Show that the following are equivalent.

   (a) $M/E_1 = M \backslash E_1$.

   (b) $M/E_2 = M \backslash E_2$.

   Conversely, let M_1 and M_2 be two matroids with (disjoint) ground sets E_1 and E_2. Show that there is a unique matroid M on the disjoint union $E_1 \sqcup E_2$ of E_1 and E_2 such that $M_1 = M/E_2 = M \backslash E_2$ and $M_2 = M/E_1 = M \backslash E_1$.

4. Let N be a minor of M, that is, there are disjoints sets $C, D \subseteq E(M)$ such that $N = M/C \backslash D$. Show that there are disjoint sets C' and D' with $N = M/C' \backslash D'$ and such that C' is M-independent and D' is M^*-independent.

5.* Let M be a matroid with two bases B_1 and B_2. Show that for any $x \in B_1$ there is some $y \in B_2$ such that both $B_1 - x + y$ and $B_2 - y + x$ are bases of M.

Hints

Concerning question 5:

Think about fundamental circuits and cocircuits.